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GALERKIN APPROXIMATIONS IN THE PROBLEM OF ONE-DIMENSIONAL UNSTEADY MOTION OF A VISCOUS COMPRESSIBLE TWO-COMPONENT FLUID

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Abstract. We study an initial-boundary value problem which describes unsteady motions of a viscous compressible two-component fluid. We formulate an approximate problem via the Galerkin method, and prove its solvability.

Keywords: Galerkin approximations, non-stationary boundary value problem, one-dimensional flow, viscous compressible fluid, homogeneous multi-velocity multifluid.

1. Introduction

The solvability theory for the equations of motions of viscous compressible fluids, mainly developed at the end of the last century, cannot be considered complete. One of the reasons for this is that the methods and results developed and obtained for the case of single-component fluids are not automatically applicable to the case of the movement of mixtures (multi-component fluids). During the heyday of the one-dimensional one-component theory, certain attention was also paid to the case of mixtures; see, for example [1], [2]. However, in these articles only the case of diagonal viscosity matrices is considered. From the mathematical point of view, this means that the equations for different components are interconnected only through the lower order terms, and from the physical point of view, that viscous friction between the components is not considered, but only the momentum exchange.

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In connection with the foregoing arguments, it is relevant to prove solvability of initial-boundary value problems for the equations of motion of mixtures of viscous compressible fluids in the case of non-diagonal viscosity matrices, which means physically taking into account the inter-component viscous friction, and from the point of view of mathematics, the presence of second derivatives of velocities of different components in the viscous terms. In the case of one-dimensional movements, for each version of the problem statement, this work involves two stages. The first stage is to prove the existence of approximate solutions, and the second is to continue them over a large time interval by proving global a priori estimates. The first stage is not trivial because the well-known works for one-component one-dimensional viscous gas flows dealt with the Lagrangian description, while for mixtures, as a rule, it is more convenient to use the Eulerian coordinates.

In the paper, we fulfill the first mentioned stage for one of the versions of stating the problem. Namely, the local solvability of the problem for Galerkin approximations is shown in the case of one-dimensional unsteady barotropic motion of a two-component mixture of viscous compressible fluids.

2. Statement of the problem

Let $Q_T = (0,1) \times (0,T)$, where $T > 0$ is an arbitrary positive number. Consider the following system of equations of dynamics of a viscous compressible two-component fluid, in which it is needed to determine the density $\rho_i > 0$ and the velocity u_i of each component [3], [4], [5]:

$$\partial_t \rho_i + \partial_x(\rho_i u_i) = 0, \quad i = 1, 2, \tag{2.1}$$

$$\rho_i \left(\partial_t u_i + u_i \partial_x u_i\right) + \partial_x p_i(\rho_i) = \sum_{j=1}^{2} \nu_{ij} \partial_{xx}^2 u_j + J_i(u_1, u_2), \quad i = 1, 2. \tag{2.2}$$

Here p_i is the pressure in the i-th component which is a known function of the density of the i-th component, more precisely, the following polytropic constitutive equations are supposed to be fulfilled:

$$p_i(\rho_i) = K_i \rho_i^{\gamma_i}, \quad i = 1, 2, \tag{2.3}$$

where $K_i > 0$ and polytropic exponents $\gamma_i > 1$ are known constants. Next, given constant viscosity coefficients $\{\nu_{ij}\}_{i,j=1}^2$ form a (asymmetric) matrix $\mathbf{N} > 0$ (the last condition is equivalent to the following requirements: $\nu_{11} > 0$, $4\nu_{11}\nu_{22} - (\nu_{12}+\nu_{21})^2 > 0$), moreover $\nu_{12} = 0$. Finally, the intensity of the momentum exchange J_i between the components of the mixture is expressed as follows:

$$J_i(u_1, u_2) = (-1)^i a(u_1 - u_2), \quad i = 1, 2, \quad a = \text{const} \geqslant 0. \tag{2.4}$$

At the initial moment of time, let the distribution for densities and velocities be given

$$\rho_i|_{t=0} = \rho_{0i}, \quad u_i|_{t=0} = u_{0i}, \quad i = 1, 2. \tag{2.5}$$

