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APPROXIMATING FORMULAE

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Abstract: The notion of a approximating formula is introduced. Links of approximating formulae and their spectra are studied. Semilattices, lattices, and Boolean algebras related to approximating formulae are found. Rank values of approximating formulae are studied. Relations between approximating formulae and positive, negative, ∀-formulae, ∃-formulae, ∃∀-formulae, ∀∃-formulae are considered. Families of consistent formulae are considered and their approximability is characterized. The notion of totally approximating sentence is introduced and families of these sentences are characterized in terms of cardinalities of models and cardinalities of signatures.

Keywords: approximating formula, approximation of theory.

1 Introduction

Formulae of first-order logic are used to reflect an information for structures and related semantic and syntactic objects including theories and their families. Some properties can be described by formulae, including properties in general [1], properties for abelian groups [2, 3], the forcing of infinite structures [4], kinds of axiomatizations for finitely axiomatizable theories

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[5], properties for pseudofinite theories [6, 7], etc. The notion of pseudofinite formula, adapted for pseudofinite theories, was introduced and studied in [8].

In the present paper we continue to study formulae and their properties. We introduce the general notion of approximating formula, generalizing the notion of pseudofinite formula, and describe various possibilities for approximating formulae, algebras on sets of these formulae. We study links between approximating and pseudofinite formulae. We consider positive, negative, $\forall$-formulae, $\exists$-formulae, $\exists\forall$-formulae, and $\forall\exists$-formulae, and describe links between these formulae and approximating ones. Connections of approximating formulae and ranks for families of theories are studied.

The paper is organized as follows. In Section 2, preliminary notions, notations, and assertions are represented.

In Section 3, the notion of approximating formula is introduced and related classes of formulae are considered. Links of approximating formulae, pseudofinite formulae and pseudofinite theories are found. A characterizations is proved, in terms of signature, showing when pseudofinite formulae are approximating, and vice versa. The set of approximating sentences is expressed in terms of accumulation points of given family of theories. The property of finite axiomatizability of a theory in given family is characterized in terms of approximating sentences.

In Section 4, semilattices, lattices, and Boolean algebras related to approximating sentences are studied and described. Their links with RS-rank are shown. Finite Boolean algebras for approximating sentences are characterized in terms of e-minimality. It is shown that these finite cardinalities are unbounded for any e-minimal family.

In Section 5, links of approximating formulae with their quantification are shown. It is proved that universal, existential and some other formulae, including positive and negative, with infinite models are approximating.

In Section 6, the notion of totally approximating formula is introduced. It is shown that these formulae have infinite RS-rank. The property of total approximability is characterized in terms of signature for formulae describing finite cardinalities of models.

2 Preliminaries

Throughout we consider complete first-order theories in a signature Σ, and that signature for a family $\mathcal{T}$ of theories is denoted by $\Sigma(\mathcal{T})$. Some results can be spread for incomplete theories and we postulate their incompleteness.

As usual we denote by $F(\Sigma)$ the set of all formulae in a signature Σ, and by $\mathrm{Sent}(\Sigma)$ the set of all sentences in the signature Σ.

Historically, the class of pseudofinite fields was defined in the work by J. Ax [6] and regardless of him in the work by Yu.L. Ershov [9] allowing to approximate in the class of fields. A general approximation approach for theories was developed in [10].

Definition. [10] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be $\mathcal{T}$*-approximated,* or *approximated by the family* $\mathcal{T}$, or a *pseudo-*$\mathcal{T}$*-theory,* if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ such that $\varphi \in T'$.

If the theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and the theories $T' \in \mathcal{T}$ are said to be *approximations* for T. We put $\mathcal{T}_\varphi = \{T \in \mathcal{T} \mid \varphi \in T\}$. Any set $\mathcal{T}_\varphi$ is called the φ*-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$. An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite. It was shown in [10] that any *e-minimal* family $\mathcal{T}$ has unique accumulation point T with respect to neighbourhoods $\mathcal{T}_\varphi$, and $\mathcal{T} \cup \{T\}$ is also called *e-minimal*.

Recall that the *E-closure* $\mathrm{Cl}_E(\mathcal{T})$ [11] for a family $\mathcal{T}$ of complete theories is characterized by the following proposition.

Proposition 1. *$\mathcal{T}$ be a family of complete theories of a signature Σ. Then $\mathrm{Cl}_E(\mathcal{T}) = \mathcal{T}$ for finite $\mathcal{T}$, and for infinite $\mathcal{T}$, a theory T belongs to $\mathrm{Cl}_E(\mathcal{T})$ if and only if T is a complete theory of the signature Σ, and $T \in \mathcal{T}$ or $T \notin \mathcal{T}$ and for any sentence $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

Definition. [6, 7] An infinite Σ-structure $\mathcal{M}$ is *pseudofinite* if for any sentence $\varphi \in \mathrm{Sent}(\Sigma)$, $\mathcal{M} \models \varphi$ implies that there is a finite structure $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$. The theory $T = \mathrm{Th}(\mathcal{M})$ of the pseudofinite structure $\mathcal{M}$ is called *pseudofinite*, too.

We denote by $\overline{\mathcal{T}}$ the class of all complete elementary theories of given signature, by $\overline{\mathcal{T}}_{\mathrm{fin}}$ the subclass of $\overline{\mathcal{T}}$ consisting of all theories with finite models, and by $\overline{\mathcal{T}}_{\mathrm{inf}}$ the class $\overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\mathrm{fin}}$.

Proposition 2. [10] *For any theory T the following conditions are equivalent:*
(1) *T is pseudofinite;*
(2) *$T \in \overline{\mathcal{T}}_{\mathrm{inf}}$ and T is $\overline{\mathcal{T}}_{\mathrm{fin}}$-approximated;*
(3) *$T \in \mathrm{Cl}_E(\overline{\mathcal{T}}_{\mathrm{fin}}) \setminus \overline{\mathcal{T}}_{\mathrm{fin}}$.*

Definition. [8] A formula φ true in an infinite model is called *pseudofinite* if it is true in a finite model, too.

We denote the set of all pseudofinite formulae (respectively, sentences) in a signature Σ by $\mathrm{PFF}(\Sigma)$ ($\mathrm{PFS}(\Sigma)$).

Following [8] we denote by $\mathrm{CF}(\Sigma)$ the set of consistent formulae in a signature Σ.

By the definition we have $\mathrm{PFF}(\Sigma) \subset \mathrm{CF}(\Sigma)$, and that inclusion is strict since there are formulae with finite models only.

Definition. [12, 13] The *spectrum* of a formula φ, $\mathrm{Spec}(\varphi)$, is the set $\{n \in \omega \setminus \{0\} \mid \text{there is } \mathcal{M} \models \varphi \text{ with } |\mathcal{M}| = n\}$.

Definition (cf. [14]). A sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called *spectrally minimal*, or *s-minimal*, if $\mathrm{Spec}(\varphi)$ is a finite or a cofinite subset of ω.

A signature Σ is called *spectrally minimal*, or *s-minimal*, if each sentence in $\mathrm{Sent}(\Sigma)$ is s-minimal.

Theorem 1. [8] *For a signature* Σ *the following conditions are equivalent:*

(1) *the set* $\mathrm{CF}(\Sigma)$ *consists of formulae with finite models;*

(2) *the set* $\mathrm{CF}(\Sigma)$ *consists of formulae with finite models including pseudo-finite formulae;*

(3) Σ *is s-minimal;*

(4) Σ *does not have functional symbols of arities* ≥ 1 *and predicate symbols of arities* ≥ 2.

Definition [15]. Let $\mathcal{T}$ be a family of theories. If $\mathcal{T}$ is empty we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, if $\mathcal{T}$ finite and nonempty we put $\mathrm{RS}(\mathcal{T}) = 0$, and for any infinite family $\mathcal{T}$ — $\mathrm{RS}(\mathcal{T}) \geq 1$.

For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.

We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.

If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

If $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Definition. [1] For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a family $\mathcal{T} \subseteq \mathcal{T}_\Sigma$, where $\mathcal{T}_\Sigma$ is the family of all complete theories in the signature Σ, we put $\mathrm{RS}_{\mathcal{T}}(\varphi) = \mathrm{RS}(\mathcal{T}_\varphi)$, and $\mathrm{ds}_{\mathcal{T}}(\varphi) = \mathrm{ds}(\mathcal{T}_\varphi)$ if $\mathrm{ds}(\mathcal{T}_\varphi)$ is defined.

