SYNCHRONIZING CLASSICAL AND DYNAMIC VARIATIONS CONFORMED ON TIME SCALES

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Abstract. We establish several generalizations of reverses of Callebaut's, Rogers–Hölder's and Cauchy–Schwarz's inequalities via reverses of Young's inequalities on time-scales. Discrete, continuous, quantum versions of results are unified and extended on time-scales.

1. Introduction

The calculus concerning intervals of time-scales was accomplished by Hilger [6]. A time-scale (closed interval) is considered a domain (arbitrary and nonempty) of real functions. $\mathbb{T}$ denotes a time-scale and $[t_1, t_2]_{\mathbb{T}} = [t_1, t_2] \cap \mathbb{T}$, while $t_1, t_2 \in \mathbb{T}$ with $t_1 < t_2$. The major aim of the calculus of time-scales is to establish results in general, comprehensive, unified, harmonized, reconciled and extended forms. This theory (hybrid in nature) is extensively utilized on dynamic variations, see [9–12]. The basic ideas about this hybrid theory are presented in the monographs [2, 3].

We state here the different versions of reverses of Callebaut's, Rogers–Hölder's and Cauchy–Schwarz's inequalities, see [5].

Let $f_k > 0$, $g_k > 0$ and $w_k \geq 0$ for any $k \in \mathbb{N} = \{1, 2, \ldots, \eta\}$ with $\sum\limits_{k\in\mathbb{N}} w_k = 1$. Then

$$\begin{aligned} 2\beta\left(\sum_{k\in\mathbb{N}} w_k f_k^2 \sum_{k\in\mathbb{N}} w_k g_k^2 - \left(\sum_{k\in\mathbb{N}} w_k f_k g_k\right)^2\right) &\\ \leq \sum_{k\in\mathbb{N}} w_k x_k^2 \sum_{k\in\mathbb{N}} w_k g_k^2 - \sum_{k\in\mathbb{N}} w_k f_k^{2(1-\delta)} g_k^{2\delta} \sum_{k\in\mathbb{N}} w_k f_k^{2\delta} g_k^{2(1-\delta)} &\\ \leq 2\gamma\left(\sum_{k\in\mathbb{N}} w_k f_k^2 \sum_{k\in\mathbb{N}} w_k g_k^2 - \left(\sum_{k\in\mathbb{N}} w_k f_k g_k\right)^2\right), \quad (1) \end{aligned}$$

where $\delta \in [0, 1]$, $\beta = \min\{1-\delta, \delta\}$ and $\gamma = \max\{1-\delta, \delta\}$.

If $\frac{f_k}{g_k} \in [m, M]$ for any $k \in \mathbb{N} = \{1, 2, \ldots, \eta\}$, then the upcoming result is satisfied:

$$\sum_{k\in\mathbb{N}} w_k f_k^2 \sum_{k\in\mathbb{N}} w_k g_k^2 - \sum_{k\in\mathbb{N}} w_k f_k^{2(1-\delta)} g_k^{2\delta} \sum_{k\in\mathbb{N}} w_k f_k^{2\delta} g_k^{2(1-\delta)} \leq \gamma(M-m)^2 \left(\sum_{k\in\mathbb{N}} w_k g_k^2\right)^2, \quad (2)$$

where $\gamma = \max\{1-\delta, \delta\}$.

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Let $p \in (1, +\infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$2\beta \left(\sqrt{\sum_{k\in\mathbb{N}} w_k f_k^p \sum_{k\in\mathbb{N}} w_k g_k^q} - \sum_{k\in\mathbb{N}} w_k f_k^{\frac{p}{2}} g_k^{\frac{q}{2}} \right) \left(\sum_{k\in\mathbb{N}} w_k f_k^p \right)^{\frac{1}{p}-\frac{1}{2}} \left(\sum_{k\in\mathbb{N}} w_k g_k^q \right)^{\frac{1}{q}-\frac{1}{2}}$$
$$\leq \left(\sum_{k\in\mathbb{N}} w_k f_k^p \right)^{\frac{1}{p}} \left(\sum_{k\in\mathbb{N}} w_k g_k^q \right)^{\frac{1}{q}} - \sum_{k\in\mathbb{N}} w_k f_k g_k$$
$$\leq 2\gamma \left(\sqrt{\sum_{k\in\mathbb{N}} w_k f_k^p \sum_{k\in\mathbb{N}} w_k g_k^q} - \sum_{k\in\mathbb{N}} w_k f_k^{\frac{p}{2}} g_k^{\frac{q}{2}} \right) \left(\sum_{k\in\mathbb{N}} w_k f_k^p \right)^{\frac{1}{p}-\frac{1}{2}} \left(\sum_{k\in\mathbb{N}} w_k g_k^q \right)^{\frac{1}{q}-\frac{1}{2}}, \quad (3)$$

where $\beta = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $\gamma = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

2. Preliminaries

First, we shortly explain $\diamond_\alpha$-derivative [1, 13].

