SPECIAL ELEMENTS ON PRINCIPALLY ORDERED REGULAR SEMIGROUPS

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Abstract. In this paper, we investigate principally ordered regular semigroups, S. In section 1, we start with some basic and technical results: (a) S is an inverse semigroup if and only if the operation $x \to x^0$ is the identity, when restricted to the the set of idempotents of S, that is, $e = e^0$ for every $e \in E(S)$; (b) If S has a zero element, then 0^* is the biggest element (in fact, idempotent) of S. We prove, in section 2, that S is naturally ordered if and only if S^* is a singleton. Then, we obtain several results in a principally ordered regular semigroup S with zero element, 0, that were previously obtained in [**13**] under stronger hypothesis. In section 3, we obtain necessary and sufficient conditions for a naturally ordered regular semigroup with a zero element, S, to have as a subalgebra a copy of N_5, the smallest naturally ordered non-orthodox regular semigroup with a biggest idempotent. We then study, in section 4, principally ordered regular semigroups S, for which the operation $x \to x^*$ satisfies $e = e^*$, for every idempotent $e \in S$. We prove that under this condition the semigroup S is dually naturally ordered, inverse and $S = S^0$. We obtain the following results: (1) S has a zero element if and only if S has a greatest element which is idempotent; (2) S has an identity element if and only if S has a smallest idempotent; (3) If $E(S)$ is a finite set, then S has a greatest idempotent and hence $E(S)$ is a band with a zero element; (4) the $*$-subsemigroup generated by a pair of idempotents is described; (5) S is completely simple if and only if S is a group.

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1. Introduction

We recall (see, for example [**1**]) that the *natural order*, $\leq_n$, on the idempotents of a regular semigroup S, is defined by

$$e \leq_n f \iff e = ef = fe$$

and that an ordered regular semigroup $(T, \leq)$ is said to be

(1) *naturally ordered* if the order extends the natural order, in the sense that if $e \leq_n f$ then $e \leq f$.

(2) *dually naturally ordered* if the order reverses the natural order, in the sense that if $e \leq_n f$ then $f \leq e$.

An ordered regular semigroup S is said to be *principally ordered* if, for every $x \in S$, there exists $x^* = \max\{y \in S | xyx \leq x\}$.

The basic properties of the operation $x \to x^*$, in principally ordered regular semigroups semigroups, were established in [**4**] and [**5**] and are listed in [**1**, Theorem 13.26].

In particular, we recall for the reader's convenience that, in such a semigroup, the following properties hold, and will be used throughout in what follows:

(P_1) $\big(\forall x \in S\big)$ $x = xx^*x$;
(P_2) every $x \in S$ has a biggest inverse, namely $x^0 = x^*xx^*$;
(P_3) $\big(\forall x \in S\big)$ $x^0 \leq x^*$;
(P_4) $\big(\forall x \in S\big)$ $xx^0 = xx^*$ and $x^0x = x^*x$;
(P_5) $\big(\forall e \in E(S)\big)$ $e \leq e^0 \leq e^*$;
(P_6) $\big(\forall x \in S\big)$ $x \leq x^{**}$;
(P_7) $\big(\forall x \in S\big)$ $x^* = x^{***}$;
(P_8) $\big(\forall x \in S\big)$ $x^{0*} = x^{**} = x^{*0}$.

In any ordered regular semigroup S, in which every $x \in S$ has a biggest inverse x^0, it was proven in [**14**], and is stated in [**1**, Theorem 13.22], that

(P_9) $\big(\forall x \in S\big)$ $(xx^0)^0 = x^{00}x^0$ and $(x^0x)^0 = x^0x^{00}$;
(P_{10}) $\big(\forall x \in S\big)$ $x^0 = x^{000}$.
(P_{11}) $(x, y) \in \mathcal{R} \iff xx^0 = yy^0$; $(x, y) \in \mathcal{L} \iff x^0x = y^0y$.

We recall that Green's relations $\mathcal{R}$ and $\mathcal{L}$ are said to be *regular* if for every $x, y \in S$, such that $x \leq y$, we have that $xx^0 \leq yy^0$ and $x^0x \leq y^0y$.

Similarly, Green's relations $\mathcal{R}$ and $\mathcal{L}$ are *weakly regular* if for every $e, f \in E(S)$, such that $e \leq f$, we have that $ee^0 \leq ff^0$ and $e^0e \leq f^0f$.

Clearly, if Green's relations $\mathcal{R}$ and $\mathcal{L}$ are regular, they are also weakly regular.

Properties (P_9), (P_{10}) and (P_{11}) hold in a principally ordered regular semigroup since by (P_2), biggest inverses in such a semigroup exist.

In the sequel we shall refer to the application $x \to x^*$ as being *antitone*, if $x \leq y$ implies that $y^* \leq x^*$.

An important property, obtained in [**5**, Theorem 3.3], that holds in a principally ordered regular semigroup S is, that the application $x \to x^*$ is antitone if and only if, S is naturally ordered.

Similarly, we say that the application $x \to x^*$ is *weakly isotone* if, for every $e, f \in E(S)$ such that $e \leq f$, we have that $e^* \leq f^*$.

Throughout this paper we consider S a principally ordered regular semigroup, with the set of idempotents $E(S)$. If S has a zero element it will be denoted by 0.

In the next result, we present a necessary and sufficient condition for a principally ordered regular semigroups to be an inverse semigroup.

