

Cohomology and deformations of compatible Leibniz algebras

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January 22, 2024

Abstract

In this paper we study a cohomology theory of compatible Leibniz algebra. We construct a bi-differential graded Lie algebra whose Maurer-Cartan elements characterize the compatible Leibniz algebra structures. Using this, we study cohomology, infinitesimal deformations, Nijenhuis operator and their relation for compatible Leibniz algebras. Finally using cohomology of compatible Leibniz algebra with coefficients in an arbitrary representation we study the abelian extensions of compatible Leibniz algebra.

Keywords: Compatible Leibniz algebra, Maurer-Cartan element, cohomology, deformation, abelian extension

AMS: 17B56, 13D10, 17A30.

1 Introduction

Leibniz algebra is a non-anti symmetric generalisation of Lie algebra. It was introduced and called D-algebra in papers by A. M. Bloch published in the 1960s to signify its relation with derivations. Later in 1993 J. L. Loday [5] introduced the same structure and called it Leibniz algebra. Cohomology theory of Leibniz algebra with coefficients in a bimodule has been studied in [4].

Algebraic deformation theory was introduced by Gerstenhaber for rings and algebra in a series of papers [7]-[11]. Subsequently algebraic deformation theory has been studied for different kind of algebras. To study deformation theory of any algebra, one needs a suitable cohomology, known as the deformation cohomology, which controls the deformation. In [6], D. Balavoine studies the formal deformation of algebras using the theory of Maurer-Cartan elements in a

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graded Lie algebra. In particular this approach is used to study the deformation of Leibniz algebra.

Here, we have defined a compatible Leibniz algebra to be a pair of Leibniz algebras such that the linear combination of their algebraic structure is also a Leibniz algebra. Recently, cohomology and infinitesimal deformations of compatible Lie algebra and compatible associative algebra has been studied in [1] and [2] respectively. Motivated by these works, in this paper we study the cohomology theory of compatible Leibniz algebra. Using the Balavoine bracket we define a graded Lie algebra whose Maurer-Cartan elements characterize the structure of compatible Leibniz algebras. We then study the cohomology of a compatible Leibniz algebra with coefficients in itself. This is then used to study infinitesimal deformation of compatible Leibniz algebra. We also establish the relation between Nijenhuis operator and the trivial infinitesimal deformation. Further we introduce the cohomology of compatible Leibniz algebra with coefficients in an arbitrary representation. Using this we study the abelian extensions of compatible Leibniz algebra.

This paper is organised as follows: In section 2 we start with some basic concepts of Leibniz algebra. We then review Balavoine bracket, some results on cohomologies and the differential graded Lie algebra that controls the deformation of Leibniz algebra. In section 3 we define compatible Leibniz algebra and compatible bimodules. We then construct the graded Lie algebra whose Maurer-Cartan elements characterize compatible Leibniz algebra structure. In section 4 infinitesimal deformation of compatible Leibniz algebra is studied using cohomology of compatible Leibniz algebra with coefficients in itself. It is shown that equivalent infinitesimal deformations are in the same cohomology group. Then the notion of Nijenhuis operator on a compatible Leibniz algebra is studied and the correspondence between Nijenhuis operator and a trivial deformation is established. In section 5, cohomology of compatible Leibniz algebra with coefficients in an arbitrary representation is introduced. Finally in section 6, using the theory developed in section 5, abelian extension of compatible Leibniz algebra is studied. We end the paper by showing that the abelian extensions are classified by the second cohomology group.

Throughout the paper we consider the underlying field K to be of characteristic 0.

2 Background

Definition 2.1. A Leibniz algebra is a vector space L together with a K -linear operation $[\cdot, \cdot] : L \otimes L \rightarrow L$ such that

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]], \quad \forall x, y, z \in L$$

Definition 2.2. A homomorphism between two Leibniz algebras $(L_1, [\cdot, \cdot]_1)$ and $(L_2, [\cdot, \cdot]_2)$ is a K -linear map $\phi : L_1 \rightarrow L_2$ satisfying

$$\phi([x, y]_1) = [\phi(x), \phi(y)]_2.$$

Definition 2.3. Let $(L, [\])$ be a Leibniz algebra. An L -bimodule is a vector space M together with two L -actions $m_l : L \otimes M \rightarrow M$, $m_r : M \otimes L \rightarrow M$ such that for any $x, y \in L$ and $m \in M$ we have

$$\begin{aligned} m_l(x, m_l(y, m)) &= m_l([x, y], m) + m_l(y, m_l(x, m)) \\ m_l(x, m_r(m, y)) &= m_r(m_l(x, m), y) + m_r(m, [x, y]) \\ m_r(m, [x, y]) &= m_r(m_r(m, x), y) + m_l(x, m_r(m, y)). \end{aligned}$$

The following is a well established result.