Let us consider $\Phi(t)$ to be Δ-differentiable and ∇-differentiable on $\mathbb{T}$. Then $\diamond_\alpha$-derivative $\Phi^{\diamond_\alpha}(t)$ for $t \in \mathbb{T}$ is formulated by

$$\Phi^{\diamond_\alpha}(t) = \alpha \Phi^{\Delta}(t) + (1-\alpha)\Phi^{\nabla}(t), \ \alpha \in [0, 1].$$

We note that $\diamond_\alpha$-derivative takes the form of Δ-derivative for $\alpha = 1$, or it becomes the ∇-derivative for $\alpha = 0$.

The following definition is given in [13].

Let $t_1, t_2 \in \mathbb{T}$. Further $\Psi : \mathbb{T} \to \mathbb{R}$. The $\int_{t_1}^{t_2} \diamond_\alpha$ is formulated by

$$\int_{t_1}^{t_2} \Psi(\theta) \diamond_\alpha \theta = \alpha \int_{t_1}^{t_2} \Psi(\theta)\Delta\theta + (1-\alpha) \int_{t_1}^{t_2} \Psi(\theta)\nabla\theta, \ 0 \leq \alpha \leq 1,$$

satisfying the conditions that Δ-integral and ∇.

The following well-known Young's inequality holds:

For $\Phi, \Psi > 0$ and $\delta \in [0, 1]$, we have

$$\Phi^{1-\delta}\Psi^{\delta} \leq (1-\delta)\Phi + \delta\Psi. \quad (4)$$

Manasrah and Kittaneh [7, 8] proved a refined and Young's reverse inequality:

$$\beta \left(\sqrt{\Phi} - \sqrt{\Psi}\right)^2 \leq (1-\delta)\Phi + \delta\Psi - \Phi^{1-\delta}\Psi^{\delta} \leq \gamma \left(\sqrt{\Phi} - \sqrt{\Psi}\right)^2, \quad (5)$$

where $\Phi, \Psi > 0$, $\delta \in [0, 1]$, $\beta = \min\{1-\delta, \delta\}$ and $\gamma = \max\{1-\delta, \delta\}$.

The following estimate is given in [5]. We deduce that, if $\Phi, \Psi \in [m, M]$, while $m, M > 0$ then $\left|\sqrt{\Phi} - \sqrt{\Psi}\right| \leq \sqrt{M} - \sqrt{m}$ and by (5). The reverse Young inequality states

$$(1-\delta)\Phi + \delta\Psi - \Phi^{1-\delta}\Psi^{\delta} \leq \gamma \left(\sqrt{M} - \sqrt{m}\right)^2. \quad (6)$$

The following inequality is given in [4]. Let $\delta \in [0, 1]$ and $\Phi, \Psi > 0$. Then

$$(1-\delta)\Phi + \delta\Psi \leq K^{\gamma}(L)\Phi^{1-\delta}\Psi^{\delta}, \quad (7)$$

where $0 < L^{-1} \leq \frac{\Phi}{\Psi} \leq L < \infty$, $L > 1$ and $\gamma = \max\{1-\delta, \delta\}$.

The following inequality is given in [4]. Let $\delta \in [0, 1]$ and $\Phi, \Psi > 0$. Then

$$(1-\delta)\Phi + \delta\Psi \leq \max\left\{K^{\gamma}(l), K^{\gamma}(L)\right\} \Phi^{1-\delta}\Psi^{\delta}, \quad (8)$$

where $0 < l^{-1} \leq \frac{\Phi}{\Psi} \leq L < \infty$, for some L, $l > 0$ with $Ll > 1$ and $\gamma = \max\{1-\delta, \delta\}$.

In the upcoming results, we shall assume the existence of all integrals and they are finite.

3. Main Results

Now, we form an extended reverse dynamic Callebaut's inequality on time-scales. Throughout the section, we assume that neither $f \equiv 0$ nor $g \equiv 0$.