If $\mathcal{T} = \mathcal{T}_\Sigma$ then we omit $\mathcal{T}$ and write $\mathrm{RS}(\varphi)$, $\mathrm{ds}(\varphi)$ instead of $\mathrm{RS}_{\mathcal{T}}(\varphi)$ and $\mathrm{ds}_{\mathcal{T}}(\varphi)$, respectively.

Theorem 2. [16] *For any signature* Σ *either* $\mathrm{RS}(\mathcal{T}_\Sigma)$ *is finite, if* Σ *consists of finitely many* 0*-ary and unary predicates, and finitely many constant symbols, or* $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$*, otherwise.*

Since for any sentence φ and families $\mathcal{T}$, $\mathcal{T}_1$, $\mathcal{T}_2$ of theories, $\mathcal{T}_\varphi \subseteq \mathcal{T}$, and $\mathcal{T}_1 \subseteq \mathcal{T}_2$ implies $\mathrm{RS}(\mathcal{T}_1) \leq \mathrm{RS}(\mathcal{T}_2)$, then $\mathrm{RS}_{\mathcal{T}}(\varphi) \leq \mathrm{RS}(\mathcal{T})$ and Theorem 2 produces the following:

Corollary 1. *For any signature* Σ *either* $\mathrm{RS}(\varphi)$ *is finite for any sentence* $\varphi \in \mathrm{Sent}(\Sigma)$*, if* Σ *consists of finitely many* 0*-ary and unary predicates, and finitely many constant symbols, or* $\mathrm{RS}(\varphi) = \infty$ *for some sentence* $\varphi \in \mathrm{Sent}(\Sigma)$*, otherwise.*

Definition. [17] For a family $\mathcal{T}$ of theories and sentences $\varphi, \psi \in \mathrm{Sent}(\Sigma(\mathcal{T}))$ we put $\varphi \vdash_{\mathcal{T}} \psi$ if $\mathcal{T}_\varphi \subseteq \mathcal{T}_\psi$. The sentences φ and ψ are called $\mathcal{T}$*-equivalent*, written $\varphi \equiv_{\mathcal{T}} \psi$, if $\varphi \vdash_{\mathcal{T}} \psi$ and $\psi \vdash_{\mathcal{T}} \varphi$.

Clearly, for any $\varphi, \psi \in \mathrm{Sent}(\Sigma)$, $\varphi \vdash_{\mathcal{T}_\Sigma} \psi$ iff $\varphi \vdash \psi$. Therefore the relations $\equiv_{\mathcal{T}_\Sigma}$ and $\equiv$ on $\mathrm{Sent}(\Sigma)$ coincide.

3 Approximating formulae and their properties

In this section, we generalize the notion of pseudofinite formula as follows.

Definition. For a family $\mathcal{T} \subseteq \mathcal{T}_\Sigma$, a formula $\varphi = \varphi(\overline{x})$ is called $\mathcal{T}$-*approximating* if φ is a formula satisfied in a model of some accumulation point of $\mathcal{T}$. The formula φ is called *approximating* if it is $\mathcal{T}_\Sigma$-approximating, where $\Sigma \supseteq \Sigma(\varphi)$.

Remark 1. Clearly, the approximability of a formula φ depends on the signature Σ. Indeed, the formula $\varphi = \forall x \forall y\, (x \approx y)$ is not approximating in the empty signature Σ_0 whereas adding infinitely many unary predicates P_i we can write that these predicates are independently empty/nonempty and obtain an infinite family $\mathcal{T}$ of theories whose models are singletons and satisfy φ.

Thus it is essential what signature $\Sigma \supseteq \Sigma(\varphi)$ is used to witness that φ is an approximating formula.

Remark 2. By the definition and Proposition 1, a formula φ is $\mathcal{T}$-approximating iff φ is satisfied in models of infinitely many theories of $\mathcal{T}$ by some tuples, i.e. a sentence $\exists \overline{x} \varphi$ belongs to infinitely many theories of $\mathcal{T}$.

By the last equivalent condition of Remark 2 we observe that approximating formulae are reduced to approximating sentences, and below we mainly consider approximating sentences.

Remark 3. In view of the definition and Proposition 1 a formula φ is $\mathcal{T}$-approximating iff φ is $\mathrm{Cl}_E(\mathcal{T})$-approximating.

We denote by $\mathrm{AF}(\mathcal{T})$ the set of all $\mathcal{T}$-approximating formulae, by $\mathrm{AF}(\Sigma)$ the set of all approximating formulae in the signature Σ, and by $\mathrm{PFS}_\infty(\Sigma)$ the set of all formulae in $\mathrm{PFS}(\Sigma)$ with infinite spectra. Restrictions of $\mathrm{AF}(\mathcal{T})$, and $\mathrm{AF}(\Sigma)$ to the set $\mathrm{Sent}(\Sigma)$ of sentences are denoted by $\mathrm{AS}(\mathcal{T})$ and $\mathrm{AS}(\Sigma)$, respectively.

Remark 4. By the definition the sets $\mathrm{AF}(\mathcal{T})$, $\mathrm{AS}(\mathcal{T})$, $\mathrm{PFS}_\infty(\Sigma)$ are closed both under $\vdash$-deducibility and $\vdash_{\mathcal{T}}$-deducibility. Thus these sets are divided into $\equiv_{\mathcal{T}}$-classes of formulae/sentences, which are divided into $\equiv$-classes.

Besides, $\mathrm{AF}(\mathcal{T}) = \mathrm{AF}(\mathrm{Cl}_E(\mathcal{T}))$ and $\mathrm{AS}(\mathcal{T}) = \mathrm{AS}(\mathrm{Cl}_E(\mathcal{T}))$.

Remark 5. The property of approximability is monotone with respect to families of theories: if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and φ is a $\mathcal{T}_1$-approximating formula, then φ is a $\mathcal{T}_2$-approximating. It is implied by $(\mathcal{T}_1)_\psi \subseteq (\mathcal{T}_2)_\psi$ for any sentence ψ. In particular, any $\mathcal{T}$-approximating formula is approximating, too.

Proposition 3. For any signature Σ, $\mathrm{PFS}_\infty(\Sigma) \neq \emptyset$.

Proof. It suffices to note that the formula $\exists x, y \neg x \approx y$ of the arbitrary signature is pseudofinite, with $\mathrm{Spec}(\varphi) = \omega \setminus 2$. □

Remark 6. Since structures of distinct finite cardinalities are not elementary equivalent, then having infinitely many finite models of a formula φ we obtain both an infinite spectrum $\mathrm{Spec}(\varphi)$ and the approximability of φ. Thus for any signature Σ,

$$\mathrm{PFF}_{\infty}(\Sigma) \subseteq \mathrm{AF}(\Sigma) \tag{1}$$

and

$$\mathrm{PFS}_{\infty}(\Sigma) \subseteq \mathrm{AS}(\Sigma), \tag{2}$$

i.e. any pseudofinite formula/sentence with an infinite spectrum is approximating. Besides, $\mathrm{PFS}(\Sigma) \subset \mathrm{PFF}(\Sigma)$, $\mathrm{PFS}_{\infty}(\Sigma) \subset \mathrm{PFF}_{\infty}(\Sigma)$, and $\mathrm{AS}(\Sigma) \subset \mathrm{AF}(\Sigma)$. These inclusions are strict since the formula $x \approx x$ is pseudofinite, with spectrum $\omega \setminus \{0\}$, and has the free variable x.

Thus, using Proposition 3, for any signature Σ we have $\mathrm{AF}(\Sigma) \neq \emptyset$ and $\mathrm{AS}(\Sigma) \neq \emptyset$.

At the same time, there are approximating formulae/sentences which are not pseudofinite. Indeed, for instance, if a structure $\mathcal{M}$ is divided into two disjoint infinite definable parts P and Q such that the part P has a finitely axiomatizable theory with an axiom φ and another part Q is pseudofinite then for any sentence ψ satisfying $\mathcal{M}|_Q$ the sentence $\varphi|_P \wedge \psi|_Q$, where $\varphi|_P$ is the relativization of φ to P and $\psi|_Q$ is the relativization of ψ to Q, is both approximating, as $\psi|_Q$ is approximating, and non-pseudofinite, since $\varphi|_P$ does not have finite models. Thus the inclusions (1) and (2) can be strict.

