S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 20, No. 2, pp. 144–145 (2023) УДК ???.??
https://doi.org/10.33048/semi.2023.20.??? MSC ??X??

M-DMP INVERSE OF RECTANGULAR MATRICES IN MINKOWSKI SPACE

D. P. SHENOY

Communicated by P.P. PETROV

Abstract: Gao, Wang and Zuo defined m-DMP inverse for a square matrix. In the same article they discussed about the sope of extending the m-DMP inverse for a rectangular matrix. Here we extend m-DMP inverse for a rectangular matrix and defines W-weighted m - DMP inverse. Some characterizations of W-weighted m - DMP inverse also has been discussed.In this paper, we develop the m-DMP inverse for a rectangular matrix and define the W-weighted m-DMP inverse. Some characterizations of W-weighted m-DMP inverse have also been discussed.

Keywords: Moore Penrose inverse, Minkowski inverse, Minkowski space, generalized inverse, determinant.

1 Introduction

Recently, many new generalized inverses have been defined by combining the Drazin inverse, core inverse and Moore Penrose inverse etc., and these inverses help to solve various system of equations. Malik and Thome [4] defined the DMP inverse of a square matrix A as $A^d AA^\dagger$ where A^d is the Drazin inverse of A and $A^\dagger$ is the Moore Penrose inverse of A. The DMP inverse, denoted by $A^{d,\dagger}$ extends the notion of core inverse. Mehdipur and Salemi [6] defined CMP inverse and its characterizations from the core part of the core nilpotent decomposition of a matrix A and the Moore Penrose inverse $A^\dagger$. There are several generalized inverses in literature, such as Star

Received , Published .

Drazin matrices [9], Theta Drazin matrices [11], Outer theta matrices [10]. The notion of DMP inverse was extended to rectangular matrices by Meng [7]. For the extension of CMP inverse over rectangular matrices, one can refer to [8].

The Minkowski inverse and s-g inverse are similar to Moore penrose inverse. However, unlike Moore Penrose inverse, the existence of Minkowski inverse is assured only when $rank(AA^{\sim}) = rank(A^{\sim}A) = rank(A)$, where $A^{\sim}$ represents the Minkowski adjoint [5]. For more properties and characterizations of the s-g inverse, refer to [13]. Gao. et. al [3] defined m-DMP inverse for a square matrix in Minkowski space. We were inspired by this idea, and extended the m-DMP inverse to rectangular matrices and termed it as W-weighted m-DMP inverse. Some properties of the W-weighted m-DMP inverse are discussed here. Also, we obtain a canonical form of W-weighted m-DMP inverse. We also demonstrate that the W-weighted m-DMP inverse is different from W-weighted DMP inverse using relevant examples.

Before moving to main results, let us examine some preliminary results and notations.

2 Preliminaries

Throughout this paper, we consider $\mathbb{C}^{m\times n}$ to be the set of all $m \times n$ matrices over a complex field. The space of complex m-tuples is denoted by $\mathbb{C}^m$. The components of a complex vector in $\mathbb{C}^m$ can be indexed from 0 to $m-1$, i.e., $v = (v_0, v_1, v_2, ..., v_{m-1})$. Let B be the Minkowski tensor defined by

$$Bv = (v_0, -v_1, -v_2, ..., -v_{m-1})$$

The Minkowski metric matrix is defined by
$B = \begin{pmatrix} 1 & 0 \\ 0 & -I_{m-1} \end{pmatrix}; B^* = B, B^2 = I_m.$
The Minowski inner product [5, 12] in $\mathbb{C}^m$ is defined by $(u, v) = [u, Bv]$ where $[.,.]$ denotes the conventional Hilbert (unitary) space inner product. A space with Minkowski inner product is called a Minkowski space and is denoted by $\mathcal{M}$.
For $A \in \mathbb{C}^{n\times n}, x, y \in \mathbb{C}^m$

$$(Ax, y) = [Ax, By] = [x, A^*By] = [x, B(BA^*B)y] = [x, BA^{\sim}y] = (x, A^{\sim}y)$$

where $A^{\sim} = BA^*B$. Here, $A^{\sim}$ is called the Minkowski adjoint of A in $\mathcal{M}$.