At the boundary of the domain $(0, 1)$, the conditions are supposed

$$u_i|_{x=0} = u_i|_{x=1} = 0, \quad i = 1, 2. \tag{2.6}$$

The initial data satisfy the conditions

$$
\begin{gathered}
(\rho_{01}, \rho_{02}, u_{01}, u_{02}) \in W_2^1(0,1), \quad \rho_{0i} > 0, \quad i = 1,2, \\
u_{0i}(0) = u_{0i}(1) = 0, \quad i = 1,2.
\end{gathered} \tag{2.7}
$$

A solution to the problem (2.1)–(2.6) is searched in the following class of functions ($i = 1,2$):

$$
\begin{gathered}
\rho_i \in L_\infty(0,T; W_2^1(0,1)), \quad \frac{\partial \rho_i}{\partial t} \in L_\infty(0,T; L_2(0,1)), \\
u_i \in L_\infty(0,T; W_2^1(0,1)) \bigcap L_2(0,T; W_2^2(0,1)), \\
\frac{\partial u_i}{\partial t} \in L_2(Q_T),
\end{gathered} \tag{2.8}
$$

moreover ρ_i are strictly positive and bounded.

3. Statement of the approximate problem

The main object of study in the paper is the initial-boundary value problem obtained from the problem (2.1)–(2.6) via the Galerkin method with respect to the spatial variable x in the momentum equation (2.2) ($i = 1,2$, $k = 1,\ldots,m$, $m \in \mathbb{N}$):

$$
\partial_t \rho_i + \partial_x(\rho_i u_i) = 0, \tag{3.1}
$$

$$
\int_0^1 \Big(\rho_i \left(\partial_t u_i + u_i \partial_x u_i\right) + \partial_x p_i(\rho_i) - \sum_{j=1}^{2} \nu_{ij} \partial_{xx}^2 u_j - J_i(u_1,u_2) \Big) \omega_k(x)\, dx = 0, \tag{3.2}
$$

$$
\rho_i|_{t=0} = \rho_{0i}(x), \tag{3.3}
$$

$$
u_i = \sum_{s=1}^{m} \psi_{is}(t)\omega_s(x), \quad u_i|_{t=0} = \sum_{s=1}^{m} \psi_{0is}\omega_s(x), \tag{3.4}
$$

where $\omega_k(x) = \sin(\pi k x)$, $\psi_{ik}(0) = \psi_{0ik} = 2 \int_0^1 u_{0i}(x)\omega_k(x)\, dx$.

4. Construction of a solving operator for the approximate problem

Let us prove the solvability of the problem (3.1)–(3.4). To do this, we choose $t_m \in (0,T]$ arbitrarily, consider the set

$$
B = \left\{ \boldsymbol{\psi} \in (C[0,t_m])^{2m} \middle|\ \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0, \ \ \|\boldsymbol{\psi}\|_{(C[0,t_m])^{2m}} \leqslant b \right\}
$$

in the space $(C[0,t_m])^{2m}$, where

$$
\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2), \quad \boldsymbol{\psi}_0 = (\boldsymbol{\psi}_{01}, \boldsymbol{\psi}_{02}), \quad \boldsymbol{\psi}_i = (\psi_{i1}, \ldots, \psi_{im}), \quad i = 1,2,
$$

$$
\boldsymbol{\psi}_{0i} = (\psi_{0i1}, \ldots, \psi_{0im}), \quad i = 1,2, \quad b^2 = e \frac{\max\limits_{1 \leqslant i \leqslant 2} \left\{ \sup\limits_{(0,1)} \rho_{0i} \right\}}{\min\limits_{1 \leqslant i \leqslant 2} \left\{ \inf\limits_{(0,1)} \rho_{0i} \right\}} \|\boldsymbol{\psi}_0\|_{\mathbb{R}^{2m}}^2 + 1,
$$

and construct the operator $\Lambda : B \to (C[0,t_m])^{2m}$, $\operatorname{Im}\Lambda \subset (C^1[0,t_m])^{2m}$, $\Lambda(\boldsymbol{\psi}) = \boldsymbol{\Psi}$, where $\boldsymbol{\Psi} = (\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2)$, $\boldsymbol{\Psi}_i = (\Psi_{i1}, \dots, \Psi_{im})$, $i = 1, 2$, via the following algorithm. First, we find the functions $\rho_i \in L_\infty\big(0, t_m; W_2^1(0,1)\big) \bigcap W_\infty^1\big(0, t_m; L_2(0,1)\big)$, $\rho_i > 0$, $i = 1, 2$, as the solutions to the Cauchy problems (3.1), (3.3), where u_i, $i = 1, 2$, are defined via the formulae (3.4), and $\boldsymbol{\psi} \in B$, which is possible since $\|\boldsymbol{\psi}_0\|_{\mathbb{R}^{2m}} \leqslant b$. Moreover, the following equalities hold