Besides, there are pseudofinite formulae which are not approximating. For instance, taking the signature of binary relational symbol $\leq$ of linear order and a sentence φ writing that $\leq$ is a singleton or a dense linear order: $\forall x, y, z(x \leq x \wedge (x \leq y \wedge y \leq x \to x \approx y) \wedge (x \leq y \wedge y \leq z \to x \leq z) \wedge (x \leq y \vee y \leq x) \wedge (x \leq y \wedge \neg x \approx y \to \exists u(x \leq u \wedge \neg x \approx u \wedge u \leq y \wedge \neg u \approx y)))$ we have five possibilities for its theories: the theory T_1 of a singleton, the theories T_2, T_3, T_4, T_5 of dense linear orders with(out) the least/greatest elements. We have $\mathrm{Spec}(\varphi) = 1$, witnessed by T_1. At the same time φ is not approximating since the family $\left(\mathcal{T}_{\{\leq\}}\right)_{\varphi}$ consists of four theories T_2, T_3, T_4, T_5 and it can not be infinite.

We have the similar effect, with RS-rank 0, for theories of dense circular and spherical orders [18].

The following folklore examples illustrate the notion of approximating formula.

Example 1. 1. Any sentence φ of the empty signature Σ_0 with infinite models has a cofinite spectrum implying $\varphi \in \mathrm{PFS}_{\infty}(\Sigma_0)$ and $\varphi \in \mathrm{AS}(\Sigma_0)$ by the inclusion (2).

2. By the description of finite fields, any sentence φ_F axiomatizing the class of fields has the spectrum $\mathrm{Spec}(\varphi_F) = \{p^n \mid p \text{ is prime}, n \in \omega \setminus \{0\}\}$. Thus

$\varphi_F \in \mathrm{PFS}_\infty(\Sigma_F)$, where Σ_F is the signature of fields, and $\varphi_F \in \mathrm{AS}(\Sigma_F)$ in view of the inclusion (2).

3. By Stone Theorem on representation of Boolean algebras, any sentence φ_{BA} axiomatizing the class of Boolean algebras has the spectrum

$$\mathrm{Spec}(\varphi_{\mathrm{BA}}) = \{2^n \mid n \in \omega \setminus \{0\}\}.$$

Thus $\varphi_{\mathrm{BA}} \in \mathrm{PFS}_\infty(\Sigma_{\mathrm{BA}})$, where Σ_{BA} is the signature of Boolean algebras, and $\varphi_{\mathrm{BA}} \in \mathrm{AS}(\Sigma_{\mathrm{BA}})$ again in view of the inclusion (2).

The arguments for the latter example in Remark 6 produce the following:

Proposition 4. *Let φ be a sentence with finite nonempty spectrum and ψ be a consistent sentence belonging to finitely many theories of given signature $\Sigma \supseteq (\Sigma(\varphi) \cup \Sigma(\psi))$ such that φ does not have infinite models and ψ does not have finite ones. Then $\varphi \vee \psi$ is pseudofinite and not approximating.*

Proof. Pseudofiniteness of $\varphi \vee \psi$ is witnessed by φ with finite models and ψ with infinite ones. The sentence $\varphi \vee \psi$ is not approximating since $(\mathcal{T}_\Sigma)_{\varphi\vee\psi}$ is finite as the union of two finite sets $(\mathcal{T}_\Sigma)_\varphi$ and $(\mathcal{T}_\Sigma)_\psi$. □

Remark 7. Since for any formula $\varphi \in F(\Sigma)$ some formula φ or $\neg\varphi$ has an infinite spectrum, i.e. $\varphi \in \mathrm{PFF}_\infty(\Sigma)$ or $\neg\varphi \in \mathrm{PFF}_\infty(\Sigma)$. Therefore, in view of Remark 6, φ is approximating or $\neg\varphi$ is approximating. It implies that each approximating sentence belongs to some theory consisting of approximating sentences.

Proposition 5. *For any pseudofinite theory T of a signature Σ, $T \subseteq \mathrm{PFS}_\infty(\Sigma)$.*

Proof. Let φ be a sentence in a pseudofinite theory T. We argue to show that φ has an infinite spectrum. Since any pseudofinite theory has a finite model, then $\mathrm{Spec}(\varphi) \neq \emptyset$, and it contains the least number, say m_0. Now we define a sequence $(\varphi_n)_{n\in\omega}$ of sentences in T such that $\varphi_0 = \varphi$, $\varphi_{n+1} \vdash \varphi_n$, and $\mathrm{Spec}(\varphi_{n+1}) \subsetneqq \mathrm{Spec}(\varphi_n)$, $n \in \omega$. For this aim, if φ_n is defined with the least m_n in $\mathrm{Spec}(\varphi_n)$, we take for φ_{n+1} the sentence $\varphi_n \wedge \psi_{m_n}$, where $\psi_{m_n} = \exists x_0, \ldots x_{m_n} \bigwedge\limits_{i\neq j} \neg x_i \approx x_j$ asserts that its model should have at least $m_n + 1$ elements. We have $\varphi_{n+1} \in T$, since T has infinite models, and $\mathrm{Spec}(\varphi_{n+1}) = \mathrm{Spec}(\varphi_n) \setminus \{m_n\}$. Thus $\mathrm{Spec}(\varphi) = \{m_n \mid n \in \omega\}$ and $\varphi \in \mathrm{PFS}_\infty(\Sigma)$. The required inclusion $T \subseteq \mathrm{PFS}_\infty(\Sigma)$ follows since $\varphi \in T$ was chosen arbitrarily. □

Inequality (2) and Proposition 5 immediately imply:

Corollary 2. *Any pseudofinite theory T consists of $\overline{\mathcal{T}}_{\mathrm{fin}}$-approximating sentences.*

The following theorem extends Theorem 1:

Theorem 3. *For any signature Σ the following conditions are equivalent:*
(1) $\mathrm{PFF}(\Sigma) = \mathrm{PFF}_\infty(\Sigma) = \mathrm{AF}(\Sigma)$;

(2) $\mathrm{PFS}(\Sigma) = \mathrm{PFS}_\infty(\Sigma) = \mathrm{AS}(\Sigma)$;

(3) *the signature* Σ *does not contain functional symbols of arities* > 0 *and predicate symbols of arities* > 1.

Proof. (1) $\Leftrightarrow$ (2) is implied since pseudofinite/approximating formulae $\varphi(\overline{x})$ are transformed to pseudofinite/approximating sentences $\exists\overline{x}\varphi(\overline{x})$, and vice versa.

(3) $\Rightarrow$ (1) follows by Theorem 1 and possibilities of cardinality descriptions of definable sets in the given signature.

(2) $\Rightarrow$ (3). Assuming on contrary that Σ contains a n-ary functional symbol f or $(n+1)$-ary predicate symbol R, for $n \geq 1$, we can reduce $f(x_1, \ldots, x_n)$ to the unary one $g(x) = f(x, \ldots, x)$, and reduce $R(x_1, \ldots, x_n, y)$ to the graph $R(x, \ldots, x, y)$ of some g by the formula $\forall x\exists^{=1}y\, R(x, \ldots, x, y)$. Now the formula $\forall y\exists^{\geq 2}x\, g(x) \approx y$ belongs to $\mathrm{AS}(\Sigma) \setminus \mathrm{PFS}(\Sigma)$ contradicting the assumption $\mathrm{PFS}(\Sigma) = \mathrm{AS}(\Sigma)$. $\Box$

Proposition 6. *For any family* $\mathcal{T}$ *of theories in a signature* Σ *the following equalities hold:*

1) $\mathrm{AS}(\mathcal{T}) = \bigcup\{T \in \mathcal{T} \mid T$ *is an accumulation point of* $\mathcal{T}\}$;

2) $\mathrm{AS}(\mathcal{T}) = \bigcup\{T \mid T$ *is an accumulation point of* $\mathrm{Cl}_E(\mathcal{T})\}$;

3) $\mathrm{AS}(\mathcal{T}) = \{\varphi \in \mathrm{Sent}(\Sigma(\mathcal{T})) \mid \mathrm{RS}_{\mathcal{T}}(\varphi) \geq 1\}$.