Theorem 1. *Let $f, g, w \in C([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ be three functions and are $\diamond_\alpha$-integrable. Let $\delta \in [0,1]$. Then*

$$\begin{aligned}
2\beta & \left[\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma - \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)||g(\varsigma)| \diamond_\alpha \varsigma\right)^2\right] \\
& \leq \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma \\
& \quad - \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta} \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^{2\delta}|g(\varsigma)|^{2(1-\delta)} \diamond_\alpha \varsigma \\
& \leq 2\gamma \left[\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma - \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)||g(\varsigma)| \diamond_\alpha \varsigma\right)^2\right], \quad (9)
\end{aligned}$$

where $\beta = \min\{1-\delta, \delta\}$ and $\gamma = \max\{1-\delta, \delta\}$.

Proof. Let $\Phi(\varsigma) = \frac{|f(\varsigma)|^2}{|g(\varsigma)|^2}$ and $\Psi(\tau) = \frac{|f(\tau)|^2}{|g(\tau)|^2}$ for any $\varsigma, \tau \in [t_1, t_2]_{\mathbb{T}}$. Then using (5), we have

$$\begin{aligned}
\beta \left(\frac{|f(\varsigma)|}{|g(\varsigma)|} - \frac{|f(\tau)|}{|g(\tau)|}\right)^2 & \leq (1-\delta)\frac{|f(\varsigma)|^2}{|g(\varsigma)|^2} + \delta\frac{|f(\tau)|^2}{|g(\tau)|^2} - \left(\frac{|f(\varsigma)|^2}{|g(\varsigma)|^2}\right)^{1-\delta}\left(\frac{|f(\tau)|^2}{|g(\tau)|^2}\right)^{\delta} \\
& \leq \gamma \left(\frac{|f(\varsigma)|}{|g(\varsigma)|} - \frac{|f(\tau)|}{|g(\tau)|}\right)^2. \quad (10)
\end{aligned}$$

If we multiply inequality (10) by $|g(\varsigma)|^2|g(\tau)|^2$, for any $\varsigma, \tau \in [t_1, t_2]_{\mathbb{T}}$, then we get

$$\begin{aligned}
\beta & \left(|f(\varsigma)|^2|g(\tau)|^2 - 2|f(\varsigma)g(\varsigma)||f(\tau)g(\tau)| + |g(\varsigma)|^2|f(\tau)|^2\right) \\
& \leq (1-\delta)|f(\varsigma)|^2|g(\tau)|^2 + \delta|g(\varsigma)|^2|f(\tau)|^2 - |f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta}|f(\tau)|^{2\delta}|g(\tau)|^{2(1-\delta)} \\
& \leq \gamma \left(|f(\varsigma)|^2|g(\tau)|^2 - 2|f(\varsigma)g(\varsigma)||f(\tau)g(\tau)| + |g(\varsigma)|^2|f(\tau)|^2\right). \quad (11)
\end{aligned}$$

Multiplying (11) by $|w(\varsigma)|$ and operating the integral over ς from t_1 to t_2, we obtain

$$\begin{aligned}
\beta & \Bigg(|g(\tau)|^2 \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma - 2|f(\tau)g(\tau)| \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma \\
& \quad + |f(\tau)|^2 \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma\Bigg) \\
& \leq (1-\delta)|g(\tau)|^2 \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma + \delta|f(\tau)|^2 \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma \\
& \quad - |f(\tau)|^{2\delta}|g(\tau)|^{2(1-\delta)} \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta} \diamond_\alpha \varsigma \\
& \leq \gamma \Bigg(|g(\tau)|^2 \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma - 2|f(\tau)g(\tau)| \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma \\
& \quad + |f(\tau)|^2 \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma\Bigg). \quad (12)
\end{aligned}$$

Again, multiplying (12) by $|w(\tau)|$ and operating the integral over τ from t_1 to t_2, we obtain the desired inequality (9). □

In the following, we generate another reconciliation of reverse dynamic Callebaut's inequality.

Theorem 2. *Let $f, g, w \in C([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ be three functions and are $\diamond_\alpha$-integrable. Consider the hypothesis $0 < m \le \frac{|f(\varsigma)|}{|g(\varsigma)|} \le M < \infty$ on $[t_1, t_2]_{\mathbb{T}}$. Let $\delta \in [0, 1]$. Then*

$$\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma - \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta} \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^{2\delta}|g(\varsigma)|^{2(1-\delta)} \diamond_\alpha \varsigma \le \gamma(M-m)^2 \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma\right)^2, \quad (13)$$

where $\gamma = \max\{1-\delta, \delta\}$.