Lemma 1. *A principally ordered regular semigroup, S, is an inverse semigroup if and only if the operation $x \to x^0$ is E-identity, that is, $e = e^0$, for every $e \in E(S)$.*

Proof. It follows from [**11**, Theorem 5.1.1] that a semigroup is inverse if and only if each element has a unique inverse, if and only if every $\mathcal{R}$-class and every $\mathcal{L}$-class have a unique idempotent.
For an idempotent e, in a principally ordered regular semigroup S, we have that e is an inverse of itself and, by (P_2), e^0 is also an inverse of e.
If, on one hand, S is an inverse semigroup we can immediately conclude that $e = e^0$. If, on the other hand, $e = e^0$, for every $e \in E(S)$, let us assume that $f, g \in E(S)$ are such that $(f, g) \in \mathcal{L}$. Then, by (P_{11}), $f^0f = g^0g$ and therefore, by hypothesis, $f = g$ which means that every $\mathcal{L}$ has a unique idempotent. Similarly, every $\mathcal{R}$ has a unique idempotent. Therefore, S is an inverse semigroup. □

In the next Lemma we generalise a result on principally ordered regular semigroup with a zero element, obtained in [**13**], with the additional condition of S being BZS (Boolean Zero Square), that is, the square of every element is either zero or itself.

Lemma 2. *If S is a principally ordered regular semigroup with a zero element 0, then $0^* = 0^{**} \in E(S)$ is the biggest element of S.*

Proof. For any $x \in S$,

$$0 \cdot x \cdot 0 = 0 \leq 0 \quad \Longrightarrow \quad x \leq 0^*$$

so, 0^* is the biggest element of S. Using this fact, it follows immediately that $0^{**} \leq 0^*$. Now,

$$0^* \cdot 0^* \cdot 0^* \leq 0^* \quad \Longrightarrow \quad 0^* \leq 0^{**}$$

and therefore $0^{**} = 0^*$.

Finally,

$$0^* \cdot 0^*0^* \cdot 0^* \leq 0^* \quad \Longrightarrow \quad 0^*0^* \leq 0^{**} = 0^*$$

and

$$0^* = (0^*0^{**})0^* \leq 0^*0^*$$

which allows us to conclude that $0^* = 0^{**} \in E(S)$. □

2. S Naturally Ordered with a Zero Element

In this section we consider S a principally ordered regular semigroup with a zero element, 0, that it is also naturally ordered.

We start with some technical properties that were partially obtained in [**13**] on a different perspective. The new proofs are included for the sake of completeness.

Lemma 3. *If S is a principally ordered naturally ordered regular semigroup with a zero element 0, then*

(1) *The zero of S is the smallest element of S;*
(2) $\big(\forall x \in S\big)$ $0 \leq x \leq 0^* = x^* \in E(S)$;
(3) $\big(\forall x \in S\big)$ $0 \leq x \leq x^0 = x^{00} = 0^*x0^* \in E(S)$;
(4) *Green's relations $\mathcal{R}$ and $\mathcal{L}$ are regular;*
(5) *If S is a monoid then the identity element is the greatest element of S.*

Proof. Assume that x is any element of S.

(1): We have that

$$0 \cdot xx^* = 0 = xx^* \cdot 0 \quad \Longrightarrow \quad 0 \leq_n xx^* \quad \Longrightarrow \quad 0 \leq xx^*$$

from which we obtain, multiplying on the right by x, that

$$0 = 0 \cdot x \leq xx^* \cdot x = x$$

which means that 0 is the smallest element of S.

(2): From (1) and Lemma 2, we have that $0 \leq x \leq 0^*$, for every x in S. Using the fact that the application $x \to x^*$ is antitone, we have

$$0 \leq x \leq 0^* \quad \Longrightarrow \quad 0^* = 0^{**} \leq x^* \leq 0^* \quad \Longrightarrow \quad x^* = 0^*$$

and therefore $0 \leq x \leq 0^* = x^*$.

(3): For $x \in S$ we have, by (P_2), (2) and Lemma 2, that

$$x^0 = x^*xx^* = 0^*x0^* = 0^*0^*x0^*0^* = (x^0)^*x^*xx^*(x^0)^* = (x^0)^*x^0(x^0)^* = x^{00}$$

By (2) we have that x^0 is an idempotent, since

$$x^0x^0 = x^*xx^*x^*xx^* = x^*xx^*xx^* = x^0$$

Finally, from, (1), (P_1) and Lemma 2, we have

$$0 \leq x = xx^* \cdot x \cdot x^*x \leq 0^* \cdot x \cdot 0^* = x^*xx^* = x^0$$

(4): Let $x, y \in S$ be such that $x \leq y$. Then, using (P_4) and (2), we have that

$$xx^0 = xx^* = x0^* \leq y0^* = yy^* = yy^0$$

and similarly, $x^0x \leq y^0y$. Therefore, $\mathcal{R}$ and $\mathcal{L}$ are regular.
(5): Let us denote the identity element of S by 1. We have, by Lemma 2, that $1 \leq 0^*$. Now, $0^* \cdot 1 = 1 \cdot 0^* = 0^*$, which means that $0^* \leq_n 1$, and since S is naturally ordered we obtain $0^* \leq 1$. Thus, $1 = 0^*$ is the greatest element of S. □

The equivalence of (1) and (2) in the next Theorem, is obtained in [**11**, Theorem 2.4], with the additional condition of being BZS.

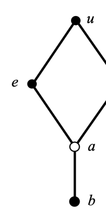
Theorem 1. *If S is a principally ordered regular semigroup with a zero element 0, then the following statements are equivalent:*
(1) *S is naturally ordered;*
(2) $\big(\forall e, f \in E(S)\big)$ $e^* = f^*$*;*
(3) *S^* is a singleton.*

Proof. (1) $\iff$ (2): This follows like in [**11**, Theorem 2.4].
(1) $\implies$ (3): This follows immediately by Lemma 3(2).
(3) $\implies$ (1): For every $x, y \in S$, we have that

$$xy(xy)^* \leq x0^*0^* = x0^* = xx^*$$

Therefore, by [**5**, Theorem 3.2], S is naturally ordered. □

Example 1. *In [3], Blyth and McFadden presented, using the following Hasse diagram and Cayley table:*



	u	e	f	a	b
u	u	u	f	f	b
e	e	e	a	a	b
f	u	b	f	b	b
a	e	b	a	b	b
b	b	b	b	b	b

an ordered semigroup N_5, which is the smallest naturally ordered non-orthodox regular semigroup with a biggest idempotent.