Definition 1. [1] *The Moore Penrose inverse $X \in \mathbb{C}^{n\times m}$ of $A \in \mathbb{C}^{m\times n}$ is a unique matrix satisfying the following four conditions:*
(1) X is an inner inverse, $AXA = A$
(2) X is an outer inverse, $XAX = X$
(3) AX is symmetric with the involution $*$, the complex conjugate operator.
(4) XA is symmetric with respect to $*$.

Definition 2. [1] *The Drazin inverse of a square matrix* $A \in \mathbb{C}^{n\times n}$ *is a unique matrix* $X = A^D$ *satisfying the following conditions*

$$(1)\ XAX = X \qquad (2)\ AX = XA \qquad (3)\ A^{k+1}X = A^k$$

where k *is index of* A.

Definition 3. [4] *The DMP inverse* $A^{D,\dagger} = X$ *for* $A \in \mathbb{C}^{n\times n}$ *is a unique solution for the system of equations*

$$(1)\ XAX = X \qquad (2)\ XA = A^D A \qquad (3)\ A^k X = A^k A^\dagger$$

where k *is the index of* A.

Definition 4. [5] *For a matrix* $A \in \mathbb{C}^{m\times n}$, *the matrix* $X \in \mathbb{C}^{n\times m}$ *satisfying the set of equations*

(1) $AXA{=}A$
(2) $XAX{=}X$
(3) $(AX)^{\sim} = AX$, *i.e.,* AX *is* $\mathcal{M}$- *symmetric*
(4) $(XA)^{\sim} = XA$, *i.e.,* XA *is* $\mathcal{M}$- *symmetric*

is known as the Minkowski inverse of A *denoted by* A^m. *Such a matrix,* X, *if exists, will be unique.*

Definition 5. [3] *Consider* $A \in \mathbb{C}_k^{n\times n}$ *with* $rk(A^{\sim}AA^{\sim}) = rk(A)$. *The m-DMP inverse of* A *in Minkowski space is denoted by* $A^{D,m}$ *and defined as* $A^{D,m} = A^D AA^m$.

Theorem 1. [12] *The matrix* P *has the decomposition of the form* $P = QDR$ *with* Q *and* R *orthogonal and* D *diagonal, if and only if the following conditions hold:*

(1) P^+P *has nonnegative, real eigen values.*
(2) P^+P *is diagonalizable.*
(3) *The null space of* P^+P *is same as the null space of* P.

Here $P^+ = BA^*B$ where B is defined as above.

3 Results

First we define the m-DMP inverse for rectangular matrices. Note that the definition is valid only when $rk(A) = rk(AA^{\sim}) = rk(A^{\sim}A)$, where $rk(A)$ represents the rank of the matrix A.

Theorem 2. *Consider* $A \in \mathbb{C}^{m\times n}$ *and* $W \in \mathbb{C}^{n\times m}$. *Whenever* $rk(A) = rk(AA^{\sim}) = rk(A^{\sim}A)$, *there exists a unique matrix* $G = WA_{D,w}WAA^m$ *that satisfies the following conditions*

$$GAG = G, \quad GA = WA_{D,w}WA \quad \text{and} \quad (WA)^{p+1}G = (WA)^{p+1}A^m \quad (1)$$

where $p = ind(AW)$.

Proof. By the definition of $A_{D,w}$,

$$WA_{D,w}WAA^{m}AWA_{D,w}WAA^{m} = WA_{D,w}WAA^{m}$$

$$WA_{D,w}WAA^{m}A = WA_{D,w}WA$$

and

$$(WA)^{p+1}WA_{D,w}WAA^{m} = W(AW)^{p+1}A_{D,w}WAA^{m} = (WA)^{p+1}A^{m}$$

Therefore, $G = WA_{D,w}WAA^{m}$ satisfies all the conditions of equation 1. Assume that G_1 and G_2 are two solutions of equation 1. Then,

$$\begin{aligned} G_1 &= G_1AG_1 = WA_{D,w}WAG_1 = (WA_{D,w})^2(WA)^2G_1 \\ &= (WA_{D,w})^{p+1}(WA)^{p+1}G_1 = (WA_{D,w})^{p+1}(WA)^{p+1}A^{m} \\ &= (WA_{D,w})^{p+1}(WA)^{p+1}G_2 = WA_{D,w}WAG_2 = G_2AG_2 = G_2. \end{aligned}$$

