The referee's report on the paper 'Properties of moments of density for nonlocal mean field game equations with a quadratic cost' by O.S. Rozanova and M.V. Inyakin

The paper is concerned with the study of the mean field game system in the case using analytical tools when the dynamics of a sampling player includes drift part and jump diffusion process. The main feature of the examined model is the assumption that only drift is controlled by the player, while that quality of the control is evaluated as

$$\mathbb{E}\int_0^T \left(\frac{|\alpha_s|^2}{2} + g(t, X_s, m_s)ds + K(X_T)\right),$$

where α_s is a control at time s, X_s stands for a state of the sapling player, while m_s is a distribution of all players. Recall that in this case the mean field game system consists of two integral differential equations where the first one is a Hamilton-Jacobi-Bellman equation (HJB) that describes the value of a sampling player, whereas the second is a Fokker-Planck-Kolmogorov (FPK) equation that determines a distribution of all players. Notice that the mean field concept implies that this is also a distribution of states of a sampling player.

Under assumption that the interaction term $g = a(t)|x|^2 + b(t)x + c(t)$, where the coefficients can depend on not only time but also on expectation and variance of X_s (denoted in the paper by $\mathbb{E}(X_s)$ and $\mathbb{V}(X_s)$ respectively), the authors use the ansaltz that the solution of HJB equation has the same form and reduces the FPK equation (and, thus, the whole MFG system) to a system of ODEs those provides the coefficients of the Fourier transform of the solution of FPK equation.

My comments are concentrated on two issues.

- The first is the dependence of coefficients of the interaction term g on expectation and variance of X_t. In fact, introducing the model, the authors say that this dependence is assumed. However, it seems to me that throughout the paper they treat the functions a, b and c depending only on time. I encourage the authors to clarify this point. Here, I am to say that, if g does not depend on m, the MFG contains no interaction between the players and this case has no meaning from the mean field game point of view.
- Secondly, the questions on existence of the boundary value problem for ODEs arising within the analysis of FPK equation bother me.

Comments

1. I suggest to indicate precisely in (5) that the coefficients a, b and c depend not only on time but also on $\mathbb{E}(X_s)$ and $\mathbb{V}(X_s)$.
2. In (8) the dependence of a and b on $\mathbb{E}(X_s)$ and $\mathbb{V}(X_s)$ should be added. Moreover, the first equation in (8) has a quadratic growth w.r.t. A. This implies generally only short time existence. Could the authors clarify it?
3. It seems to me that the function c is missed in formula for C (see the third equation in (8)).

4. Please, indicate in (9) that A and B depend on $\mathbb{E}(X_s)$ and $\mathbb{V}(X_s)$. The same concerns also (10).

5. In (11), (12) and (13), the coefficients should also depend on expectation and variance those take the following forms:

$$\mathbb{E}(X_s) = \int_{\mathbb{R}^d} x m_s(x) dx, \quad \mathbb{V}(X_s) = \int_{\mathbb{R}^d} x^2 m_s(x) dx - (\mathbb{E}(X_s))^2.$$

   This makes the equation examined below nonlinear.

6. It is not clear to me why boundary problems (16), (17) and (18) have solutions. Could the authors add some explanation?

7. I think that the example on page 150 gives a good opportunity to discuss the influence of the interaction (that now should be in the function c) on the solution of the MFG system. I think that this discussion will strengthen the paper.

Evaluating the whole paper, I think that the paper provides an interesting case of analytically solved MFG system. However, I encourage the authors to clarify the issues concerning the interaction terms and existence of solution of ODEs.