Proof. The equality 1) is satisfied by the definition of $\mathcal{T}$-approximating formula. The equality 2) is true in view of Proposition 1. The equality 3) follows by Remark 2, since a sentence φ is $\mathcal{T}$-approximating iff $\mathcal{T}_\varphi$ is infinite, i.e. $\mathrm{RS}_{\mathcal{T}}(\varphi) \geq 1$. $\Box$

Proposition 6 immediately implies:

Corollary 3. *For any family* $\mathcal{T}$ *of theories in a signature* Σ *the set of sentences* φ *separating finite nonempty subsets* $\mathcal{T}_\varphi$ *in* $\mathcal{T}$ *equals* $\mathrm{CF}(\mathcal{T}) \cap (\mathrm{Sent}(\Sigma) \setminus \mathrm{AS}(\mathcal{T}))$.

Definition. A theory $T \in \mathcal{T}$ is called *finitely axiomatizable* in $\mathcal{T}$, or $\mathcal{T}$-*finitely axiomatizable*, if there is a sentence φ such that $\mathcal{T}_\varphi = \{T\}$. Here the sentence φ is called the $\mathcal{T}$-*complete axiom* for T.

By the definition any finitely axiomatizable theory T with its axiom φ is $\mathcal{T}_{\Sigma(\varphi)}$-finitely axiomatizable with the $\mathcal{T}_{\Sigma(\varphi)}$-complete axiom φ.

Since families $\mathcal{T}_\varphi$ are Hausdorff, any finite family $\mathcal{T}_\varphi$ is divided into singletons $\mathcal{T}_{\varphi_1}, \ldots, \mathcal{T}_{\varphi_n}$, where $\varphi \equiv_{\mathcal{T}} \varphi_1 \vee \ldots \vee \varphi_n$. Thus, Corollary 3 implies the following modification:

Corollary 4. *For any family* $\mathcal{T}$ *of theories in a signature* Σ *the set of sentences* φ *separating singletons* $\mathcal{T}_\varphi$ *in* $\mathcal{T}$ *equals the subset of* $\mathrm{CF}(\mathcal{T}) \cap (\mathrm{Sent}(\Sigma) \setminus \mathrm{AS}(\mathcal{T}))$ *consisting of* $\mathcal{T}$*-complete axioms such that each sentence in* $\mathrm{CF}(\mathcal{T}) \cap (\mathrm{Sent}(\Sigma) \setminus \mathrm{AS}(\mathcal{T}))$ *is* $\mathcal{T}$*-equivalent to a disjunction of these axioms.*

We denote the set of $\mathcal{T}$-complete axioms in $\mathrm{CF}(\mathcal{T}) \cap (\mathrm{Sent}(\Sigma) \setminus \mathrm{AS}(\mathcal{T}))$ by $\mathrm{CA}(\mathcal{T})$. Thus $\mathrm{CA}(\mathcal{T})$ is the set of all sentences in $\mathrm{Sent}(\Sigma)$ separating singletons in $\mathcal{T}$.

Now Corollary 4 implies the following:

Corollary 5. *For any finite signature* Σ *the family* $\mathcal{T}_{\rm fa}\subset\mathcal{T}_\Sigma$ *of all finitely axiomatized theories in the signature* Σ *consists of theories axiomatized by sentences in* ${\rm CA}(\mathcal{T}_\Sigma)$.

Remark 8. By the definition a theory $T\notin\mathcal{T}$ is $\mathcal{T}$-approximated if and only if each sentence $\varphi\in T$ is $\mathcal{T}$-approximating. Therefore the approximated theory property is closed under restrictions $T_0=T|_{\Sigma_0}$ of T to subsignatures Σ_0 such that T_0 does not belong to the Σ_0-restriction of $\mathcal{T}$. Respectively, the non-approximated theory property is closed under expansions of theories.

4 Algebras for approximating formulae

Remark 9. Since ${\rm AF}(\mathcal{T})$ and ${\rm AS}(\mathcal{T})$ are closed under deducibility then, for any infinite $\mathcal{T}$, ${\rm AF}(\mathcal{T})$ and ${\rm AS}(\mathcal{T})$ form upper semilattices with respect to the operation $\vee$, denoted by $\mathcal{AF}(\mathcal{T})$ and $\mathcal{AS}(\mathcal{T})$, respectively: $\mathcal{AF}(\mathcal{T})=\langle{\rm AF}(\mathcal{T});\vee\rangle$, $\mathcal{AS}(\mathcal{T})=\langle{\rm AS}(\mathcal{T});\vee\rangle$. Here $\mathcal{T}$ should be infinite since the universes ${\rm AF}(\mathcal{T})$ and ${\rm AS}(\mathcal{T})$ are nonempty iff $\mathcal{T}$ is infinite, i.e. $\mathcal{T}$ has an accumulation point.

At the same time ${\rm AF}(\mathcal{T})$ and ${\rm AS}(\mathcal{T})$ are not closed under conjunctions and negations, in general. Indeed, if $\varphi,\psi\in{\rm AF}(\mathcal{T})$ then both $\varphi\wedge\psi$ and $\neg\varphi$ can be outside ${\rm AF}(\mathcal{T})$. For instance, these formulae can be inconsistent or can belong to finitely many theories in $\mathcal{T}$. In fact, ${\rm AF}(\mathcal{T})$ and, in particular, ${\rm AS}(\mathcal{T})$ are closed under conjunctions iff ${\rm AS}(\mathcal{T})=T$ for unique accumulation point T of $\mathcal{T}$, i.e. $\mathcal{T}$ is e-minimal, that is ${\rm RS}(\mathcal{T})=1$ and ${\rm ds}(\mathcal{T})=1$, since having at least two accumulation points T_1 and T_2 of $\mathcal{T}$ we can find $\varphi\in T_1$ and $\psi\in T_2$ with inconsistent $\varphi\wedge\psi$.

Clearly, if ${\rm AF}(\mathcal{T})$ and ${\rm AS}(\mathcal{T})$ are closed under conjunctions then they form distributive lattices, as parts of lattices on $F(\Sigma(\mathcal{T}))$.

In view of Remark 9 we have the following:

Theorem 4. *For any infinite family* $\mathcal{T}$ *the following conditions are equivalent:*

(1) *the semilattice* $\mathcal{AF}(\mathcal{T})$ *is expandable till the* (*distributive*) *lattice*

$$\langle{\rm AF}(\mathcal{T});\vee,\wedge\rangle;$$

(2) *the semilattice* $\mathcal{AS}(\mathcal{T})$ *is expandable till the* (*distributive*) *lattice*

$$\langle{\rm AS}(\mathcal{T});\vee,\wedge\rangle;$$

(3) $\mathcal{T}$ *is e-minimal;*

(4) ${\rm RS}(\mathcal{T})=1$ *and* ${\rm ds}(\mathcal{T})=1$.

Using restrictions of the semilattice $\mathcal{AF}(\mathcal{T})$ and its restriction $\mathcal{AS}(\mathcal{T})$ to the set ${\rm Sent}(\Sigma)$ we obtain a series of Boolean algebras in the following way.

Let $\varphi\in{\rm AF}(\mathcal{T})$ (respectively, $\varphi\in{\rm AS}(\mathcal{T})$), ${\rm AF}(\mathcal{T})_\varphi=\{\psi\in{\rm AF}(\mathcal{T})\mid\ \vdash\varphi\to\psi\}$ (${\rm AS}(\mathcal{T})_\varphi=\{\psi\in{\rm AS}(\mathcal{T})\mid\ \vdash\varphi\to\psi\}$). Clearly, ${\rm AF}(\mathcal{T})_\varphi$, and

$\mathrm{AS}(\mathcal{T})_\varphi$ for $\varphi \in \mathrm{AS}(\mathcal{T})$, are closed under conjunctions $\psi \wedge \psi'$ and under relative negations $\neg_\varphi \psi = \neg\psi \vee \varphi$, where ψ and ψ' belong to $\mathrm{AF}(\mathcal{T})_\varphi$ and $\mathrm{AS}(\mathcal{T})_\varphi$, respectively.