$$
\begin{aligned}
&\Big(\inf_{(0,1)} \rho_{0i}\Big) \exp\Big\{-\int_0^t \sup_{(0,1)} |\partial_x u_i| \, d\tau\Big\} \leqslant \rho_i(x,t) \\
&\leqslant \Big(\sup_{(0,1)} \rho_{0i}\Big) \exp\Big\{\int_0^t \sup_{(0,1)} |\partial_x u_i| \, d\tau\Big\}, \quad i = 1, 2,
\end{aligned}
\tag{4.1}
$$

which, due to $\boldsymbol{\psi} \in B$, give the estimates

$$
\Big(\inf_{(0,1)} \rho_{0i}\Big) e^{-\pi m^2 bt} \leqslant \rho_i(x,t) \leqslant \Big(\sup_{(0,1)} \rho_{0i}\Big) e^{\pi m^2 bt}, \quad i = 1, 2.
\tag{4.2}
$$

Secondly, we define the function $\boldsymbol{\Psi}$ from the following Cauchy problem for the system of $2m$ ODEs of the first order:

$$
\begin{aligned}
&\int_0^1 \Big(\rho_i \partial_t U_i + \rho_i u_i \partial_x U_i + \partial_x p_i(\rho_i) - \sum_{j=1}^2 \nu_{ij} \partial_{xx}^2 U_j \\
&-J_i(U_1, U_2)\Big) \omega_k \, dx = 0, \quad i = 1, 2, \quad k = 1, \dots, m,
\end{aligned}
\tag{4.3}
$$

$$
\boldsymbol{\Psi}(0) = \boldsymbol{\psi}_0,
\tag{4.4}
$$

where $U_i = \sum\limits_{s=1}^m \Psi_{is}(t) \omega_s(x)$, $i = 1, 2$. The relations

$$
A > 0 \quad \Longrightarrow \quad \det A \neq 0,
$$

where

$$
A(t) = \begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix}, \quad A_i(t) = \Big\{\int_0^1 \rho_i \omega_k \omega_s \, dx\Big\}_{k,s=1}^m, \quad i = 1, 2,
$$

which are valid due to the positiveness of ρ_i, $i = 1, 2$, allow to resolve the system (4.3) with respect to the derivatives, which justifies the existence of a function $\boldsymbol{\Psi} \in (C^1[0,t_m])^{2m}$. Thus, for arbitrary $t_m \in (0, T]$, the operator $\Lambda : B \to (C^1[0,t_m])^{2m} \subset (C[0,t_m])^{2m}$, $\Lambda(\boldsymbol{\psi}) = \boldsymbol{\Psi}$, is defined, and its fixed point, together with the corresponding functions ρ_i, $i = 1, 2$, gives a solution to the problem (3.1)–(3.4).

Let us show that, for sufficiently small t_m, the operator Λ satisfies the conditions of the Schauder theorem (see [6], P. 31).

5. Mapping the ball into itself

We demonstrate first that $\Lambda(B) \subset B$. Let us multiply the equations (4.3) by $\Psi_{ik}(t)$, $i = 1, 2$, $k = 1, \ldots, m$, sum in $i = 1, 2$ and $k = 1, \ldots, m$, so we obtain via (3.1) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{2}\int_0^1 \rho_i U_i^2\,dx + \sum_{i,j=1}^{2}\nu_{ij}\int_0^1 (\partial_x U_i)(\partial_x U_j)\,dx \\ +a\int_0^1 (U_1-U_2)^2\,dx = \sum_{i=1}^{2}\int_0^1 p_i(\rho_i)\partial_x U_i\,dx.\end{aligned} \tag{5.1}$$