Proposition 2.1. Let $(L, [\])$ be a Leibniz algebra and (M, m_l, m_r) an L -bimodule. Then $L \oplus M$ is a Leibniz algebra with the Leibniz bracket defined as

$$[(x, u), (y, v)]_{\times} = ([x, y], m_l^1(x, v) + m_r^1(u, y)) \quad \forall x, y \in L \text{ and } u, v \in M.$$

This is known as the semi-direct product.

Definition 2.4. A permutation $\sigma \in S_n$ is called an $(i, n - i)$ -shuffle if $\sigma(1) < \sigma(2) < \dots < \sigma(i)$ and $\sigma(i + 1) < \sigma(i + 2) < \dots < \sigma(n)$. If $i = 0$ or n , we assume $\sigma = id$. $S_{(i, n - i)}$ denotes the set of all $(i, n - i)$ -shuffles.

Definition 2.5. Let $(\mathfrak{g} = \oplus_{k \in \mathbb{Z}} \mathfrak{g}^k, [\], d)$ be a differential graded Lie algebra. A degree 1 element $x \in \mathfrak{g}^1$ is called a Maurer-Cartan element of \mathfrak{g} if it satisfies

$$dx + \frac{1}{2}[x, x] = 0.$$

Theorem 2.1. [3] Let $(\mathfrak{g} = \oplus_{k \in \mathbb{Z}} \mathfrak{g}^k, [\])$ be a graded Lie algebra and $\mu \in \mathfrak{g}^1$ be a Maurer-Cartan element. Then the map

$$d_{\mu} : \mathfrak{g} \rightarrow \mathfrak{g}, \quad d_{\mu}(u) := [\mu, u], \quad \forall u \in \mathfrak{g},$$

is a differential on \mathfrak{g} .

Further, for any $v \in \mathfrak{g}^1$, the sum $\mu + v$ is a Maurer-Cartan element of the graded Lie algebra $(\mathfrak{g} = \oplus_{k \in \mathbb{Z}} \mathfrak{g}^k, [\])$ iff v is a Maurer-Cartan element of the differential graded Lie algebra $(\mathfrak{g} = \oplus_{k \in \mathbb{Z}} \mathfrak{g}^k, [\], d_{\mu})$.

2.1 The Balavoine bracket

[3] Let \mathfrak{g} be a vector space. We denote $C^n(\mathfrak{g}, \mathfrak{g}) = \text{Hom}(\otimes^n \mathfrak{g}, \mathfrak{g})$ and set $C^*(\mathfrak{g}, \mathfrak{g}) = \oplus_{n \in \mathbb{N}} C^n(\mathfrak{g}, \mathfrak{g})$.

We assume the degree of an element in $C^n(\mathfrak{g}, \mathfrak{g})$ is $n - 1$.

For $P \in C^{p+1}(\mathfrak{g}, \mathfrak{g}), Q \in C^{q+1}(\mathfrak{g}, \mathfrak{g})$ we define the **Balavoine bracket** as

$$[P, Q]_B = P \circ Q - (-1)^{pq} Q \circ P$$

where $P \circ Q \in C^{p+q+1}$ is defined as

$$(P \circ Q)(x_1, x_2, \dots, x_{p+q+1}) = \sum_{k=1}^{p+1} (-1)^{(k-1)q} P \circ_k Q,$$

and

$$P \circ_k Q(x_1, x_2, \dots, x_{p+q+1}) = \sum_{\sigma \in S(k-1, q)} (-1)^\sigma P(x_{\sigma(1)}, \dots, x_{\sigma(k-1)}, Q(x_{\sigma(k)}, \dots, x_{\sigma(k+q-1)}, x_{k+q}), x_{k+q+1}, \dots, x_{p+q+1}).$$

Theorem 2.2. *The graded vector space $\mathbb{C}^*(\mathfrak{g}, \mathfrak{g})$ equipped with the Balavoine bracket given above is a graded Lie algebra.*

In particular for $\pi \in C^1(\mathfrak{g}, \mathfrak{g})$, we have $[\pi, \pi]_B \in C^3(\mathfrak{g}, \mathfrak{g})$ such that

$$[\pi, \pi]_B = \pi \circ \pi - (-1)^{1 \cdot 1} \pi \circ \pi = 2\pi \circ \pi = 2 \sum_{k=1}^2 (-1)^{k-1} \pi \circ_k \pi = 2(\pi \circ_1 \pi - \pi \circ_2 \pi)$$

$$\pi \circ_1 \pi(x, y, z) = \pi(\pi(x, y), z) \text{ and } \pi \circ_2 \pi(x, y, z) = \pi(x, \pi(y, z)) - \pi(y, \pi(x, z))$$

Thus we have the following corollary.