Thus, the structures $\langle \mathrm{AF}(\mathcal{T})_\varphi; \vee, \wedge, \neg_\varphi\rangle$ for $\varphi \in \mathrm{AF}(\Sigma)$, and $\langle \mathrm{AS}(\mathcal{T})_\varphi; \vee, \wedge, \neg_\varphi\rangle$ for $\varphi \in \mathrm{AS}(\mathcal{T})$, have $\equiv$-quotients $\mathcal{BAF}_{\mathcal{T}}(\varphi)$ and $\mathcal{BAS}_{\mathcal{T}}(\varphi)$ being Boolean algebras iff φ is not identically true and $\mathcal{T}$ is infinite. We also consider $\equiv_{\mathcal{T}}$-quotients $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ for $\mathcal{BAS}_{\mathcal{T}}(\varphi)$, with the greatest $\equiv_{\mathcal{T}}$-classes $\{\psi \mid \mathcal{T}_\psi = \mathcal{T}\}$.

In view of Remark 9 we have the following:

Theorem 5. *Let $\mathcal{T}$ be an infinite family of theories in a signature Σ. Then the following conditions hold:*

(1) *For any formula $\varphi \in \mathrm{AF}(\mathcal{T})$ the structure $\mathcal{BAF}_{\mathcal{T}}(\varphi)$ is a Boolean algebra iff φ is not identically true.*

(2) *For any sentence $\varphi \in \mathrm{AS}(\mathcal{T})$ the structure $\mathcal{BAS}_{\mathcal{T}}(\varphi)$ is a Boolean algebra iff φ is not identically true.*

(3) *For any sentence $\varphi \in \mathrm{AS}(\mathcal{T})$ the structure $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ is a Boolean algebra iff φ does not define the greatest $\equiv_{\mathcal{T}}$-class, i.e. $\mathcal{T}_\varphi \neq \mathcal{T}$.*

Remark 10. The Boolean algebras $\mathcal{BAF}_{\mathcal{T}}(\varphi)$, $\mathcal{BAS}_{\mathcal{T}}(\varphi)$, $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ and their lattice restrictions $\mathcal{LF}_{\mathcal{T}}(\varphi)$, $\mathcal{LS}_{\mathcal{T}}(\varphi)$, $\mathcal{LS}^*_{\mathcal{T}}(\varphi)$ form hierarchies with respect to approximating formulae φ defined in view of the following properties:

i) if $\varphi \vdash \psi$, for non-identically true formulae $\varphi, \psi \in \mathrm{AF}(\mathcal{T})$, then $\mathcal{LF}_{\mathcal{T}}(\psi)$ is a restriction of $\mathcal{LF}_{\mathcal{T}}(\varphi)$ to $\{\equiv(\chi) \mid \psi \vdash \chi\}$; if additionally $\varphi, \psi \in \mathrm{AS}(\mathcal{T})$ then $\mathcal{LS}_{\mathcal{T}}(\psi)$ is a restriction of $\mathcal{LS}_{\mathcal{T}}(\varphi)$, and $\mathcal{LS}^*_{\mathcal{T}}(\psi)$ is a restriction of $\mathcal{LS}^*_{\mathcal{T}}(\varphi)$ to $\{\equiv_{\mathcal{T}}(\chi) \mid \psi \vdash_{\mathcal{T}} \chi\}$;

ii) if $\mathrm{RS}_{\mathcal{T}}(\varphi) = \alpha \geq 1$ and φ is not identically true then the Boolean algebras $\mathcal{BAS}_{\mathcal{T}}(\varphi)$ and $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ is composed by sentences ψ with $\varphi \vdash \psi$ satisfying $\alpha \leq \mathrm{RS}_{\mathcal{T}}(\psi) \leq \mathrm{RS}(\mathcal{T})$.

Clearly the operations of relative negations in distinct $\mathcal{BAF}_{\mathcal{T}}(\varphi)$ and $\mathcal{BAF}_{\mathcal{T}}(\psi)$, $\mathcal{BAS}_{\mathcal{T}}(\varphi)$ and $\mathcal{BAS}_{\mathcal{T}}(\psi)$, $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ and $\mathcal{BAS}^*_{\mathcal{T}}(\psi)$ do not correlate since for an identically true sentence φ_1, $\neg_\varphi \varphi_1 \equiv \varphi$, $\neg_\psi \varphi_1 \equiv \psi$, $\neg_\varphi \varphi_1 \equiv_{\mathcal{T}} \varphi$, $\neg_\psi \varphi_1 \equiv_{\mathcal{T}} \psi$.

The constructions above show that $\equiv$-quotients and $\equiv_{\mathcal{T}}$-quotients on the sets $\mathrm{AF}(\mathcal{T})$ and $\mathrm{AS}(\mathcal{T})$ form lattices with relative complements, i.e. Ershov algebras [19].

Remark 10 allows to show the following propositions.

Proposition 7. *Let $\mathcal{T}$ be an infinite family of theories, $\varphi, \psi \in \mathrm{AS}(\mathcal{T})$ be sentences which are not identically true. Then:*

1. *$\mathcal{LS}_{\mathcal{T}}(\psi)$ is a sublattice of $\mathcal{LS}_{\mathcal{T}}(\varphi)$ iff $\varphi \vdash \psi$, and $\mathcal{LS}_{\mathcal{T}}(\psi)$ is a proper sublattice of $\mathcal{LS}_{\mathcal{T}}(\varphi)$ iff $\varphi \vdash \psi$ and $\psi \nvdash \varphi$. In particular, $\mathcal{LS}_{\mathcal{T}}(\psi)$ is a proper sublattice of $\mathcal{LS}_{\mathcal{T}}(\varphi)$ if $\mathrm{RS}_{\mathcal{T}}(\varphi) < \mathrm{RS}_{\mathcal{T}}(\psi)$, or $\mathrm{RS}_{\mathcal{T}}(\varphi) = \mathrm{RS}_{\mathcal{T}}(\psi)$ and $\mathrm{ds}_{\mathcal{T}}(\varphi) < \mathrm{ds}_{\mathcal{T}}(\psi)$;*

2. *$\mathcal{LS}^*_{\mathcal{T}}(\psi)$ is a sublattice of $\mathcal{LS}^*_{\mathcal{T}}(\varphi)$ iff $\varphi\vdash_{\mathcal{T}}\psi$, i.e. $\mathcal{T}_\varphi\subseteq\mathcal{T}_\psi$, and $\mathcal{LS}^*_{\mathcal{T}}(\psi)$ is a proper sublattice of $\mathcal{LS}^*_{\mathcal{T}}(\varphi)$ iff $\mathcal{T}_\varphi\subsetneqq\mathcal{T}_\psi$. In particular, $\mathcal{LS}^*_{\mathcal{T}}(\psi)$ is a proper sublattice of $\mathcal{LS}^*_{\mathcal{T}}(\varphi)$ if $\mathrm{RS}_{\mathcal{T}}(\varphi)<\mathrm{RS}_{\mathcal{T}}(\psi)$, or $\mathrm{RS}_{\mathcal{T}}(\varphi)=\mathrm{RS}_{\mathcal{T}}(\psi)$ and $\mathrm{ds}_{\mathcal{T}}(\varphi)<\mathrm{ds}_{\mathcal{T}}(\psi)$.*

Proof. 1. Let $\mathcal{LS}_{\mathcal{T}}(\psi)$ be a sublattice of $\mathcal{LS}_{\mathcal{T}}(\varphi)$. Then $\equiv(\psi)$ belongs to $\mathcal{LS}_{\mathcal{T}}(\varphi)$ implying $\varphi\vdash\psi$. Conversely, let $\varphi\vdash\psi$. If $\equiv(\chi)\in\mathrm{LS}_{\mathcal{T}}(\psi)$, i.e. $\psi\vdash\chi$, then $\equiv(\chi)\in\mathrm{LS}_{\mathcal{T}}(\varphi)$, since $\varphi\vdash\psi$ and $\psi\vdash\chi$ implying $\varphi\vdash\chi$. Thus $\mathrm{LS}_{\mathcal{T}}(\psi)\subseteq\mathcal{BAS}_{\mathcal{T}}(\varphi)$. Since $\equiv$-classes depend on signature only and operations $\vee$, $\wedge$ agree on common elements of $\mathcal{LS}_{\mathcal{T}}(\varphi)$ and $\mathcal{LS}_{\mathcal{T}}(\psi)$, $\mathcal{LS}_{\mathcal{T}}(\psi)$ is a sublattice of $\mathcal{LS}_{\mathcal{T}}(\varphi)$.