Proof. For any $\varsigma, \tau \in [t_1, t_2]_{\mathbb{T}}$, it is clear that

$$m^2 \le \frac{|f(\varsigma)|^2}{|g(\varsigma)|^2}, \frac{|f(\tau)|^2}{|g(\tau)|^2} \le M^2.$$

Let $\Phi(\varsigma) = \frac{|f(\varsigma)|^2}{|g(\varsigma)|^2}$ and $\Psi(\tau) = \frac{|f(\tau)|^2}{|g(\tau)|^2}$ for any $\varsigma, \tau \in [t_1, t_2]_{\mathbb{T}}$. Then using the inequality (6), we have

$$(1-\delta)\frac{|f(\varsigma)|^2}{|g(\varsigma)|^2} + \delta\frac{|f(\tau)|^2}{|g(\tau)|^2} - \left(\frac{|f(\varsigma)|^2}{|g(\varsigma)|^2}\right)^{1-\delta}\left(\frac{|f(\tau)|^2}{|g(\tau)|^2}\right)^{\delta} \le \gamma(M-m)^2. \quad (14)$$

If we multiply inequality (14) by $|g(\varsigma)|^2|g(\tau)|^2$, for any $\varsigma, \tau \in [t_1, t_2]_{\mathbb{T}}$, then we get

$$(1-\delta)|f(\varsigma)|^2|g(\tau)|^2 + \delta|g(\varsigma)|^2|f(\tau)|^2 - |f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta}|f(\tau)|^{2\delta}|g(\tau)|^{2(1-\delta)} \le \gamma(M-m)^2|g(\varsigma)|^2|g(\tau)|^2. \quad (15)$$

Multiplying (15) by $|w(\varsigma)|$ and operating the integral over ς from t_1 to t_2, we obtain

$$(1-\delta)|g(\tau)|^2 \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma + \delta|f(\tau)|^2 \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma - |f(\tau)|^{2\delta}|g(\tau)|^{2(1-\delta)} \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^{2(1-\delta)}|g(\varsigma)|^{2\delta} \diamond_\alpha \varsigma \le \gamma(M-m)^2|g(\tau)|^2 \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma. \quad (16)$$

Again, multiplying (16) by $|w(\tau)|$ and operating the integral over τ from t_1 to t_2, we arrive (13). □

Now, we generate a unification of reverse dynamic Cauchy–Shwarz's inequality.

Corollary 1. *Let $f, g, w \in C([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ be three functions and are $\diamond_\alpha$-integrable. Consider the hypothesis $0 < m \le \frac{|f(\varsigma)|}{|g(\varsigma)|} \le M < \infty$ on the set $[t_1, t_2]_{\mathbb{T}}$. Then*

$$\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma \int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma - \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)||g(\varsigma)| \diamond_\alpha \varsigma\right)^2 \le \frac{1}{2}(M-m)^2 \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma\right)^2. \quad (17)$$

Proof. Put $\delta = \frac{1}{2}$ in Theorem 2 and hence the result is obvious. □

Remark 1. ($\mathbb{T}=\mathbb{N}$). *With suitable substitutions on t_1, t_2 and α, we arrive at*

$$\int_1^{\eta+1} \cdot \diamond_1 \lambda = \sum_{k=1}^{\eta} \cdot.$$

Let $f(k)=f_k>0$, $g(k)=g_k>0$ and $w(k)=w_k\geq 0$ for any $k\in\mathbb{N}=\{1,2,\dots,\eta\}$ with $\sum_{k\in\mathbb{N}} w_k=1$. Then (9) recaptures (1) and inequality (13) recaptures (2).

Now, we give an extension of reverse Rogers–Hölder's dynamic inequality.

Theorem 3. *Let $f,g,w\in C([t_1,t_2]_{\mathbb{T}},\mathbb{R})$ be three functions and are $\diamond_\alpha$-integrable. Consider $\frac{1}{p}+\frac{1}{q}=1$ with $p\in(1,+\infty)$. Then*

$$\begin{aligned}
&2\beta\left[\sqrt{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}-\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{\frac{p}{2}}|g(\varsigma)|^{\frac{q}{2}}\diamond_\alpha\varsigma\right]\\
&\qquad\times\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\right)^{\frac{1}{p}-\frac{1}{2}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma\right)^{\frac{1}{q}-\frac{1}{2}}\\
&\leq\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma\right)^{\frac{1}{q}}-\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)g(\varsigma)|\diamond_\alpha\varsigma\\
&\leq 2\gamma\left[\sqrt{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}-\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^{\frac{p}{2}}|g(\varsigma)|^{\frac{q}{2}}\diamond_\alpha\varsigma\right]\\
&\qquad\times\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\right)^{\frac{1}{p}-\frac{1}{2}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma\right)^{\frac{1}{q}-\frac{1}{2}},
\end{aligned}\tag{18}$$