From [4] we know that N_5 is a naturally ordered principally ordered regular semigroup with a zero element b. In fact, b is the smallest element (idempotent) of N_5, with $u = b^*$ the biggest element (idempotent) of N_5 and that ${N_5}^* = \{u\}$, which is a singleton as stated in Theorem 1.

We now generalise several results of [**13**] without the hypothesis of being BZS. More specifically, we obtain Theorem 2.2, Theorem 2.3, Corollary 2.6, Theorem 2.13 and Theorem 2.14, obtained in [**13**], in an ordered regular semigroup with a zero element which is, simultaneously, principally ordered and naturally ordered. Theorem 2.5 and Theorem 2.12 are also obtained removing the BZS hypothesis.

In the next two Theorems, we consider the set S^0, which is the set of the biggest inverses of the elements of S. It follows, from Lemma 3(3), that S^0 is a subset of $E(S)$.

In [**4**, Theorem 4] Blyth and Pinto proved that if S is a naturally ordered principally ordered regular semigroup, then S^0 is a subsemigroup of S. We can now prove that if S has a zero element then S^0 is, in fact, a semilattice.

Theorem 2. *If S is a naturally ordered principally ordered regular semigroup with a zero element 0, then the subsemigroup S^0 of S is a semillatice, for which*
$$\big(\forall x, y \in S\big)\ x^0 y^0 = (x0^* y)^0 = y^0 x^0.$$

Proof. The proof follows like in [**13**, Theorem 2.2]. □

To answer the question: when is S itself a semillatice, we need first to remind the definition of several types of bands.

A band B is said to be *left regular* (see for example [**10**]) if $xyx = xy$, for all $x, y \in B$. Similarly, B is a *right regular* band if $xyx = yx$, for all $x, y \in B$. A band B for which $xy = yx$, for all $x, y \in B$, is said to be a *semilattice*.

Theorem 3. *Let S be a naturally ordered principally ordered regular semigroup with a zero element 0. The following statements are equivalent:*
(1) *$S = S^0$;*
(2) *$\big(\forall x \in S\big)\ x = x^0$;*
(3) *S is a left regular band and a right regular band;*
(4) *S is a semillatice.*

Proof. The proof follows like in [**13**, Theorem 2.3]. □

A *strong Dubreil-Jacotin semigroup* (see, for example, [**1**]) is an ordered semigroup S for which there exists an ordered group G and an epimorphism $f : S \to G$ that is *residuated*, in the sense that the pre-image under f of every principal order ideal of G is a principal order ideal of S. In particular, the pre-image of the negative cone $N(G) = \{x \in G | x \leq 1\}$ is a principal order ideal $\xi^{\downarrow} = \{x \in S | x \leq \xi\}$ of S. To the top element ξ we call *bimaximum element*. This element is said to be *equiresidual* if, for every $x \in S$, the order ideals $\{y \in S | xy \leq \xi\}$ and $\{y \in S | yx \leq \xi\}$ coincide and have a greatest element denoted by $\xi : x$. When S is regular, the bimaximum element ξ is the biggest idempotent of S and if, e is an idempotent then $\xi : e = \xi$. The set of those elements $x \in S$, for which $x = x(\xi : x)x$ is called the set of *perfect* elements.

In [**1**, Theorem 13.28] it is proved that a naturally ordered regular semigroup S is a strong Dubreil-Jacotin semigroup if and only if S is, simultaneously, principally ordered and has a biggest idempotent. In the case where S is a naturally ordered regular semigroup S with a zero element then, this result can be rephrased in the following way.

Theorem 4. *Let S be a naturally ordered regular semigroup with a zero element 0. The following statements are equivalent:*
(1) *S is principally ordered;*
(2) *S is a strong Dubreil-Jacotin semigroup.*

Proof. The proof follows like in [**13**, Theorem 2.5]. □

An obvious conclusion of the previous results, can be summarise in the following:

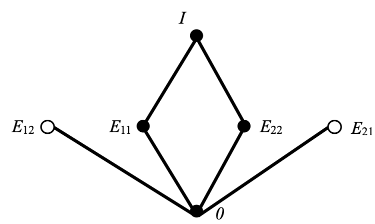
Corollary 1. *If S is naturally ordered principally ordered regular semigroup with a zero element 0, then S is a strong Dubreil-Jacotin semigroup with a biggest and a smallest elements (in fact, idempotents).*

Let us now present some examples to illustrate this type of semigroups.

Example 2. *The set of* 2×2 *real matrices* $S = \{I, O, E_{11}, E_{12}, E_{21}, E_{22}\}$*, where* O *is the zero matrix and*

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

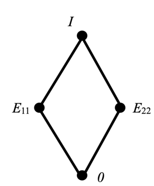
with the usual product of matrices is an inverse ordered semigroup with a partial order defined by the following Hasse diagram



We obtain a naturally ordered inverse semigroup. It can be checked directly from the Hasse diagram that S *has a biggest idempotent* I*, that is not a biggest element. Now,* S *is not principally ordered since for example,* E_{12}^{*} *does not exist. Therefore by Theorem 4,* S *is not a strong Dubreil-Jacotin semigroup.*

Also, the identity element of S *is not the greatest element of* S*. This does not contradicts Lemma 3(5), since* S *is not principally ordered.*