Next, since $\mathbf{N} > 0$, the following pointwise inequality holds

$$\sum_{i,j=1}^{2}\nu_{ij}(\partial_x U_i)(\partial_x U_j) \geqslant C_0\sum_{i=1}^{2}|\partial_x U_i|^2 \tag{5.2}$$

with a certain positive constant $C_0 = C_0(\mathbf{N})$ (here and below, let the letter C with indices denote positive quantities depending on the values indicated in brackets or sometimes indicated otherwise). The right hand side of (5.1), due to the Cauchy inequality and (4.2), is estimated as follows:

$$\sum_{i=1}^{2}\int_0^1 p_i(\rho_i)(\partial_x U_i)\,dx \leqslant \frac{C_0}{2}\sum_{i=1}^{2}\int_0^1 |\partial_x U_i|^2\,dx + C_1,$$

where $C_1 = \dfrac{1}{2C_0}\displaystyle\sum_{i=1}^{2}K_i^2\Big(\sup_{(0,1)}\rho_{0i}\Big)^{2\gamma_i}e^{2\pi\gamma_i m^2 b t_m}$. Thus, we have the estimate

$$\frac{d}{dt}\sum_{i=1}^{2}\int_0^1 \rho_i U_i^2\,dx + C_0\sum_{i=1}^{2}\int_0^1 |\partial_x U_i|^2\,dx \leqslant 2C_1, \tag{5.3}$$

from which it follows that

$$\sum_{i=1}^{2}\int_0^1 \rho_i U_i^2\,dx \leqslant \sum_{i=1}^{2}\int_0^1 \rho_{0i}U_{0i}^2\,dx + 2C_1 t_m, \tag{5.4}$$

where $U_{0i} = \displaystyle\sum_{s=1}^{m}\Psi_{is}(0)\omega_s(x) = \sum_{s=1}^{m}\psi_{0is}\omega_s(x)$, $i = 1, 2$. Using (4.2) once more, we obtain from (5.4) the inequality

$$\|\mathbf{\Psi}\|^2_{(C[0,t_m])^{2m}} \leqslant e^{\pi m^2 b t_m}\frac{\max\limits_{1\leqslant i\leqslant 2}\Big\{\sup\limits_{(0,1)}\rho_{0i}\Big\}}{\min\limits_{1\leqslant i\leqslant 2}\Big\{\inf\limits_{(0,1)}\rho_{0i}\Big\}}\|\boldsymbol{\psi}_0\|^2_{\mathbb{R}^{2m}} + \frac{4C_1 e^{\pi m^2 b t_m}}{\min\limits_{1\leqslant i\leqslant 2}\Big\{\inf\limits_{(0,1)}\rho_{0i}\Big\}}t_m. \tag{5.5}$$

Choosing

$$t_m < \min\left\{\frac{1}{\pi m^2 b}, \frac{\min\limits_{1\leqslant i\leqslant 2}\Big\{\inf\limits_{(0,1)}\rho_{0i}\Big\}}{2eC_2}, T\right\}, \tag{5.6}$$

where $C_2 = \dfrac{1}{C_0}\sum_{i=1}^{2} K_i^2\Big(\sup_{(0,1)} \rho_{0i}\Big)^{2\gamma_i} e^{2\gamma_i}$, we obtain that $C_1 \leqslant \dfrac{C_2}{2}$, and arrive at the needed estimate

$$\|\mathbf{\Psi}\|_{(C[0,t_m])^{2m}} \leqslant b. \tag{5.7}$$

Thus, provided (5.6), the operator Λ maps the set B into itself.

6. Compactness of the operator

Let us show the compactness of the operator Λ. Multiplying (4.3) by $\dfrac{d\Psi_{ik}(t)}{dt}$, $i = 1, 2$, $k = 1, \dots, m$, summing in $i = 1, 2$ and $k = 1, \dots, m$, we deduce the relation

$$\begin{gathered}\sum_{i=1}^{2}\int_0^1 \rho_i |\partial_t U_i|^2\,dx = \sum_{i=1}^{2}\int_0^1 \Big(-\sum_{j=1}^{2} \nu_{ij} (\partial^2_{xt} U_i)(\partial_x U_j) \\ -\rho_i (\partial_x U_i)(\partial_t U_i) u_i + p_i(\rho_i)\partial^2_{xt} U_i + J_i(U_1, U_2)(\partial_t U_i)\Big)dx.\end{gathered} \tag{6.1}$$