where $\beta=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $\gamma=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Proof. Let $\delta=\frac{1}{q}$, $\Phi(\varsigma)=\frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma}$ and $\Psi(\varsigma)=\frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}$ on the set $\in[t_1,t_2]_{\mathbb{T}}$. Then using the inequality (5), we have

$$\begin{aligned}
&\beta\left(\frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma}+\frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}\right.\\
&\qquad\left.-2\frac{|w(\varsigma)||f(\varsigma)|^{\frac{p}{2}}|g(\varsigma)|^{\frac{q}{2}}}{\sqrt{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}}\right)\\
&\leq\frac{1}{p}\frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma}+\frac{1}{q}\frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}\\
&\quad-\frac{|w(\varsigma)||f(\varsigma)||g(\varsigma)|}{\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma\right)^{\frac{1}{q}}}\\
&\leq\gamma\left(\frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma}+\frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}\right.\\
&\qquad\left.-2\frac{|w(\varsigma)||f(\varsigma)|^{\frac{p}{2}}|g(\varsigma)|^{\frac{q}{2}}}{\sqrt{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}}\right).
\end{aligned}\tag{19}$$

Integrating (19) with respect to ς from t_1 to t_2, we obtain the desired inequality (18). $\square$

Remark 2. ($\mathbb{T}=\mathbb{N}$). *With suitable substitutions on t_1, t_2 and α, we arrive at*

$$\int_1^{\eta+1} \cdot \diamond_1 \lambda = \sum_{k=1}^{\eta} \cdot .$$

Let $f(k)=f_k>0$, $g(k)=g_k>0$ and $w(k)=w_k\geq 0$ for any $k\in\mathbb{N}=\{1,2,\ldots,\eta\}$ with $\sum_{k\in\mathbb{N}} w_k = 1$. Then inequality (18) recaptures (3).

Now, we give another extension of reverse Rogers–Hölder's inequality on time-scales.

Theorem 4. *Let $f,g,w\in C([t_1,t_2]_{\mathbb{T}},\mathbb{R})$ be three functions and are $\diamond_\alpha$-integrable, while satisfying $\int_{t_1}^{t_2}|w(\varsigma)|\diamond_\alpha\varsigma=1$. Consider the hypothesis $0<m\leq|f(\varsigma)|\leq M<\infty$ and $0<n\leq|g(\varsigma)|\leq N<\infty$ on the set $[t_1,t_2]_{\mathbb{T}}$. Let $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. Then*

$$\begin{aligned}
0 &\leq \left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma\right)^{\frac{1}{q}}-\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)g(\varsigma)|\diamond_\alpha\varsigma\\
&\leq \gamma\left(\max\left\{\left(\frac{M}{m}\right)^{\frac{p}{2}},\left(\frac{N}{n}\right)^{\frac{q}{2}}\right\}-\min\left\{\left(\frac{m}{M}\right)^{\frac{p}{2}},\left(\frac{n}{N}\right)^{\frac{q}{2}}\right\}\right)^2\\
&\quad\times\left(\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma\right)^{\frac{1}{p}}\left(\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma\right)^{\frac{1}{q}}, \qquad (20)
\end{aligned}$$

where $\gamma=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Proof. Using the given conditions, for all $\varsigma\in[t_1,t_2]_{\mathbb{T}}$, we have

$$m^p\leq|f(\varsigma)|^p\leq M^p \text{ and } n^q\leq|g(\varsigma)|^q\leq N^q,$$

which imply that

$$\left(\frac{m}{M}\right)^p\leq\frac{|f(\varsigma)|^p}{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma}\leq\left(\frac{M}{m}\right)^p$$

and

$$\left(\frac{n}{N}\right)^q\leq\frac{|g(\varsigma)|^q}{\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}\leq\left(\frac{N}{n}\right)^q.$$

Therefore,

$$\begin{aligned}
\min\left\{\left(\frac{m}{M}\right)^p,\left(\frac{n}{N}\right)^q\right\}&\leq\frac{|f(\varsigma)|^p}{\int_{t_1}^{t_2}|w(\varsigma)||f(\varsigma)|^p\diamond_\alpha\varsigma},\frac{|g(\varsigma)|^q}{\int_{t_1}^{t_2}|w(\varsigma)||g(\varsigma)|^q\diamond_\alpha\varsigma}\\
&\leq\max\left\{\left(\frac{M}{m}\right)^p,\left(\frac{N}{n}\right)^q\right\}. \qquad (21)
\end{aligned}$$