Hence the uniqueness. □

Definition 6. *For any $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$, the W-weighted m-DMP inverse is denoted by $A_W^{D,\sim}$ and is defined as $A_W^{D,\sim} = WA_{D,w}WAA^{m}$.*

Note that the W-weighted m-DMP inverse which we define here is different from the W-weighted DMP invere defined by [7]. The existence of the former is not assured whereas the latter always exists.
Consider the example:

Example 1. *Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Here, $A_{D,w} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A^{\dagger} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A^{\sim} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$*

The W-weighted DMP inverse is $A_w^{D,\dagger} = WA_{D,w}WAA^{\dagger} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
However, in this case, the W-weighted m-DMP inverse does not exists, since $rk(AA^{\sim}) \neq rk(A^{\sim}A)$.
The following example demostrates the existence of the W-weighted m-DMP inverse that is different from the W-weighted DMP inverse.
Consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ and $W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
Now, $A_{D,w} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$, $A^{m} = \begin{pmatrix} \frac{-1}{3} & \frac{2}{3} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A^{\dagger} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.
Observe that the W-weighted DMP inverse is $A_w^{D,\dagger} = WA_{D,w}WAA^{\dagger} =$

$\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ *whereas the W-weighted m-DMP inverse is* $A_w^{D,\sim} = WA_{D,w}WAA^{\sim} =$
$\begin{pmatrix} \frac{-1}{3} & \frac{2}{3} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

Renardy [12] has obtained the singular value decomposition of a matrix over a Minkowski space. This particular result helps us to obtain a canonical form of W-weighted m-DMP inverse of A.
The matrix A and W has a singular value decomposition only when
(1) The eigen values of $A^{\sim}A$ as well as $W^{\sim}W$ are real and non negative.
(2) Both $A^{\sim}A$ and $W^{\sim}W$ are diagonalizable.
(3) The null space $\mathcal{N}(A^{\sim}A) = \mathcal{N}(A)$ and $\mathcal{N}(W^{\sim}W) = \mathcal{N}(W)$.
We assume that these three results hold true.

Consider $A \in \mathbb{C}_r^{m\times n}$ and $W \in \mathbb{C}_s^{n\times m}$ respectively with the following singular value decompositions

$$A = P\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}Q \quad \text{and} \quad W = U\begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix}V \tag{2}$$

where $P = (P_1, P_2), V = (V_1, V_2) \in \mathbb{C}^{m\times m}$ and $Q = (Q_1, Q_2), U = (U_1, U_2) \in \mathbb{C}^{n\times n}$ are orthogonal matrices. Here, $P_1 \in \mathbb{C}^{m\times r}, V_1 \in \mathbb{C}^{m\times s}, Q_1 \in \mathbb{C}^{n\times r}, U_1 \in \mathbb{C}^{n\times s}, D_1 = diag(\sigma_1, ...\sigma_r),\ \sigma_1 \geq ... \geq \sigma_r > 0$ and $D_2 = diag(\mu_1, ...\mu_s), \mu_1 \geq ... \geq \mu_s > 0$.

Theorem 3. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *with the singular value decompositions as above. Then*

$$A_w^{D,\sim} = U\begin{pmatrix} D_2S_{11}\Lambda D_2S_{11} & 0 \\ 0 & 0 \end{pmatrix}P^{\sim} \tag{3}$$

where $S_{11} = V_1^{\sim}P_1$ *and* $\Lambda = (D_1T_{11})_{d,D_2S_{11}}$ *with* $T_{11} = Q_1^{\sim}U_1$. *Here,* $(D_1T_{11})_{d,D_2S_{11}}$ *is the* D_1S_{11} *weighted Drazin inverse of matrix* D_1T_{11}.