We estimate the terms in the right hand side of (6.1) using (4.2), the Cauchy inequality, and the inequalities $\|\boldsymbol{\psi}\|_{(C[0,t_m])^{2m}} \leqslant b$, $\|\mathbf{\Psi}\|_{(C[0,t_m])^{2m}} \leqslant b$, $\|\partial_x U_i\|_{L_2(0,1)} \leqslant C_3(m)\|U_i\|_{L_2(0,1)}$, $\|\partial^2_{xt} U_i\|_{L_2(0,1)} \leqslant C_3\|\partial_t U_i\|_{L_2(0,1)}$, $i = 1, 2$:

$$\left|\sum_{i,j=1}^{2} \nu_{ij} \int_0^1 (\partial^2_{xt} U_i)(\partial_x U_j)\,dx\right| \leqslant \frac{1}{8}\sum_{i=1}^{2}\int_0^1 \rho_i |\partial_t U_i|^2\,dx + C_4,$$

$$\left|\sum_{i=1}^{2}\int_0^1 \rho_i (\partial_x U_i)(\partial_t U_i) u_i\,dx\right| \leqslant \frac{1}{8}\sum_{i=1}^{2}\int_0^1 \rho_i |\partial_t U_i|^2\,dx + C_5,$$

$$\left|\sum_{i=1}^{2}\int_0^1 p_i(\rho_i)\partial^2_{xt} U_i\,dx\right| \leqslant \frac{1}{8}\sum_{i=1}^{2}\int_0^1 \rho_i |\partial_t U_i|^2\,dx + C_6,$$

$$\left|\sum_{i=1}^{2}\int_0^1 J_i(U_1, U_2)(\partial_t U_i)\,dx\right| \leqslant \frac{1}{8}\sum_{i=1}^{2}\int_0^1 \rho_i |\partial_t U_i|^2\,dx + C_7,$$

where $C_4 = C_4\left(C_3, \left\{\inf_{(0,1)} \rho_{0i}\right\}, \mathbf{N}, b, m, t_m\right)$, $C_5 = C_5\left(C_3, \left\{\sup_{(0,1)} \rho_{0i}\right\}, b, m, t_m\right)$, $C_6 = C_6\left(C_3, \left\{\inf_{(0,1)} \rho_{0i}\right\}, \left\{\sup_{(0,1)} \rho_{0i}\right\}, \{K_i\}, \{\gamma_i\}, b, m, t_m\right)$, $C_7 = C_7\left(\left\{\inf_{(0,1)} \rho_{0i}\right\}, a, b, m, t_m\right)$. Thus, from (6.1) we obtain the inequality

$$\frac{1}{2}\sum_{i=1}^{2}\int_0^1 \rho_i |\partial_t U_i|^2\,dx \leqslant C_4 + C_5 + C_6 + C_7, \tag{6.2}$$

which, after integration in time and application of (4.2), provides the estimate

$$\sum_{i=1}^{2} \|\partial_t U_i\|_{L_2(Q_{t_m})} \leqslant C_8\left(C_4, \dots, C_7, \left\{\inf_{(0,1)} \rho_{0i}\right\}, b, m, t_m\right), \tag{6.3}$$

where $Q_{t_m} = (0,1)\times(0,t_m)$. Thus, the estimate $\mathbf{\Psi}$ in $(W_2^1(0,t_m))^{2m}$ is obtained. Therefore, Λ is a compact operator.