Example 3. *The set of* 2×2 *real matrices* $T = \{I, O, E_{11}, E_{22}\}$*, is a subsemigroup of the semigroup* S *of Example 2, where the partial order is defined by the following Hasse diagram*



We obtain a principally ordered naturally ordered regular semigroup, with $T^{*} = \{I\}$*. It can be checked directly from the Hasse diagram that the identity element of* T *is the greatest element of* T*, which illustrates property* (5) *of Lemma 3.*

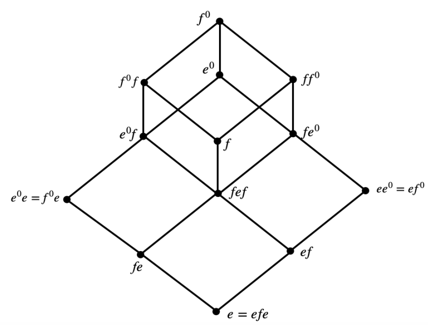
In [**1**, Exercise 13.19] it is mentioned that in a strong Dubreil-Jacotin regular semigroup S, the set of perfect elements $P(S) = \{x \in S | x(\xi : x)x = x\}$, is a regular subsemigroup of S. With this in mind, we can prove the following result.

Theorem 5. *If* S *is a strong Dubreil-Jacotin regular semigroup with a zero element* 0*, then the set of perfect elements* $P(S)$*, is a strong Dubreil-Jacotin if and only if, the following statements hold:*
(1) $P(S)$ *is naturally ordered;*
(2) $P(S)$ *is principally ordered.*

Proof. The proof follows like in [**13**, Theorem 2.12]. □

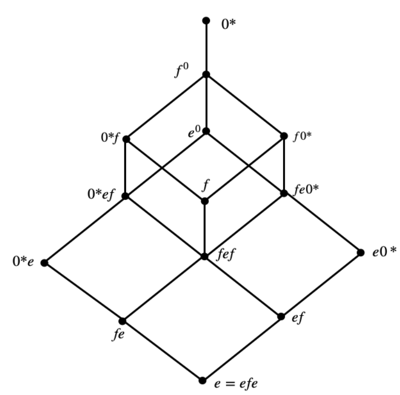
In the next two theorems we describe in a naturally ordered principally ordered regular semigroup S with a zero element, what is the structure generated by a pair of comparable idempotents of S. More specifically, if e, f are two comparable idempotents of S, we present the Hasse diagrams of the subsemigroup generated by the set $\{e, f, e^0, f^0\}$, and of the subsemigroup of S generated by $\{e, f\}$ that preserves the $*$−operation. In order to prove these statements, we shall use results of Blyth and Pinto, proved in [**6**] and [**7**].

Theorem 6. *Let S be a naturally ordered principally ordered regular semigroup with a zero element, 0. If $e, f \in E(S)$ are such that $e \leq f$ then the subsemigroup generated by the set $\{e, f, e^0, f^0\}$ has the Hasse diagram*



Proof. The proof follows like in [**13**, Theorem 2.13]. □

Theorem 7. *Let S be a naturally ordered principally ordered regular semigroup with a zero element, 0. If $e, f \in E(S)$ are such that $e < f$ then the subsemigroup T of S generated by $\{e, f\}$ that preserves the $*$−operation is a band having at most 14 elements. In the case where T has precisely 14 elements it is represented by the Hasse diagram*



where lines with positive slope are $\mathcal{R}$ related, lines with negative slopes are $\mathcal{L}$ related, and vertical lines also denote the natural order.

Proof. The proof follows like in [**13**, Theorem 2.14]. □

3. N_5 as a Subalgebra of a Naturally Ordered Principally Regular Semigroup with a Zero Element

In order to obtain necessary and sufficient conditions, for a naturally ordered principally ordered regular semigroup, S, with a zero element, 0, to have a subalgebra N, isomorphic to N_5, that is,

(a) if $x, y \in N$ then xy calculated in S is an element of N;

(b) if $x \in N$ then x^* calculated in S is an element of N;

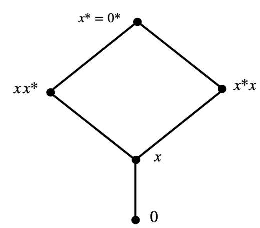
(c) the zero element of S belongs to N.

we introduce two preliminary lemmas.

Lemma 4. *Let S be a naturally ordered principally ordered regular semigroup with a zero element 0. For any $x \in S$, that is not an idempotent, the elements of the set*

$$N = \{0, x, xx^*, x^*x, x^* = 0^*\}$$

are distinct with the following Hasse diagram



Proof. Let x be a non-idempotent element of S. We have, by Lemma 3, that

$$0 \leq x \leq x^0 \leq x^* = 0^*$$

Now considering the set $N = \{0, x, xx^*, x^*x, x^* = 0^*\}$, we have that

$$0 \leq x = x \cdot x^*x \leq x0^* = xx^* \leq 0^*$$

and

$$0 \leq x = xx^* \cdot x \leq 0^*x = x^*x \leq 0^*$$

If $xx^* \leq x^*x$ then

$$x = xx^*x = xx^* \cdot x^*x^*x \leq x^*x \cdot x^*x^*x = x^*x \cdot x^*x = x^*x \quad \Longrightarrow \quad x^2 \leq x$$

and

$$x = xx^* \cdot xx^* \cdot x \leq xx^* \cdot x^*x \cdot x = xx^*xx = x^2 \quad \Longrightarrow \quad x \leq x^2$$

from which we obtain that x is an idempotent, which is a contradiction. Similarly, $x^*x \leq xx^*$ lead us into a contradiction. Therefore, xx^* and x^*x are non comparable. To complete the proof we have to verify that these five elements are distinct.
The fact that x is not idempotent immediately tell us that

$$0 < x < xx^* \qquad \text{and} \qquad 0 < x < x^*x.$$

If $xx^* = x^*$ then

$$x \cdot xx^*x = xx^*x \quad \Longrightarrow \quad x^2 = x$$

which is a contradiction, and therefore, $xx^* < x^*$. Similarly, $x^*x < x^*$.