Using the inequality (6) for $\delta = \frac{1}{q}$, $\Phi(\varsigma) = \frac{|f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma}$, $\Psi(\varsigma) = \frac{|g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}$, we get

$$\begin{aligned}
&\frac{1}{p}\frac{|f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma} + \frac{1}{q}\frac{|g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma} \\
&\qquad - \frac{|f(\varsigma)g(\varsigma)|}{\left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma\right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma\right)^{\frac{1}{q}}} \\
&\qquad \le \gamma \left(\max\left\{ \left(\frac{M}{m}\right)^{\frac{p}{2}}, \left(\frac{N}{n}\right)^{\frac{q}{2}} \right\} - \min\left\{ \left(\frac{m}{M}\right)^{\frac{p}{2}}, \left(\frac{n}{N}\right)^{\frac{q}{2}} \right\} \right)^2. \qquad (22)
\end{aligned}$$

Multiplying (22) by $|w(\varsigma)|$ and operating the integral over ς from t_1 to t_2, we obtain

$$\begin{aligned}
&1 - \frac{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma}{\left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma\right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma\right)^{\frac{1}{q}}} \\
&\qquad \le \gamma \left(\max\left\{ \left(\frac{M}{m}\right)^{\frac{p}{2}}, \left(\frac{N}{n}\right)^{\frac{q}{2}} \right\} - \min\left\{ \left(\frac{m}{M}\right)^{\frac{p}{2}}, \left(\frac{n}{N}\right)^{\frac{q}{2}} \right\} \right)^2. \qquad (23)
\end{aligned}$$

This completes the proof of Theorem 4. □

In the upcoming result, we deduce reverse time-scales Cauchy–Schwarz's inequality.

Corollary 2. *Let $f, g, w \in C([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ be three functions and are $\diamond_\alpha$-integrable, while satisfying $\int_{t_1}^{t_2} |w(\varsigma)| \diamond_\alpha \varsigma = 1$. Consider the hypothesis $0 < m \le |f(\varsigma)| \le M < \infty$ and $0 < n \le |g(\varsigma)| \le N < \infty$ on the set $[t_1, t_2]_{\mathbb{T}}$. Then the following inequalities hold true:*

$$\begin{aligned}
0 &\le \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma\right)^{\frac{1}{2}} - \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma \\
&\le \frac{1}{2} \left(\max\left\{ \frac{M}{m}, \frac{N}{n} \right\} - \min\left\{ \frac{m}{M}, \frac{n}{N} \right\} \right)^2 \\
&\quad \times \left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma\right)^{\frac{1}{2}}. \qquad (24)
\end{aligned}$$

Proof. Take $p = q = 2$ in Theorem 4 and the result is obvious. □

Remark 3. ($\mathbb{T} = \mathbb{N}$). *With suitable substitutions on t_1, t_2 and α, we arrive at*

$$\int_1^{\eta+1} \cdot \diamond_1 \lambda = \sum_{k=1}^{\eta} \cdot.$$

Let $f(k) = f_k > 0$, $g(k) = g_k > 0$ and $w(k) = w_k \ge 0$ for any $k \in \mathbb{N} = \{1, 2, \ldots, \eta\}$. Then (20) recaptures [5]

$$\begin{aligned}
0 &\le \left(\sum_{k\in\mathbb{N}} w_k f_k^p\right)^{\frac{1}{p}} \left(\sum_{k\in\mathbb{N}} w_k g_k^q\right)^{\frac{1}{q}} - \sum_{k\in\mathbb{N}} w_k f_k g_k \\
&\le \gamma \left(\max\left\{ \left(\frac{M}{m}\right)^{\frac{p}{2}}, \left(\frac{N}{n}\right)^{\frac{q}{2}} \right\} - \min\left\{ \left(\frac{m}{M}\right)^{\frac{p}{2}}, \left(\frac{n}{N}\right)^{\frac{q}{2}} \right\} \right)^2 \left(\sum_{k\in\mathbb{N}} w_k f_k^p\right)^{\frac{1}{p}} \left(\sum_{k\in\mathbb{N}} w_k g_k^q\right)^{\frac{1}{q}} \qquad (25)
\end{aligned}$$

and (24) *reduces to inequality [5]*

$$0 \leq \left(\sum_{k\in\mathbb{N}} w_k f_k^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{N}} w_k g_k^2\right)^{\frac{1}{2}} - \sum_{k\in\mathbb{N}} w_k f_k g_k$$
$$\leq \gamma \left(\max\left\{\frac{M}{m}, \frac{N}{n}\right\} - \min\left\{\frac{m}{M}, \frac{n}{N}\right\}\right)^2 \left(\sum_{k\in\mathbb{N}} w_k f_k^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{N}} w_k g_k^2\right)^{\frac{1}{2}}. \quad (26)$$

Upcoming version of time-scales reverse Rogers–Hölder's inequality states.