7. Continuity of the operator

Let us show the continuity of the operator Λ from B to $(C[0,t_m])^{2m}$. Let $\boldsymbol{\psi}^{(1,2)} \in B$, $\mathbf{\Psi}^{(1,2)} = \Lambda(\boldsymbol{\psi}^{(1,2)})$, $u_i^{(1,2)} = \sum\limits_{s=1}^{m} \psi_{is}^{(1,2)}\omega_s$, $U_i^{(1,2)} = \sum\limits_{s=1}^{m} \Psi_{is}^{(1,2)}\omega_s$, $i=1,2$. Next, let $\rho_i^{(1,2)}$ be the solutions to the Cauchy problems (3.1), (3.3), where $u_i^{(1,2)}$ is substituted instead of u_i. Denote $\rho_i = \rho_i^{(1)} - \rho_i^{(2)}$, $u_i = u_i^{(1)} - u_i^{(2)}$, $U_i = U_i^{(1)} - U_i^{(2)}$, $i=1,2$, $\rho = \rho^{(1)} - \rho^{(2)}$, where $\rho^{(1,2)} = \rho_1^{(1,2)} + \rho_2^{(1,2)}$. When we differentiate the equations (3.1) for $\rho_i^{(2)}$, $i=1,2$ $\Big($i. e. the equations $\partial_t\rho_i^{(2)} + \partial_x\left(\rho_i^{(2)}u_i^{(2)}\right) = 0$, $i=1,2\Big)$ via the variable x, using the initial conditions $\rho_i^{(2)}|_{t=0} = \rho_{0i}$, $i=1,2$, and Gronwall's inequality, we obtain (for all $i=1,2$) the estimates

$$\left\|\partial_x\rho_i^{(2)}\right\|_{L_2(0,1)} \leqslant C_9\left(\left\{\|\rho_{0i}\|_{W_2^1(0,1)}\right\}, b, m, t_m\right). \tag{7.1}$$

Let us observe that (3.1), (3.3) provide the following equalities

$$\partial_t\rho_i + \partial_x\left(\rho_i u_i^{(1)}\right) + \partial_x\left(\rho_i^{(2)}u_i\right) = 0, \quad \rho_i|_{t=0} = 0, \quad i=1,2. \tag{7.2}$$

Multiplying (7.2) by ρ_i, $i=1,2$, and integrating in $x\in(0,1)$, we obtain for all $i=1,2$ that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_i^2\,dx = -\int_0^1\left(\frac{1}{2}\rho_i^2\left(\partial_x u_i^{(1)}\right) + \rho_i^{(2)}\rho_i(\partial_x u_i) + \left(\partial_x\rho_i^{(2)}\right)\rho_i u_i\right)dx$$

$$\leqslant \frac{1}{2}\left(\sup_{(0,1)}\left|\partial_x u_i^{(1)}\right|\int_0^1\rho_i^2\,dx + \sup_{(0,1)}\rho_i^{(2)}\int_0^1\left(\rho_i^2 + (\partial_x u_i)^2\right)dx\right. \tag{7.3}$$

$$\left. + \sup_{(0,1)} u_i^2\int_0^1\left(\partial_x\rho_i^{(2)}\right)^2 dx + \int_0^1\rho_i^2\,dx\right) \leqslant C_{10}\left(\int_0^1\rho_i^2\,dx + \int_0^1 u_i^2\,dx\right),$$

where $C_{10} = C_{10}\left(C_9, \left\{\sup\limits_{(0,1)}\rho_{0i}\right\}, b, m, t_m\right)$. Hence, applying Gronwall's inequality and taking into account the initial conditions in (7.2), we derive the inequalities for all $t\in(0,t_m]$

$$\int_0^1\rho_i^2\,dx \leqslant C_{11}(C_{10},t_m)\int_0^t\int_0^1 u_i^2\,dxd\tau, \quad i=1,2. \tag{7.4}$$

Next, the equations for $U_i^{(1,2)}$, $i=1,2$ (see (4.3)), provide the following relation for all $t\in(0,t_m]$

$$\frac{1}{2}\sum_{i=1}^{2}\int_0^1\rho_i^{(1)}U_i^2\,dx + \sum_{i,j=1}^{2}\nu_{ij}\int_0^t\int_0^1(\partial_x U_i)(\partial_x U_j)\,dxd\tau \tag{7.5}$$

$$+a\int_0^t\int_0^1 |U_1-U_2|^2\,dxd\tau = \sum_{i=1}^2\int_0^t\int_0^1 \left(p_i\left(\rho_i^{(1)}\right)-p_i\left(\rho_i^{(2)}\right)\right)(\partial_x U_i)\,dxd\tau$$

$$-\sum_{i=1}^2\int_0^t\int_0^1 \rho_i U_i\left(\partial_\tau U_i^{(2)}\right)dxd\tau-\sum_{i=1}^2\int_0^t\int_0^1 \rho_i^{(1)}u_iU_i\left(\partial_x U_i^{(2)}\right)dxd\tau$$

$$-\sum_{i=1}^2\int_0^t\int_0^1 \rho_i u_i^{(2)}U_i\left(\partial_x U_i^{(2)}\right)dxd\tau.$$