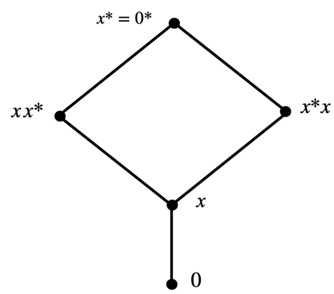
The result now follows. □

Lemma 5. *Let S be a naturally ordered principally ordered regular semigroup with a zero element 0. For any $x \in S$, that is not an idempotent such that $x^2 = 0$ and $x^0 = 0^*$, the elements of the set*

$$N = \{0, x, xx^*, x^*x, x^* = 0^*\}$$

with the induced order form a subalgebra of S, which is isomorphic to N_5.

Proof. We have, by Lemma 4, that $N = \{0, x, xx^*, x^*x, x^* = 0^*\}$ is closed under the multiplication defined in S, with Hasse diagram and multiplication table as follows:



	0^*	xx^*	$x^*\ x$	x	0
0^*	0^*	0^*	$x^*\ x$	$x^*\ x$	0
xx^*	$x\ x^*$	$x\ x^*$	x	x	0
x^*x	0^*	0	$x^*\ x$	0	0
x	$x\ x^*$	0	x	0	0
0	0	0	0	0	0

which are precisely the same Hasse diagram and the same table than the ones of N_5 described in Example 1, with the elements b, a, e, f, u replaced respectively, by $0, x, xx^*, x^*x, x^* = 0^*$.
As an example, note that, by hypothesis, $x^* \cdot xx^* = x^0 = 0^*$.
By Lemma 3, we have in N, like in S that

$$0^* = x^* = (xx^*)^* = (x^*x)^* = (0^*)^*.$$

which means that the *-operation coincide in N and in S. Also, the zero of N is, by construction, the same as the zero of S.
Therefore,

$$N = \{0, x, xx^*, x^*x, x^* = 0^*\}$$

is a subalgebra of S isomorphic to N_5. □

With these Lemmas we can now obtain necessary and sufficient conditions for a naturally ordered principally ordered regular semigroup with a zero element to have, as a subalgebra, a copy of N_5.

Theorem 8. *Let S be a naturally ordered principally ordered regular semigroup with a zero element 0. The following statements are equivalent:*
(1) *S has a subalgebra that is algebraic isomorphic and order isomorphic to N_5.*
(2) *There exists a non-idempotent, x in S for which $x^2 = 0$ and $x^0 = 0^*$.*

Proof. (1) $\Longrightarrow$ (2): This follows immediately from Example 1. In fact, a subalgebra of S has to preserve the *-operation and the zero element must be the zero of S.

If we consider in this subalgebra, the element that covers zero, that is, the one element playing the role of a in N_5, we obtain that $a^2 = b = 0$ and $a^* = u = 0^*$, which means that (2) holds.
(2) $\Longrightarrow$ (1): This follows immediately by Lemma 5. □

Since N_5 is a non-orthodox semigroup it follows, immediately, from the previous Theorem the following result.

Corollary 2. *If S be a naturally ordered principally ordered regular semigroup with a zero element 0 such that there exists a non-idempotent, x in S for which $x^2 = 0$ and $x^0 = 0^*$, then S is not orthodox.*

4. $x \to x^*$ is E-identity

Let us consider a principally ordered regular semigroups S. We say that the operation $x \to x^*$, in S, is *E-identity* when $e = e^*$ for all $e \in E(S)$,

We start this section by observing that under these conditions such a semigroup is an inverse one.

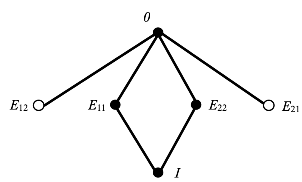
Lemma 6. *Let S be a principally ordered regular semigroup for which the operation $x \to x^*$ is E-identity. Then, S is an inverse semigroup.*

Proof. Let e be any element of $E(S)$. By (P_5), we have that $e \leq e^0 \leq e^*$, which by hypothesis, gives us that $e = e^0 = e^*$. Then, from Lemma 1, we can conclude that S is an inverse semigroup. □

Example 4. *It can be seen in [**2**, Example 1] that the set of 2×2 real matrices $S_6 = \{I, O, E_{11}, E_{12}, E_{21}, E_{22}\}$, where O is the zero matrix and*

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with the usual product of matrices (the same as in Example 2) is an inverse ordered semigroup with a partial order (the dual order of the one considered in Example 2) defined by the following Hasse diagram



Routine calculations show that S_6 is principally ordered with

$$E_{11}^* = E_{11},\quad E_{22}^* = E_{22},\quad E_{12}^* = E_{21},\quad E_{21}^* = E_{12},\quad I^* = I,\quad O^* = O$$

and

$$E_{11}^0 = E_{11},\quad E_{22}^0 = E_{22},\quad E_{12}^0 = E_{21},\quad E_{21}^0 = E_{12},\quad I^0 = I,\quad O^0 = O$$

which tells us that $S_6 = S_6^* = S_6^0$.

Also, S_6 is dually naturally ordered and, since $E(S_6) = \{E_{11}, E_{22}, O, I\}$, it follows immediately that both operations $x \to x^*$ and $x \to x^0$ are E-identity.

The following example shows that the converse of Lemma 6 does not hold.