The lattice $\mathcal{LS}_{\mathcal{T}}(\psi)$ is a proper sublattice of $\mathcal{LS}_{\mathcal{T}}(\varphi)$ iff $\varphi\vdash\psi$ and $\equiv_\varphi\notin\mathrm{LS}_{\mathcal{T}}(\psi)$, i.e. $\psi\not\vdash\varphi$. These conditions are satisfied if $\mathrm{RS}_{\mathcal{T}}(\varphi)<\mathrm{RS}_{\mathcal{T}}(\psi)$, or $\mathrm{RS}_{\mathcal{T}}(\varphi)=\mathrm{RS}_{\mathcal{T}}(\psi)$ and $\mathrm{ds}_{\mathcal{T}}(\varphi)<\mathrm{ds}_{\mathcal{T}}(\psi)$.

2. We repeat the arguments above replacing $\vdash$ by $\vdash_{\mathcal{T}}$. □

Proposition 8. *Let $\mathcal{T}$ be an infinite family of theories, $\varphi\in\mathrm{AS}(\mathcal{T})$, $\mathrm{RS}(\mathcal{T})$ be an ordinal. Then for any positive ordinal $\alpha\leq\mathrm{RS}(\mathcal{T})$ there is a sentence $\varphi_\alpha\in\mathrm{AS}(\mathcal{T})$ with $\mathrm{RS}_{\mathcal{T}}(\varphi_\alpha)=\alpha$ defining the Boolean algebras $\mathcal{BAS}_{\mathcal{T}}(\varphi_\alpha)$ and $\mathcal{BAS}^*_{\mathcal{T}}(\varphi_\alpha)$ combined by sentences with all permissible $\mathrm{RS}_{\mathcal{T}}$-ranks $\beta\geq\alpha$, $\beta\leq\mathrm{RS}(\mathcal{T})$.*

Proof. By the definition of RS-rank the condition $\mathrm{RS}(\mathcal{T})\in\mathrm{Ord}$ implies that for any $\alpha\leq\mathrm{RS}(\mathcal{T})$ there is a sentence φ'_α such that $\mathrm{RS}(\mathcal{T}_{\varphi'_\alpha})=\alpha$. Moreover, since $\mathcal{T}$ is infinite and the topology on $\mathcal{T}$ is Hausdorff, we can choose φ_α with $\mathcal{T}_{\varphi_\alpha}\subsetneqq\mathcal{T}$. Indeed, as for any distinct theories $T_1,T_2\in\mathcal{T}$ there are sentences $\varphi_i\in T_i\setminus T_{3-i}$, $i=1,2$, $\varphi_2=\neg\varphi_1$, we have $\mathrm{RS}(\varphi'_\alpha\wedge\varphi_1)=\alpha$ or $\mathrm{RS}(\varphi'_\alpha\wedge\varphi_2)=\alpha$ and we can take some $\varphi'_\alpha\wedge\varphi_i$ for φ_α. Now $\mathrm{RS}_{\mathcal{T}}(\varphi_\alpha)=\alpha$ and φ_α is not identically true producing the Boolean algebras $\mathcal{BAS}_{\mathcal{T}}(\varphi_\alpha)$ and $\mathcal{BAS}^*_{\mathcal{T}}(\varphi_\alpha)$ combined by sentences with all permissible $\mathrm{RS}_{\mathcal{T}}$-ranks $\beta\geq\alpha$, $\beta\leq\mathrm{RS}(\mathcal{T})$, in view of Theorem 5 and Proposition 7. □

Proposition 9. *Let $\mathcal{T}$ be an infinite family of theories, $\varphi\in\mathrm{AS}(\mathcal{T})$. If $\mathrm{RS}(\mathcal{T})$ is an ordinal and $\mathcal{T}_\varphi\neq\mathcal{T}$ then the algebra $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ is finite iff $\mathcal{T}\setminus\mathcal{T}_\varphi$ is finite. In particular, in such a case $\mathrm{RS}_{\mathcal{T}}(\varphi)=\mathrm{RS}(\mathcal{T})$ and $\mathrm{ds}_{\mathcal{T}}(\varphi)=\mathrm{ds}(\mathcal{T})$.*

Proof. If $\mathcal{T}\setminus\mathcal{T}_\varphi$ is finite then there are finitely many possibilities to divide $\mathcal{T}\setminus\mathcal{T}_\varphi$ by sentences. Since distinct elements of $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ correspond to distinct parts of $\mathcal{T}\setminus\mathcal{T}_\varphi$, $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ is finite.

Conversely, if $\mathcal{T}\setminus\mathcal{T}_\varphi$ is infinite then by its Hausdorffness there are infinitely many distinct parts of $\mathcal{T}\setminus\mathcal{T}_\varphi$ defined by sentences. Then $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ is infinite.

Since $\mathcal{T}$ is infinite, sentences φ with cofinite $\mathcal{T}_\varphi$ preserve RS-rank and degree, i.e. $\mathrm{RS}_{\mathcal{T}}(\varphi)=\mathrm{RS}(\mathcal{T})$ and $\mathrm{ds}_{\mathcal{T}}(\varphi)=\mathrm{ds}(\mathcal{T})$. □

Proposition 9 allows to extend the list of characterizations for e-minimality given in Theorem 4:

Corollary 6. *Let $\mathcal{T}$ be an infinite family of theories. Then for any $\varphi\in\mathrm{AS}(\mathcal{T})$ with $\mathcal{T}_\varphi\neq\mathcal{T}$ the algebra $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ is finite iff $\mathcal{T}$ is e-minimal.*

Proof follows by Proposition 9 since for any $\varphi \in \mathrm{AS}(\mathcal{T})$, $\mathcal{T}_\varphi$ is infinite. Indeed, either $\mathcal{T}_\varphi = \mathcal{T}$ or φ defines a Boolean algebra $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ which is finite for cofinite $\mathcal{T}_\varphi$. Now if all Boolean algebras $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ are finite then each infinite $\mathcal{T}_\varphi$ have finite complements, i.e. $\mathcal{T}$ is e-minimal. Conversely, if $\mathcal{T}$ is e-minimal then each $\varphi \in \mathrm{AS}(\mathcal{T})$ with $\mathcal{T}_\varphi \neq \mathcal{T}$ has cofinite $\mathcal{T}_\varphi$ implying finiteness of $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ by Proposition 9. □

Remark 11. Let $\mathcal{T}$ be an infinite family of theories and T be its accumulation point. By Proposition 1 each neighbourhood $\mathcal{T}_\varphi$ of T is infinite. Therefore there is an infinite chain $\mathcal{T}_{\varphi_n}$, $n \in \omega$, of T-neighbourhoods such that $\mathcal{T}_{\varphi_0} \neq \mathcal{T}$ and $\mathcal{T}_{\varphi_{n+1}} \subsetneqq \mathcal{T}_{\varphi_n}$, $n \in \omega$. It implies that $\mathcal{BAS}^*_{\mathcal{T}}(\varphi_{n+1})$ is a proper extension of $\mathcal{BAS}^*_{\mathcal{T}}(\varphi_n)$ for any $n \in \omega$. Thus $\mathcal{T}$ does not have the largest Boolean algebra $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$.

In view of Corollary 6 and Remark 11 we have the following:

Corollary 7. *For any e-minimal family $\mathcal{T}$ of theories all cardinalities of Boolean algebras $\mathcal{BAS}^*_{\mathcal{T}}(\varphi)$ are finite and the set of these cardinalities is unbounded, with* $\sup\{|\mathcal{BAS}^*_{\mathcal{T}}(\varphi)| \mid \varphi \in \mathrm{AS}(\mathcal{T}), \mathcal{T}_\varphi \neq \mathcal{T}\} = \omega$.

5 Quantifiers in formulae and approximability via pseudofiniteness

Definition. [20] A formula φ is called a $\forall$-*formula* (respectively, $\exists$-*formula,* $\forall\exists$-*formula,* $\exists\forall$-*formula*) if

$$\varphi = \forall x_1 \ldots \forall x_k \psi,$$
$$(\varphi = \exists x_1 \ldots \exists x_k \psi,$$
$$\varphi = \forall x_1 \ldots \forall x_k \exists y_1 \ldots \exists y_n \psi,$$
$$\varphi = \exists x_1 \ldots \exists x_k \forall y_1 \ldots \forall y_n \psi),$$

where ψ is a quantifier free formula. A class K of Σ-structures is called $\forall$-*axiomatizable* ($\exists$-*axiomatizable,* $\forall\exists$-*axiomatizable,* $\exists\forall$-*axiomatizable*) if $K = K_\Sigma(Z)$, where Z is the set of $\forall$-sentences ($\exists$-sentences, $\forall\exists$-sentences, $\exists\forall$-sentences) of signature Σ, $K_\Sigma(Z)$ is the class of all Σ-structures satisfying Z.