Proof. Denote $S = VP,\ T = QU$. Let us assume that S and T can be represented in the form of block matrices as given below.

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

and

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

where $S_{11} \in \mathbb{C}^{s\times r}$ and $T_{11} \in \mathbb{C}^{r\times s}$. Note that $S^{\sim}S$ is an identity matrix of order m and $T^{\sim}$ is an identity matrix of order m. i.e., S and T are orthogonal matrices with the involution operation Minkowski adjoint. Then we have

$$A = P\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} Q = P\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} QUU^{\sim} = P\begin{pmatrix} D_1T_{11} & D_1T_{12} \\ 0 & 0 \end{pmatrix} U^{\sim}$$

and

$$W = U\begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix} V = P\begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix} VPP^{\sim} = U\begin{pmatrix} D_2S_{11} & D_2S_{12} \\ 0 & 0 \end{pmatrix} P^{\sim}$$

Let

$$G = P\begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} U^{\sim}$$

be the W-weighted Drazin inverse of A with the involution operator as Minkowski adjoint. Here $G_1 \in \mathbb{C}^{r\times s}$. Then, from the conditions of W-weighted Drazin inverse,

$$G_1 = G_1D_2S_{11}D_1T_{11}D_2S_{11}G_2, G_1D_2S_{11}D_1T_{11} = D_1T_{11}D_2S_{11}G_1$$

$$G_2 = G_1D_2S_{11}G_1D_2S_{11}D_1T_{12}, G_3 = 0, G_4 = 0,$$

$$(D_1T_{11}D_2S_{11})^{p+1}G_1D_2S_{11} = (D_1T_11D_2S_{11})^p$$

and

$$(D_1T_{11}D_2S_{11})^{p+1}G_1D_2S_{12} = (D_1T_{11}D_2S_{11})^{p-1}D_1T_{11}D_2S_{12}.$$

These equations prove that $A_{D,w}$ as given below is W-weighted Drazin inverse of A.

$$A_{D,w} = P\begin{pmatrix} \Lambda & \Lambda D_2S_{11}\Lambda D_2S_{11}D_1T_{12} \\ 0 & 0 \end{pmatrix} U^{\sim}$$

where $\Lambda = (D_1T_{11})_{d,D_2S_{11}}$.

Now, it can be verified that, $A^m = U\begin{pmatrix} T^{\sim}D_1^{-1} & 0 \\ T^{\sim}D_1^{-1} & 0 \end{pmatrix} P^{\sim}$.

Thus, we have the W-weighted m-DMP inverse

$$A_w^{D,\sim} = WA_{D,w}WAA^m = U\begin{pmatrix} D_2S_{11}\Lambda D_2S_{11} & 0 \\ 0 & 0 \end{pmatrix} P^{\sim} \tag{4}$$

□

4 Properties of W-weighted m-DMP inverse

Theorem 4. *Consider two matrices, $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. Then the following statements hold whenever $A^{\sim}$ exists.*

(1) $AA_w^{D,\sim}$ *is a projector onto* $\mathcal{R}(AWA_{D,w})$ *along* $\mathcal{N}(A_{D,w}A^{\sim})$.

(2) $A_w^{D,\sim}A$ *is a projector onto* $\mathcal{R}((WA)^p)$ *along* $\mathcal{N}((WA)^p)$ *where* $p = ind(WA)$.

Proof. (1) $A_w^{D,\sim}$ is an 2-inverse of A since

$$A_w^{D,\sim} A A_w^{D,\sim} = WA_{D,w}WAA^m AWA_{D,w}WAA^m = A_w^{D,\sim} A A_w^{D,\sim}.$$

Hence, $AA_w^{D,\sim}$ is a projector.