Example 5. *Let G be a discretely ordered group. Adjoin to G an element z and add the single relation $z < 1_G$. Extend the multiplication of G to $S = G \cup \{z\}$ by defining $z^2 = z$ and $xz = x = zx$, for all $x \in G$.*
*From [**1**, Exercise 13.2] we can say that S is an inverse semigroup. In [**9**, Example 2] it is stated that this semigroup is principally ordered. Routine calculations give us that $x^* = x^0 = x^{-1}$, for every $x \in G$. Also, $z^* = 1_G$ and $z^0 = z$, which means that $x \to x^0$ is E-identity, but $x \to x^*$ is not E-identity. We obtain that $S = S^0$ but $S \neq S^*$.*
Since $1_G \leq_n z$ and $z \leq 1_G$, we can conclude that S is dually naturally ordered.

Like Example 4 and Example 5 suggest, it is true in general that if S is a principally ordered regular semigroups for which $x \to x^*$ is E-identity, then S is dually naturally ordered and $S = S^0$. That is the content of the next two Theorems.

Theorem 9. *If S is a principally ordered regular semigroup for which the operation $x \to x^*$ is E-identity, then, S is dually naturally ordered.*

Proof. Let $e, f \in E(S)$ be such that $e \leq_n f$, that is, $ef = e = fe$. Then,

$$efe = e \quad \Longrightarrow \quad f \leq e^* = e$$

which means that S is dually naturally ordered. □

Theorem 10. *If S is a principally ordered regular semigroup for which the operation $x \to x^*$ is E-identity, then, $S = S^0$.*

Proof. For any $x \in S$ we have that $xx^*, x^*x \in E(S)$ and, by hypothesis, we can say that $(xx^*)^* = xx^*$ and $(x^*x)^* = x^*x$. Now, using (P_1), (P_4), (P_9) and (P_2), we have that

$$x = xx^*x = (xx^*)^*x = (xx^0)^*xx^0x = (xx^0)^0xx^0x = x^{00}x^0x.$$

and, similarly, $x = xx^0x^{00}$.
Therefore,

$$x = xx^0x^{00} = x^{00}x^0xx^0x^{00} = x^{00}x^0x^{00} = x^{00}$$

which allows to conclude that $S = S^0$. □

In both Example 4 and Example 5 we can see that we have an identity element which is the smallest idempotent of the semigroup.

In Example 4 we have the presence of a zero element which is the greatest element of the semigroup, but in Example 2 none of them (neither a greatest element nor a zero element) exist.

This is not a coincidence. In the next two Theorems we relate the existence of a zero element with the existence of a greatest element, and the existence of an identity element with the existence of a smallest idempotent.

Theorem 11. *Let S be a principally ordered regular semigroup for which the operation $x \to x^*$ is E-identity. The following statements are equivalent*

(1) *S has a greatest element (in fact an idempotent) ξ.*

(2) *ξ is the zero element of S.*

Proof. (2) $\Longrightarrow$ (1): Let us assume that ξ is the zero element of S. For any $x \in S$, we have that

$$\xi \cdot xx^* = \xi = xx^* \cdot \xi \quad \Longrightarrow \quad \xi \leq_n xx^*$$

and since, by Theorem 9, S is dually naturally ordered, we obtain that $xx^* \leq \xi$. Multiplying on the right by x gives

$$x = xx^*x \leq \xi x = \xi \quad \Longrightarrow \quad x \leq \xi.$$

Thus, ξ is the greatest element of S.
(1) $\Longrightarrow$ (2): Let us assume that ξ is the greatest element of S. We have that

$$\xi \cdot \xi \leq \xi = \xi\xi^*\xi = (\xi\xi^*)\xi \leq \xi \cdot \xi \quad \Longrightarrow \quad \xi \in E(S).$$

For any $e \in E(S)$, we have

$$\xi e = \xi eee \leq \xi e\xi e \leq \xi\xi\xi e = \xi e \quad \Longrightarrow \quad \xi e \in E(S).$$

Similarly, $e\xi \in E(S)$ and $e\xi e \in E(S)$. Now,

$$\xi e\xi \cdot \xi e\xi \leq \xi e\xi\xi\xi\xi = \xi e\xi = \xi eeee\xi \leq \xi e\xi \cdot \xi e\xi$$

which means that $\xi e\xi \in E(S)$ is such that $\xi e\xi \leq_n \xi$ and, using Theorem 9, we can say that

$$\xi \leq \xi e\xi \leq \xi\xi\xi = \xi$$

and therefore $\xi e\xi = \xi$.
Thus, $\xi \in V(e\xi e)$ and since $(e\xi e)^0$ is the greatest inverse of $e\xi e$, we have that

$$\xi \leq (e\xi e)^0 \leq (e\xi e)^* = e\xi e$$

using the fact that $x \to x^*$ is E-identity. Then, since ξ is the greatest element of S,

$$\xi = e\xi e \quad \Longrightarrow \quad \xi\xi = \xi e\xi e \implies \quad \xi = \xi e$$

and, similarly, $\xi = e\xi$.

Now, for any $x \in S$, we have, since x^*x is an idempotent, that

$$\xi x \leq \xi\xi = \xi = \xi x^* x \leq \xi\xi x = \xi x \quad \Longrightarrow \quad \xi x = \xi$$

and, similarly, $x\xi = \xi$.
Therefore, ξ is the zero element of S. □

Next example shows that the equivalence in the previous Theorem does not hold, if we do not consider the hypothesis that $x \to x^*$ is E-identity.

Example 6. *Let $S = \{e, f, g\}$ be a band where g is a zero element and $ef = f = fe$, chain-ordered in the following way: $f < g < e$.*
Routine calculations allows to conclude that S is a principally ordered inverse semigroup where $f^* = g^* = e^* = e$, hence $x \to x^*$ is not E-identity. We have that e is the greatest element of S, but it is not the zero element of S.
Therefore, we can conclude, from this example, that in the previous Theorem the hypothesis that the operation $x \to x^*$ is E-identity, is essential.