A formula φ is called *positive* if its prenex normal form does not have negations. A formula φ is called *negative* if its prenex normal form is obtained from a positive prenex normal form by replacements of all atomic subformulae by their negations.

Proposition 10. [20] 1. *Let $\varphi(x_1, \ldots, x_k)$ be a $\forall$-formula ($\exists$-formula) of a signature Σ, $\mathcal{A} \subseteq \mathcal{B}$ be Σ-structures, $a_1, \ldots, a_k \in A$. Then the truth $\varphi(a_1, \ldots, a_k)$ in $\mathcal{B}$ (in $\mathcal{A}$) implies the truth of $\varphi(a_1, \ldots, a_k)$ in $\mathcal{A}$ (in $\mathcal{B}$).*

2. *Let $\{\mathcal{A}_i \mid i \in I\}$ be an upward directed family of be Σ-structures and the $\forall\exists$-sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is true in all $\mathcal{A}_i$, $i \in I$. Then φ is true in $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$.*

On a base of Proposition 10 the following assertion holds:

Proposition 11. [8] *Any positive* $\forall$*-formula* ($\exists$*-formula,* $\exists\forall$*-formula*) φ *with an infinite model is pseudofinite.*

Now we argue to show that the formulae φ in Proposition 11 have infinite spectra. Indeed, if the signature Σ is relational then an infinite model $\mathcal{M}$ admits finite restrictions containing "existing" elements written in φ, and these finite restrictions can be arbitrarily large witnessing $\varphi \in \mathrm{PFF}_\infty(\Sigma)$.

If $\Sigma(\varphi)$ contains functional symbols then the prenex normal form of φ is composed as positive Boolean combination of atomic formulae $P(t_1, \ldots, t_n)$ and $q_1 \approx q_2$, where $t_1, \ldots, t_n, q_1, q_2$ are terms, with a quantifier prefix $\forall$, $\exists$, or $\exists\forall$. Here $\exists$, and $\exists\forall$ are reduced to $\forall$ replacing "existing" elements by new constants. For the positive $\forall$-formula φ we take its infinite model $\mathcal{M}$ producing arbitrarily large finite sets X of generators for the substructure $\mathcal{M}(X)$. Since φ is universal it admits identifications of values of terms in $\mathcal{M}(X)$ transforming this structure into a finite model $\mathcal{A}(X)$ of φ with the following rule relative to the relation E of that identification:

$$P(t_1, \ldots, t_n) \wedge E(t_1, t'_1) \wedge \ldots \wedge E(t_n, t'_n) \to P(t'_1, \ldots, t'_n).$$

Since finite cardinalities of X are unbounded satisfying φ we again obtain $\varphi \in \mathrm{PFF}_\infty(\Sigma)$.

Since each pseudofinite formula with infinite spectrum is approximating by Remark 6, we have the following:

Corollary 8. *Any positive* $\forall$*-formula* ($\exists$*-formula,* $\exists\forall$*-formula*) $\varphi \in F(\Sigma)$ *with an infinite model belongs to* $\mathrm{PFF}_\infty(\Sigma)$ *and it is approximating.*

Corollary 9. *Any* (*incomplete*) *theory* T *with positive axioms and infinite models consists of sentences in* $\mathrm{PFS}_\infty(\Sigma)$ *which are approximating.*

Proposition 12. [8] *Let* Σ *be a signature without functional symbols of positive arities. Then any* $\forall$*-formula* ($\exists$*-formula,* $\exists\forall$*-formula*) $\varphi \in F(\Sigma)$ *with an infinite model is pseudofinite.*

Since formulae $\neg P$, for predicate symbols P, correspond to complements of interpretations of P, the arguments for Corollary 8 are valid for formulae φ in Proposition 12 producing $\varphi \in \mathrm{PFS}_\infty(\Sigma)$ and we have the following:

Corollary 10. *Let* Σ *be a signature without functional symbols of positive arities. Then any* $\forall$*-formula* ($\exists$*-formula,* $\exists\forall$*-formula*) $\varphi \in F(\Sigma)$ *with an infinite model belongs to* $\mathrm{PFF}_\infty(\Sigma)$ *and it is approximating.*

Corollary 11. [8] *Any* $\forall$*-axiomatizable* ($\exists$*-axiomatizable,* $\exists\forall$*-axiomatizable*) (*incomplete*) *theory* T *with infinite models and having a signature without functional symbols of positive arities consists of sentences in* $\mathrm{PFS}_\infty(\Sigma)$ *and they are approximating.*

Theorem 6. [8] *Any negative* $\forall$*-formula* ($\exists$*-formula,* $\exists\forall$*-formula*) $\varphi \in F(\Sigma)$ *with an infinite model is pseudofinite.*

As for pseudofinite formulae in [8] a natural **question** arises on the possibility of approximability of non-positive $\forall$-formulae, $\exists$-formulae, and $\exists\forall$-formulae in a language Σ with functional symbols having arities ≥ 1.

Remark 12. Partially answering the question, we consider a quantifier free formula φ of form $\neg t \approx t'$ for terms $t, t' \in T(\Sigma)$, where a sentence $\forall \overline{x} \neg t \approx t'$ is satisfied in an infinite model.

We realize φ by an unboundedly large finite model $\mathcal{M}$ with distinct elements for distinct subterms of t and t'. In such a case operations for Σ can be continued in M satisfying φ.

Taking a formula $\varphi = \neg f(t_1, \ldots, t_k) \approx g(t'_1, \ldots, t'_m)$ we introduce distinct elements a and a' with $f(t_1, \ldots, t_k) \equiv a$ and $g(t'_1, \ldots, t'_m) \equiv b$.

If $t = f(t_1, \ldots, t_k)$ and $t' = f(t'_1, \ldots, t'_k)$ we find distinct tuples of elements $a_1, \ldots, a_k$ for values of $t_1, \ldots, t_k$ and $a'_1, \ldots, a'_m$ for values of $t'_1, \ldots, t'_m$ such that $f(a_1, \ldots, a_k) = a$ for all tuples $a_1, \ldots, a_k$ related to $t_1, \ldots, t_k$ and for some new element a, as well as $f(a'_1, \ldots, a'_m) = a'$ for all tuples of elements $a'_1, \ldots, a'_m$ related to $t'_1, \ldots, t'_m$ and for some new element a'.

If t or t' is a variable x or a constant c then the opposite term t' or t takes a distinct value with respect to the value of x or c.

Since a finite model $\mathcal{M}$ can be chosen arbitrarily large, $\varphi \in \mathrm{PFF}_\infty(\Sigma)$ implying $\varphi \in \mathrm{AF}_\infty(\Sigma)$.

These finite realizations can be spread for conjunctions and disjunctions of $\neg t \approx t'$ and $\neg R(t_1, \ldots, t_n)$, where R are relational symbols, as well as for their $\forall$-, $\exists$- and $\exists\forall$-quantifications.

Thus we have the following modification of Theorem 6:

Theorem 7. *Any negative $\forall$-formula ($\exists$-formula, $\exists\forall$-formula) $\varphi \in F(\Sigma)$ with an infinite model is approximating.*

As noticed in [8] the pseudofiniteness for $\forall\exists$-formulae can be either satisfied or violated. Thus there is a border for the pseudofiniteness of a formula φ with respect to sequences of quantifiers represented by $\forall\exists$-formulae. This border for φ depends on the signature and the syntax of that formula. Similarly there is a border for approximating formulae.

6 Totally approximating sentences

As shown above, the family of $\mathcal{T}$-approximating formulae admits a classification with respect to its RS-rank, to related algebras, to syntactic, semantic and topological complexity, etc. It can be adapted both for natural classes of elementary theories and of their models, as well as for related topologies. Here we consider a topological influence to characteristics of "totally" approximating formulae and characterize this approximability for families of theories of finite models, in terms of their signature.