Theorem 5. *Let $f, g, w \in C([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ be three functions and are $\diamond_\alpha$-integrable, while $\int_{t_1}^{t_2} |w(\varsigma)| \diamond_\alpha \varsigma = 1$. Consider the hypothesis $0 < m \leq |f(\varsigma)| \leq M < \infty$ and $0 < n \leq |g(\varsigma)| \leq N < \infty$ on the set $[t_1, t_2]_{\mathbb{T}}$. Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Then*

$$\left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma\right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma\right)^{\frac{1}{q}}$$
$$\leq K^\gamma \left(\left(\frac{M}{m}\right)^p \left(\frac{N}{n}\right)^q\right) \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma, \quad (27)$$

where $\gamma = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Proof. Using the given conditions, for all $\varsigma \in [t_1, t_2]_{\mathbb{T}}$, we have

$$m^p \leq |f(\varsigma)|^p \leq M^p \text{ and } n^q \leq |g(\varsigma)|^q \leq N^q,$$

which imply that

$$\left(\frac{m}{M}\right)^p \leq \frac{|f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma} \leq \left(\frac{M}{m}\right)^p \quad (28)$$

and

$$\left(\frac{n}{N}\right)^q \leq \frac{|g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma} \leq \left(\frac{N}{n}\right)^q. \quad (29)$$

Therefore,

$$\left[\left(\frac{M}{m}\right)^p \left(\frac{N}{n}\right)^q\right]^{-1} \leq \left(\frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma}\right) \left(\frac{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}{|w(\varsigma)||g(\varsigma)|^q}\right) \leq \left(\frac{M}{m}\right)^p \left(\frac{N}{n}\right)^q. \quad (30)$$

Using the inequality (7) for $\delta = \frac{1}{q}$, $\Phi(\varsigma) = \frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma}$, $\Psi(\varsigma) = \frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}$ and $L = \left(\frac{M}{m}\right)^p \left(\frac{N}{n}\right)^q$, we get

$$\frac{1}{p} \frac{|w(\varsigma)||f(\varsigma)|^p}{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma} + \frac{1}{q} \frac{|w(\varsigma)||g(\varsigma)|^q}{\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma}$$
$$\leq K^\gamma(L) \frac{|w(\varsigma)||f(\varsigma)g(\varsigma)|}{\left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma\right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma\right)^{\frac{1}{q}}}. \quad (31)$$

Integrating (31) with respect to ς from t_1 to t_2, we obtain

$$1 \leq K^\gamma(L) \frac{\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma}{\left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma\right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma\right)^{\frac{1}{q}}}. \quad (32)$$

This completes the proof of Theorem 5. □

We deduce reverse time-scales Cauchy–Schwarz's inequality.

Corollary 3. *Let $f, g, w \in C([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ be three functions and are $\diamond_\alpha$-integrable, while satisfying $\int_{t_1}^{t_2} |w(\varsigma)| \diamond_\alpha \varsigma = 1$. Consider the hypothesis $0 < m \leq |f(\varsigma)| \leq M < \infty$ and $0 < n \leq |g(\varsigma)| \leq N < \infty$ on the set $[t_1, t_2]_{\mathbb{T}}$. Then*

$$\left(\int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)|^2 \diamond_\alpha \varsigma\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |w(\varsigma)||g(\varsigma)|^2 \diamond_\alpha \varsigma\right)^{\frac{1}{2}} \leq K^{\frac{1}{2}} \left(\left(\frac{MN}{mn}\right)^2\right) \int_{t_1}^{t_2} |w(\varsigma)||f(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma. \quad (33)$$