The first term in the left hand side of (7.5), in view of (4.2), admits the estimate

$$\frac{1}{2}\sum_{i=1}^2\int_0^1 \rho_i^{(1)}U_i^2\,dx \geqslant \frac{1}{2}\min_{1\leqslant i\leqslant 2}\left\{\inf_{(0,1)}\rho_{0i}\right\}e^{-\pi m^2 bt_m}\sum_{i=1}^2\int_0^1 U_i^2\,dx; \tag{7.6}$$

for the second term we have the inequality (see (5.2))

$$\sum_{i,j=1}^2\int_0^t\int_0^1 \nu_{ij}(\partial_x U_i)(\partial_x U_j)\,dxd\tau \geqslant C_0\sum_{i=1}^2\int_0^t\int_0^1 |\partial_x U_i|^2\,dxd\tau; \tag{7.7}$$

using the Cauchy inequality, for the first term in the right hand side of (7.5) we get the relation

$$\sum_{i=1}^2\int_0^t\int_0^1 \left(p_i\left(\rho_i^{(1)}\right)-p_i\left(\rho_i^{(2)}\right)\right)(\partial_x U_i)\,dxd\tau \leqslant \frac{C_0}{2}\sum_{i=1}^2\int_0^t\int_0^1 |\partial_x U_i|^2\,dxd\tau \tag{7.8}$$

$$+\frac{1}{2C_0}\sum_{i=1}^2\int_0^t\int_0^1 \left(p_i\left(\rho_i^{(1)}\right)-p_i\left(\rho_i^{(2)}\right)\right)^2 dxd\tau,$$

from which, due to (see (4.2))

$$p_i(s_1)-p_i(s_2)=p_i'(s)(s_1-s_2),\quad s=\lambda s_1+(1-\lambda)s_2,\quad \lambda\in[0,1],\quad s_{1,2}>0,$$

$$\left(\inf_{(0,1)}\rho_{0i}\right)e^{-\pi m^2 bt_m}\leqslant \lambda\rho_i^{(1)}+(1-\lambda)\rho_i^{(2)}\leqslant\left(\sup_{(0,1)}\rho_{0i}\right)e^{\pi m^2 bt_m},\ i=1,2,$$

and (7.4), we find that

$$\begin{gathered}\sum_{i=1}^2\int_0^t\int_0^1 \left(p_i\left(\rho_i^{(1)}\right)-p_i\left(\rho_i^{(2)}\right)\right)(\partial_x U_i)\,dxd\tau\\ \leqslant \frac{C_0}{2}\sum_{i=1}^2\int_0^t\int_0^1 |\partial_x U_i|^2\,dxd\tau + C_{12}\sum_{i=1}^2\int_0^t\int_0^1 u_i^2\,dxd\tau,\end{gathered} \tag{7.9}$$

where $C_{12}=C_{12}\left(C_0, C_{11}, \left\{\sup_{(0,1)}\rho_{0i}\right\}, \{K_i\}, \{\gamma_i\}, b, m, t_m\right)$; for the second term in the right hand side of (7.5), using (6.3), (7.4) and the Cauchy inequality with a small factor $\varepsilon > 0$, we obtain the inequality

$$
\begin{aligned}
&-\sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 \rho_i U_i \left(\partial_\tau U_i^{(2)}\right) dxd\tau \leqslant \frac{C_{11}t}{2\varepsilon^2}\sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 u_i^2\,dxd\tau \\
&+\frac{\varepsilon^2}{2}\max_{1\leqslant i\leqslant 2}\left\{\sup_{(0,1)\times(0,t)} U_i^2\right\}\sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 \left|\partial_\tau U_i^{(2)}\right|^2 dxd\tau \qquad (7.10)\\
&\leqslant \frac{C_{11}t}{2\varepsilon^2}\sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 u_i^2\,dxd\tau + 2m\varepsilon^2 C_8^2\sum_{i=1}^{2}\sup_{(0,t)}\int\limits_0^1 U_i^2\,dx \\
&= \left[2m\varepsilon^2 C_8^2 = \frac{1}{8}\min_{1\leqslant i\leqslant 2}\left\{\inf_{(0,1)}\rho_{0i}\right\}e^{-\pi m^2 b t_m}\right] \\
&= C_{13}\sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 u_i^2\,dxd\tau + \frac{1}{8}\min_{1\leqslant i\leqslant 2}\left\{\inf_{(0,1)}\rho_{0i}\right\}e^{-\pi m^2 b t_m}\sum_{i=1}^{2}\sup_{(0,t)}\int\limits_0^1 U_i^2\,dx,
\end{aligned}
$$