Now,

$AA_w^{D,\sim} = AWA_{D,w}WAA^m$ and $AWA_{D,w} = (AWA_{D,w}AA^m)AWA_{D,w}$ gives that $\mathcal{R}(AA_w^{D,\sim}) \subseteq \mathcal{R}(AWA_{D,w})$. Also $rk(AA_w^{D,\sim}) = rk(AWA_{D,w})$ which together implies $\mathcal{R}(AA_w^{D,\sim}) = \mathcal{R}(AWA_{D,w})$. Similarly, $\mathcal{N}(A)A_w^{D,\sim} = \mathcal{N}(A_w^{D,\sim}A^m)$ can be proved.
(2) This can be proved using the result $WA_{D,w}WA = P_{\mathcal{R}((WA)^p, \mathcal{N}(WA)^p)}$ [2] and $A_w^{D,\sim}A = WA_{D,w}WA$.
Hence the proof. □

Theorem 5. *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. Assume that $rk(AA^\sim) = rk(A) = rk(A^\sim A)$. Then there exists a matrix*

$$AG = P_{\mathcal{R}(AWA_{D,w}), \mathcal{N}(A_{D,w}A^m)}, \mathcal{R}(G) \subseteq \mathcal{R}((WA)^p) \tag{5}$$

where $p = ind(WA)$.

Proof. By the previous theorem, $A_w^{D,\sim}$ is a solution to equation (5). To prove the theorem, it is enough to prove that (5) has a unique solution.
Assume that G_1 and G_2 are solutions of (5).
Now, $A(G_1 - G_2) = 0$, $\mathcal{R}(G_1) \subseteq \mathcal{R}((WA)^p)$ and $\mathcal{R}(G_2) \subseteq \mathcal{R}((WA)^p)$.
Hence, $\mathcal{R}(G_1 - G_2) \subseteq \mathcal{R}((WA)^p) \cap \mathcal{N}((WA)^p) = \{0\}$. Thus $G_1 = G_2$. □

5 Conclusion

This paper defines W-weighted m-DMP inverse in Minkowski space and discusses its properties. It will be interesting to explore the determinantal representation as well as the iterative methods for W-weighted m-DMP inverse. Applications of this inverse in solving linear system of equations are worth studying.

References

[1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Springer, New York, NY, USA, 2003.

[2] R.E. Cline, T.N.E. Greville, *A Drazin inverse for rectangular matrices*, Linear Algebra Appl. **29** (1980), 53—62.

[3] J. Gao, Q. Wang, K. Zuo , *The m-DMP inverse in Minkowski space and its applications*, arXiv:2303.13944, (2023).

[4] S. B. Malik and N. Thome, *On a new generalized inverse for matrices of an arbitrary index*, Applied Mathematics and Computation, **226** (2014), 575–580.

[5] A. R. Meenakshi, *Range symmetric matrices in Minkowski space*, Bull. Malays. Math. Sci. Soc. (Second series), **23(1)** (2000), 45–52.

[6] M. Mehdipour and A. Salemi, *On a new generalized inverse of matrices*, Linear and Multilinear Algebra, **66(5)**(2017), 1046 – 1053

[7] L. Meng, *The DMP inverse for rectangular matrices*, Filomat, **31(19)** (2017), 6015—6019.

[8] D. Mosič, *The CMP inverse for rectangular matrices*, Aequationes mathematicae,**92(6)** (2018), 649 – 659.

[9] D. Mosič, *Drazin-Star and Star-Drazin Matrices*, Results in Mathematics, **75(2)**, (2020), 1 – 21.

[10] D. P. Shenoy, *Outer theta and theta outer inverses*, IAENG International Journal of Applied Mathematics, **52(4)** (2022), 1020–1024.

[11] D. P. Shenoy, *Drazin theta and theta Drazin matrices*, Numerical Algebra, Control and Optimization, (Article in press) d

[12] M. Renardy, *Singular value decomposition in Minkowski space*, Linear Algebra and Its Applications, **236** (1996), 53–58.

[13] S. Varkady, D. P. Shenoy, U. Kelathaya and R. B. Bapat, *Secondary transpose of matrix and generalized inverses*, Journal of Algebra and Its Applications, **23(3)**, 2450052, (2024),. (Article in press).

Divya Shenoy Purushothama
Department of Mathematics,
Manipal Institute of Technology, Manipal Academy of Higher Education, manipal-576104, Karnataka, India
Email address: divya.shenoy@manipal.edu