Theorem 12. *Let S be a principally ordered regular semigroup for which the operation $x \to x^*$ is E-identity. The following statements are equivalent*
(1) *S has a smallest idempotent α.*
(2) *α is the identity element of S.*

Proof. (2) $\implies$ (1): Assuming that α is the identity element of S, we have for $e \in E(S)$, using Theorem 9, that

$$\alpha \cdot e = e = e \cdot \alpha \quad \Longrightarrow \quad e \leq_n \alpha \quad \Longrightarrow \quad \alpha \leq e$$

and α is the smallest idempotent of S.
(1) $\implies$ (2): Let us now suppose that α is the smallest idempotent of S. For any $e \in E(S)$

$$\alpha e = \alpha\alpha\alpha e \leq \alpha e \cdot \alpha e \leq \alpha eee = \alpha e \quad \Longrightarrow \quad \alpha e \in E(S)$$

and, similarly, $e\alpha, e\alpha e, \alpha e\alpha \in E(S)$
Since $e\alpha e \leq_n e$ we conclude, using Theorem 9, that

$$e \leq e\alpha e \leq eee = e \quad \Longrightarrow \quad e\alpha e = e.$$

Now, $e \in V(\alpha e\alpha)$ and therefore, using the hypothesis that the operation $x \to x^*$ is E-identity, we obtain

$$e \leq (\alpha e\alpha)^0 \leq (\alpha e\alpha)^* = \alpha e\alpha \leq \alpha ee = \alpha e \leq ee = e.$$

Thus, $\alpha e = e$ and, similarly, $e\alpha = e$.
Finally, for any $x \in S$,

$$\alpha x = \alpha(xx^*x) = (\alpha \cdot xx^*)x = xx^* \cdot x = x$$

and, similarly, $x\alpha = x$. Therefore, α is the identity element of S □

We can rephrase the previous Theorem saying that a principally ordered regular monoid such that $x \to x^*$ is E-identity, has a smallest idempotent.

Example 4, shows that the existence of an identity element α, in a principally ordered regular semigroup S, for which the operation $x \to x^*$ is E-identity, does not imply that α is the smallest element of S.

The following example illustrate that, in the previous Theorem, the hypothesis that $x \to x^*$ is E-identity is crucial.

Example 7. *Let $S = \{z, e, f\}$ be a band where z is a zero element, and $ef = e = fe$, with the following order relations $e < f < z$.*
Straightforward calculations give us that S is a principally ordered semigroup with $e^* = f$, $f^* = f$ and $z^* = z$. Also, $e^0 = e$, $f^0 = f$ and $z^0 = z$, which means, by Lemma 1, that S is an inverse semigroup.
Now, S has a smallest idempotent, e, which is not an identity element. This does not contradicts Theorem 12 since the operation $x \to x^*$ is not E-identity (in fact, $e \neq e^*$). Also, $e \leq_n f$ and $z \leq_n f$, but $e \leq f$ and $f \leq z$, from which we can conclude that S is neither naturally ordered nor dually naturally ordered.

We now explore what happens when the set of idempotents is finite.

Theorem 13. *Let S be a principally ordered regular semigroup, with a finite set of idempotents, for which the operation $x \to x^*$ is E-identity. Then S has a greatest idempotent.*

Proof. Let us assume that $E(S) = \{e_1, e_2, \cdots, e_n\}$ and consider the multiplication of its elements $z = e_1 e_2 \cdots e_n$. By Lemma 6, S is inverse and therefore the idempotents commute. Then,

$$z^2 = (e_1 e_2 \cdots e_n)(e_1 e_2 \cdots e_n) = {e_1}^2 {e_2}^2 \cdots {e_n}^2 = e_1 e_2 \cdots e_n = z$$

which means that z is an idempotent. Now, for every $i \in \{1, 2, \cdots, n\}$,

$$z \cdot e_i = e_1 e_2 \cdots e_n \cdot e_i = e_1 \cdots e_i e_i \cdots e_n = e_1 \cdots e_i \cdots e_n = z$$

and, similarly, $e_i \cdot z = z$. Then, $z \leq_n e_i$ which, by Theorem 9, tells us that $e_i \leq z$ and therefore z is the greatest idempotent of S. □

Corollary 3. *If S is a principally ordered regular semigroup, with a finite set of idempotents, for which the operation $x \to x^*$ is E-identity, then $E(S)$ is a band with a zero element.*

Proof. Let us assume that $E(S) = \{e_1, e_2, ..., e_n\}$. By Lemma 6, S is an inverse semigroup and therefore its idempotents commute, which implies that $E(S)$ is a subsemigroup of S and, in fact, a band.
By the proof of Theorem 13, $z = e_1 e_2 \cdots e_n$ is the greatest idempotent of S and, in particular, is the greatest idempotent of $E(S)$.
Finally, by Theorem 11, z is the zero element of S and since it belongs to $E(S)$, it has to be the zero element of $E(S)$. □

Note that in Example 4, $E(S) = \{I, E_{11}, E_{22}, O\}$ is a finite band with a zero element, O.

In the next Theorem we present a partial converse of the previous result.

Theorem 14. *Let S be a principally ordered regular semigroup, such that $E(S)$ is a band with a zero element, z, which is the greatest element of S.*
If S has no chain of idempotents with length bigger than 2, then the operation $x \to x^*$ is E-identity.

Proof. Since $E(S)$ is a band we can say that S is an inverse semigroup and therefore, by Lemma 6, $e = e^0 \leq e^* \leq z$. By hypothesis we must have that $e^* = z$ or $e^* = e$. If $e^* = z$ then, using the fact that z is the zero element of S,

$$e = e^0 = e^* e e^* = zez = z$$

and therefore $e = z = e^*$.