where $C_{13} = C_{13}\left(C_8, C_{11}, \left\{\inf\limits_{(0,1)}\rho_{0i}\right\}, b, m, t_m\right)$; the third term in the right hand side of (7.5) is estimated using (4.2), the Cauchy inequality and the inequality $\|\mathbf{\Psi}^{(2)}\|_{(C[0,t_m])^{2m}} \leqslant b$:

$$
\begin{aligned}
&-\sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 \rho_i^{(1)} u_i U_i \left(\partial_x U_i^{(2)}\right) dxd\tau \\
&\leqslant C_{14}\left(\sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 u_i^2\,dxd\tau + \sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 U_i^2\,dxd\tau\right),
\end{aligned}
\qquad (7.11)
$$

where $C_{14} = C_{14}\left(\left\{\sup\limits_{(0,1)}\rho_{0i}\right\}, b, m, t_m\right)$; finally, due to (7.4) and the estimates $\|\boldsymbol{\psi}^{(2)}\|_{(C[0,t_m])^{2m}} \leqslant b$, $\|\mathbf{\Psi}^{(2)}\|_{(C[0,t_m])^{2m}} \leqslant b$, for the last term in the right hand side of (7.5) we get the relation

$$
\begin{aligned}
&-\sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 \rho_i u_i^{(2)} U_i \left(\partial_x U_i^{(2)}\right) dxd\tau \\
&\leqslant C_{15}\left(\sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 u_i^2\,dxd\tau + \sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 U_i^2\,dxd\tau\right),
\end{aligned}
\qquad (7.12)
$$

where $C_{15} = C_{15}\left(C_{11}, b, m, t_m\right)$. Thus, from (7.5), using (7.6)—(7.12), we obtain the inequality

$$
\sum_{i=1}^{2}\int\limits_0^1 U_i^2\,dx \leqslant C_{16}\left(\sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 u_i^2\,dxd\tau + \sum_{i=1}^{2}\int\limits_0^t\int\limits_0^1 U_i^2\,dxd\tau\right), \qquad (7.13)
$$

where $C_{16} = C_{16}\left(C_{12}, \dots, C_{15}, \left\{\inf_{(0,1)} \rho_{0i}\right\}, b, m, t_m\right)$, from which, using Gronwall's inequality, we obtain the following estimate for all $t \in (0, t_m]$

$$\sum_{i=1}^{2} \int_0^1 U_i^2\,dx \leqslant C_{17}(C_{16}, t_m) \sum_{i=1}^{2} \int_0^t \int_0^1 u_i^2\,dx d\tau, \tag{7.14}$$

and finally the inequality

$$\|\mathbf{\Psi}\|_{(C[0,t_m])^{2m}} \leqslant C_{18}(C_{17}, t_m) \|\boldsymbol{\psi}\|_{(C[0,t_m])^{2m}}, \tag{7.15}$$

justifying the continuity of the operator Λ in B.

8. Conclusions and further construction

Since the operator Λ satisfies the conditions of the Schauder theorem, there exists a fixed point $\boldsymbol{\psi}$ of this operator in B, which defines (together with the corresponding functions ρ_i, $i = 1, 2$) a solution to the problem (3.1)—(3.4) on the time interval $(0, t_m)$.

Next, we need to pass to the limit as $m \to +\infty$, simultaneously providing an estimate of the form $t_m \geqslant t_0 = \text{const} > 0$, thereby proving the local solvability of the problem (2.1)—(2.6) on the interval $(0, t_0)$. This is done in a standard way, similar to the one-component case.

Finally, to prove the global solvability of the problem (2.1)—(2.6), it is necessary to obtain the corresponding global a priori estimates of its solutions. This is to be done in the next work.

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