Definition. For a family $\mathcal{T}$, a sentence φ is called *totally $\mathcal{T}$-approximating* if $\mathcal{T}_\varphi$ consists of accumulation points for itself, i.e. $\mathcal{T}_\varphi$ does not contain isolated points.

The set of totally $\mathcal{T}$-approximating sentences is denoted by $\mathrm{TAS}(\mathcal{T})$.

Remark 13. By the definition and Proposition 1, $\mathrm{TAS}(\mathcal{T}) = \mathrm{TAS}(\mathrm{Cl}_E(\mathcal{T}))$.

Example 2. Let $\mathcal{T}$ be the family of theories in a signature of 0-ary predicate symbols Q_n, $n \in \omega$, such that for any sequence $\delta \in \omega^\omega$, there is $T_\delta \in \mathcal{T}$ with $Q_n^{\delta(n)} \in T_\delta$, $n \in \omega$. Clearly, each T_δ is an accumulation point in $\mathcal{T}$ implying that each consistent Boolean combination of sentences Q_n belong to $\mathrm{TAS}(\mathcal{T})$.

If $\mathcal{T}'$ is a restriction of $\mathcal{T}$ such that all theories except one, containing some chosen Q_i, say T_0, have been removed then $(\mathcal{T}')_{Q_i} = \{T_0\}$, $Q_i \notin \mathrm{TAS}(\mathcal{T}')$ whereas $\neg Q_i \notin \mathrm{TAS}(\mathcal{T}')$, and $Q_j \in \mathrm{TAS}(\mathcal{T}')$, where $j \neq i$, iff $Q_j \notin T_0$.

Proposition 13. *If $\varphi \in \mathrm{TAS}(\mathcal{T})$ then $\mathrm{RS}(\mathcal{T}_\varphi) = \infty$, moreover, $\mathcal{T}_\varphi$ does not contain e-totally transcendental neighbourhoods of the form $\mathcal{T}_\psi$.*

Proof. Since $\varphi \in \mathrm{TAS}(\mathcal{T})$ then $\mathcal{T}$ is infinite implying $\mathrm{RS}(\mathcal{T}_\varphi) \geq 1$. If $\mathrm{RS}(\mathcal{T}_\varphi) < \infty$ then $\mathrm{RS}(\mathcal{T}_\varphi)$ is a positive ordinal α which produces neighbourhoods $\mathcal{T}_\psi$, where $\psi \vdash \varphi$, of all lesser ordinals $\beta < \alpha$ including $\beta = 0$. The latter case $\mathrm{RS}(\mathcal{T}_{\psi_0}) = 0$ means that ψ_0 belongs to finitely many theories $T_1, \ldots, T_n$ of $\mathcal{T}$ and $\mathcal{T}_{\psi_0}$ is divided into one-element neighbourhoods. That possibility contradicts that all $T_i \in \mathcal{T}_\varphi$ should be accumulation points with respect to $\mathcal{T}_\varphi$.

These arguments also show that $\mathcal{T}_\varphi$ does not contain e-totally transcendental $\mathcal{T}_\psi$. $\square$

The following proposition gives a description of signatures producing totally approximating sentences with one-element models. It also illustrates approximating formulae for the class of one-element models.

Theorem 8. *For any signature Σ and the sentence $\varphi_1 = \forall x \forall y\, (x \approx y)$ the following conditions are equivalent:*

(1) $\varphi_1 \in \mathrm{TAS}(\mathcal{T}_\Sigma)$;

(2) $(\mathcal{T}_\Sigma)_{\varphi_1}$ *is infinite;*

(3) Σ *contains infinitely many predicate symbols.*

Proof. By the choice of φ_1 we consider theories in $\mathcal{T}_\Sigma$ with one-element models only. In such a case operations on models $\mathcal{M}$ of φ_1, including constants, are uniquely defined. Therefore possibilities for theories in $(\mathcal{T}_\Sigma)_{\varphi_1}$ are defined by possibilities for $P_i = \emptyset$ and $P_i \neq \emptyset$ of predicate symbols P_i in Σ.

(1) $\Rightarrow$ (2) follows since each approximating sentence has an infinite neighbourhood.

(2) $\Rightarrow$ (3). Let Σ contain finitely many predicate symbols P_i. Then there are finitely many possibilities for $P_i = \emptyset$ and $P_i \neq \emptyset$ of predicate symbols P_i in Σ. Therefore $(\mathcal{T}_\Sigma)_{\varphi_1}$ is finite.

(3) $\Rightarrow$ (1). If Σ contains infinitely many predicate symbols P_i then, in any considered structure, P_i is uniquely defined by the condition $P_i = \emptyset$ or $P_i \neq \emptyset$ and there exists at least continuum many possibilities for non-elementary equivalent Σ-structures producing at least continuum many theories T containing φ_1. Moreover, each T is an accumulation point in

$\mathcal{T}_\Sigma$ since the possibilities $P_i = \emptyset$ and $P_i \neq \emptyset$ are independent. Thus $\varphi_1 \in \mathrm{TAS}(\mathcal{T}_\Sigma)$. □

The arguments for consistent sentences ψ implying φ_1 stay valid, since each sentence contains finitely many signature symbols. Hence the following corollary holds.

Corollary 12. *For any signature Σ and a sentence $\psi \in \mathrm{CF}(\Sigma)$ implying φ_1 the following conditions are equivalent:*
(1) $\psi \in \mathrm{TAS}(\mathcal{T}_\Sigma)$*;*
(2) $(\mathcal{T}_\Sigma)_\psi$ *is infinite;*
(3) Σ *contains infinitely many predicate symbols.*

These characterizations are spread till theories with n-element structures only, for $n \in \omega \setminus \{0, 1\}$.

Theorem 9. *For any signature Σ and a sentence φ_n expressing that there are exactly n elements, for $n \in \omega \setminus \{0, 1\}$, the following conditions are equivalent:*
(1) $\varphi_n \in \mathrm{TAS}(\mathcal{T}_\Sigma)$*;*
(2) $(\mathcal{T}_\Sigma)_{\varphi_n}$ *is infinite;*
(3) Σ *is infinite.*

Proof. (1) ⇒ (2) follows as in the proof of Theorem 8.

(2) ⇒ (3). It is well known that any theory of finite model in a finite signature is finitely axiomatizable by a sentence describing all operations and relations up to isomorphism. Thus if Σ is finite then $(\mathcal{T}_\Sigma)_{\varphi_n}$ can not be infinite.

(3) ⇒ (1). Let Σ be infinite. Since each sentence contains finitely many signature symbols it can not control the behavior of cofinitely many signature operations and relations. Then we can construct an infinite binary tree marking each new predicate P_i to be empty/nonempty and new function f_j to be constant with the value a_1 or a_2 in a structure with the universe $\{a_1, \ldots, a_n\}$. The existence of that tree implies that theories in $\mathcal{T}_\Sigma$ with n-element models can not be isolated by sentences, i.e. $\varphi_n \in \mathrm{TAS}(\mathcal{T}_\Sigma)$. □

Arguments for Theorem 9 immediately imply:

Corollary 13. *For any signature Σ and a sentence $\psi \in \mathrm{CF}(\Sigma)$ implying φ_n, where φ_n expresses that there are exactly n elements, for $n \in \omega \setminus \{0, 1\}$, the following conditions are equivalent:*
(1) $\psi \in \mathrm{TAS}(\mathcal{T}_\Sigma)$*;*
(2) $(\mathcal{T}_\Sigma)_\psi$ *is infinite;*
(3) Σ *is infinite.*

Clearly, the arguments above can be extended for families of theories of models in arbitrary cardinalities.

7 Conclusion

We introduced the notion of approximating formula. We studied links between approximating and pseudofinite formulae as well as their spectra.

Boolean algebras, lattices and semilattices related to approximating formulae are found. Relations between approximating formulae and positive, negative, $\forall$-formulae, $\exists$-formulae, $\exists\forall$-formulae, $\forall\exists$-formulae are described. The family of all consistent formulae in a given signature is considered and the existence of approximating non-pseudofinite consistent formulae in that signature is characterized. Connections of approximating formulae and ranks for families of theories are studied, and approximations for the class of finite models are characterized. It would be interesting to consider application of a general theory of approximating formulae to families in natural classes of theories.

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