Thus, in either case, we may conclude that $e = e^*$ which means that $x \to x^*$ is E-identity. □

In [**7**, Theorem 2] Blyth and Pinto proved that if S is a principally ordered inverse semigroup such that $x \to x^*$ is *weakly isotone*, that is, for every $e, f \in E(S)$ such that $e \leq f$, we obtain $e^* \leq f^*$, then the $*$-subsemigroup generated by $\{e, f\}$ with $e < f$ and $e^* < f^*$ is a band with at most seven elements, in which every connection in the Hasse diagram also indicate the natural order or the dual natural order.

In this context of a principally ordered regular semigroup for which the operation $x \to x^*$ is E-identity, we can formulate the following result.

Theorem 15. *Let S be a principally ordered regular semigroup for which the operation $x \to x^*$ is E-identity. For $e, f \in E(S)$, we have that*

(1) *if $e < f$ then the $*$-subsemigroup generated by $\{e, f\}$ is a band with exactly two elements e, f, where $ef = f$ is the zero element.*

(2) *if e and f are incomparable, then the $*$-subsemigroup generated by $\{e, f\}$ is a band with exactly three elements e, f, ef, that verify $e < ef$ and $f < ef$, where ef is the zero element.*

Proof. Firstly, note that, by Lemma 6, S is an inverse semigroup and, for idempotents e, f, such that $e \leq f$, we have that $e^* = e \leq f = f^*$, which means that $x \to x^*$ is weakly isotone.

Secondly, by Theorem 1, we have that

$$ef \cdot e = ef = e \cdot ef \quad \Longrightarrow \quad ef \leq_n e \quad \Longrightarrow \quad e \leq ef$$

and, similarly, $f \leq ef$.

(1) Now, if $e < f$, then the hypothesis of [**7**, Theorem 2] hold and we can conclude that the $*$-subsemigroup, T, generated by $\{e, f\}$ is a band with the seven elements: $e, ef, e^*f, f, f^*, e^*, e^*f^*$. We have that

$$e = ee \leq ef \leq ff = f$$

and therefore, since $f \leq ef$, we have that $ef = f$, which implies that $T = \{e, f\}$ is a two-element chain, with $e < f$. Also,

$$e \cdot f = ef = f \qquad \text{and} \qquad f \cdot e = fe = ef = f$$

which means that $ef = f$ is the zero element of T.

(2) Let us now assume that $e \nleq f$ and $f \nleq e$. If $e = ef$, then $f \leq ef = e$, which is a contradiction. Similarly, we obtain a contradiction assuming that $f = ef$. Therefore, we have that $e < ef$ and $f < ef$.

Thus, we can immediately conclude that e, f, ef are all distinct and that they form a band with ef as its zero element. □

In Example 4, if we take idempotents I, E_{11}, then the $*$-subsemigroup T_1 generated by $\{I, E_{11}\}$ is equal to $T_1 = \{I, E_{11}\}$, where E_{11} is the zero element of T_1, since $I \cdot E_{11} = E_{11} = E_{11} \cdot I$.

Taking now, still in Example 4, the incomparable idempotents E_{11} and E_{22}, then the $*$-subsemigroup T_2 generated by $\{E_{11}, E_{22}\}$ is equal to $T_2 = \{E_{11}, E_{22}, O\}$, where $E_{11} < O$, $E_{22} < O$ and O is the zero element of T_2.

Example 8. *Consider the set $\mathbb{Z}$ of integer numbers as a join semilattice under the definition*

$$m \vee n = max\{m, n\}$$

It is easy to verify that we obtain a principally ordered inverse semigroup, with $m^* = m$, and therefore $x \to x^*$ is E-identity.

Let us now take S_6 as in Example 4 and consider the cartesian ordered set $S_6 \times \mathbb{Z}$, with the multiplication

$$(A,m)(B,n) = (AB, m \vee n).$$

Then, we can see that we obtain a principally ordered regular (in fact, inverse) semigroup, where the operation $x \to x^*$ is E-identity, with an infinite set of idempotents.

We have verified in Theorem 9 that a principally ordered regular semigroup S, for which the operation $x \to x^*$ is E-identity, is dually naturally ordered. In the next result we answer the question: what happens if in such semigroup the natural order reduces to equality, that is, for any $e, f \in E(S)$,

$$e \leq_n f \quad \Longrightarrow \quad e = f.$$

Theorem 16. *Let S be a principally ordered regular semigroup for which the operation $x \to x^*$ is E-identity. The following statements are equivalent*

(1) *S is completely simple.*

(2) *S is a group.*

Proof. Note that, by Lemma 6, S is an inverse semigroup, and therefore its idempotents commute.

(1) $\Longrightarrow$ (2): Assume that S is a completely simple semigroup. For any $e, f \in E(S)$, we have that

$$ef \cdot e = ef = e \cdot ef \quad \text{and} \quad ef \cdot f = ef = f \cdot ef$$

which means that $ef \leq_n e$ and $ef \leq_n f$. Since, S is completely simple, the natural order reduces to equality (see for example [**11**, Theorem 3.3.3]), which means that $ef = e$ and $ef = f$.

Then, S has a unique idempotent, u which is at the same time, the greatest and the smallest idempotent. By Theorem 12, u is the identity element of S.

Since S is principally ordered, for any $x \in S$, there exist x^* such that the elements xx^*, x^*x are idempotents of S. Then, $xx^* = u = x^*x$, which means that x^* is the inverse of x, and S is a group.

(2) $\Longrightarrow$ (1): This follows immediately since a group has a unique idempotent and, therefore the natural order on the idempotents reduces to equality. Thus, S is completely simple. □

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