Proof. Take $p = q = 2$ in Theorem 5 and the result is obvious. □

Remark 4. $(\mathbb{T} = \mathbb{N})$. *With suitable substitutions on t_1, t_2 and α, we arrive at*

$$\int_1^{\eta+1} \cdot \diamond_1 \lambda = \sum_{k=1}^{\eta} \cdot.$$

Let $f(k) = f_k > 0$, $g(k) = g_k > 0$ and $w(k) = w_k \geq 0$ for any $k \in \mathbb{N} = \{1, 2, \ldots, \eta\}$. Then (27) *recaptures [4]*

$$\left(\sum_{k\in\mathbb{N}} w_k f_k^p\right)^{\frac{1}{p}} \left(\sum_{k\in\mathbb{N}} w_k g_k^q\right)^{\frac{1}{q}} \leq K^\gamma \left(\left(\frac{M}{m}\right)^p \left(\frac{N}{n}\right)^q\right) \sum_{k\in\mathbb{N}} w_k f_k g_k, \quad (34)$$

and, in particular, inequality (33) *reduces to inequality*

$$\left(\sum_{k\in\mathbb{N}} w_k f_k^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{N}} w_k g_k^2\right)^{\frac{1}{2}} \leq K^{\frac{1}{2}} \left(\left(\frac{MN}{mn}\right)^2\right) \sum_{k\in\mathbb{N}} w_k f_k g_k. \quad (35)$$

Another version of time-scales reverse Rogers–Hölder's inequality states.

Theorem 6. *Let $w, u_1, u_2, f, g \in C([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ be five functions and are $\diamond_\alpha$-integrable. Consider the hypothesis $0 < m \leq |f(\varsigma)| \leq M < \infty$ and $0 < n \leq |g(\varsigma)| \leq N < \infty$ on the set $[t_1, t_2]_{\mathbb{T}}$. Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Then the following inequalities hold true:*

$$\begin{aligned}
&\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)f(\varsigma)| \diamond_\alpha \varsigma\right) \left(\int_{t_1}^{t_2} |w(\varsigma)||u_2(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma\right) \\
&\leq \frac{1}{p} \left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma\right) \left(\int_{t_1}^{t_2} |w(\varsigma)||u_2(\varsigma)| \diamond_\alpha \varsigma\right) \\
&\quad + \frac{1}{q} \left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)| \diamond_\alpha \varsigma\right) \left(\int_{t_1}^{t_2} |w(\varsigma)||u_2(\varsigma)||g(\varsigma)|^q \diamond_\alpha \varsigma\right) \\
&\leq \max\left\{K^\gamma \left(\frac{N^q}{m^p}\right), K^\gamma \left(\frac{M^p}{n^q}\right)\right\} \left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)f(\varsigma)| \diamond_\alpha \varsigma\right) \left(\int_{t_1}^{t_2} |w(\varsigma)||u_2(\varsigma)g(\varsigma)| \diamond_\alpha \varsigma\right),
\end{aligned} \quad (36)$$

where $\gamma = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Proof. For any $\varsigma, \tau \in [t_1, t_2]_{\mathbb{T}}$, it is clear that

$$\frac{m^p}{N^q} \leq \frac{|f(\varsigma)|^p}{|g(\tau)|^q} \leq \frac{M^p}{n^q}. \quad (37)$$

Let $\delta = \frac{1}{q}$, $\Phi(\varsigma) = |f(\varsigma)|^p$, $\Psi(\tau) = |g(\tau)|^q$ for any $\varsigma, \tau \in [t_1, t_2]_{\mathbb{T}}$, $l = \frac{N^q}{m^p}$ and $L = \frac{M^p}{n^q}$. Then using the inequalities (4) and (8), respectively, we have

$$|f(\varsigma)||g(\tau)| \le \frac{1}{p}|f(\varsigma)|^p + \frac{1}{q}|g(\tau)|^q \le \max\left\{K^\gamma\left(\frac{N^q}{m^p}\right), K^\gamma\left(\frac{M^p}{n^q}\right)\right\}|f(\varsigma)||g(\tau)|. \tag{38}$$

Multiplying by $|w(\varsigma)||u_1(\varsigma)|$ and integrating (38) with respect to ς from t_1 to t_2, we obtain

$$\begin{aligned}
&\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)f(\varsigma)| \diamond_\alpha \varsigma\right)|g(\tau)| \\
&\qquad \le \frac{1}{p}\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)||f(\varsigma)|^p \diamond_\alpha \varsigma\right) + \frac{1}{q}\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)| \diamond_\alpha \varsigma\right)|g(\tau)|^q \\
&\qquad \le \max\left\{K^\gamma\left(\frac{N^q}{m^p}\right), K^\gamma\left(\frac{M^p}{n^q}\right)\right\}\left(\int_{t_1}^{t_2} |w(\varsigma)||u_1(\varsigma)f(\varsigma)| \diamond_\alpha \varsigma\right)|g(\tau)|. \qquad (39)
\end{aligned}$$

Multiplying by $|w(\tau)||u_2(\tau)|$ and integrating (39) with respect to τ from t_1 to t_2, we obtain the desired inequality (36). □

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