Corollary 2.1. *π defines a Leibniz algebra structure on \mathfrak{g} iff π is a Maurer-Cartan element of the graded Lie algebra $(\mathbb{C}^*(\mathfrak{g}, \mathfrak{g}), [\]_B)$.*

Theorem 2.3. *Let (\mathfrak{g}, π) be a Leibniz algebra. Then $(\mathbb{C}^*(\mathfrak{g}, \mathfrak{g}), [\], d_\pi)$ becomes a differential graded Lie algebra (dgLa), where $d_\pi := [\pi, \cdot]_B$.*

Further given $\pi' \in C^2(\mathfrak{g}, \mathfrak{g})$, $\pi + \pi'$ defines a Leibniz algebra structure on \mathfrak{g} iff π' is a Maurer-Cartan element of the dgLa $(\mathbb{C}^(\mathfrak{g}, \mathfrak{g}), [\], d_\pi)$.*

3 Compatible Leibniz algebra

Definition 3.1. *A Compatible Leibniz algebra is a triple $(L, [\], \{ \})$ such that $(L, [\])$ and $(L, \{ \})$ are Leibniz algebras such that*

$$[x, \{y, z\}] + \{x, [y, z]\} = [\{x, y\}, z] + \{[x, y], z\} + [y, \{x, z\}] + \{y, [x, z]\}, \quad \forall x, y, z \in L \quad (1)$$

Proposition 3.1. *A triple $(L, [\], \{ \})$ is a compatible Leibniz algebra iff $(L, [\])$ and $(L, \{ \})$ are Leibniz algebras such that for any k_1, k_2 in K , the bilinear operation*

$$\llbracket x, y \rrbracket = k_1[x, y] + k_2\{x, y\}, \quad \forall x, y \in L$$

defines a Leibniz algebra structure on L .

Proof. Let $(L, [\], \{ \})$ be a compatible Leibniz algebra. Then by definition

itself $(L, [., .])$ and $(L, \{., .\})$ are Leibniz algebras. Further,

$$\begin{aligned}
& \llbracket [x, y], z \rrbracket + \llbracket y, [x, z] \rrbracket \\
= & \llbracket k_1[x, y] + k_2\{x, y\}, z \rrbracket + \llbracket y, k_1[x, z] + k_2\{x, z\} \rrbracket \\
= & k_1[k_1[x, y] + k_2\{x, y\}, z] + k_2\{k_1[x, y] + k_2\{x, y\}, z\} + \\
& k_1[y, k_1[x, z] + k_2\{x, z\}] + k_2\{y, k_1[x, z] + k_2\{x, z\}\} \\
= & k_1k_1[[x, y], z] + k_1k_2[\{x, y\}, z] + k_2k_1\{[x, y], z\} + k_2k_2\{\{x, y\}, z\} + \\
& k_1k_1[y, [x, z]] + k_1k_2[y, \{x, z\}] + k_2k_1\{y, [x, z]\} + k_2k_2\{y, \{x, z\}\} \\
= & k_1^2(\llbracket [x, y], z \rrbracket + [y, [x, z]]) + k_2^2(\{\{x, y\}, z\} + \{y, \{x, z\}\}) \\
& k_1k_2(\{\{x, y\}, z\} + \{[x, y], z\} + [y, \{x, z\}] + \{y, [x, z]\}) \\
= & k_1^2[x, [y, z]] + k_2^2\{x, \{y, z\}\} + k_1k_2([x, \{y, z\}] + \{x, [y, z]\}) \\
= & k_1(k_1[x, [y, z]] + k_2[x, \{y, z\}]) + k_2(k_2\{x, \{y, z\}\} + k_1\{x, [y, z]\}) \\
= & k_1[x, k_1[y, z] + k_2\{y, z\}] + k_2\{x, k_2\{x, k_2y, z\} + k_1[y, z]\} \\
= & k_1[x, \llbracket y, z \rrbracket] + k_2\{x, \llbracket y, z \rrbracket\} \\
= & \llbracket x, \llbracket y, z \rrbracket \rrbracket. \tag{2}
\end{aligned}$$

The converse is straight forward. \square

Definition 3.2. A homomorphism between two compatible Leibniz algebras $(L_1, []_1, \{ \}_1)$ and $(L_2, []_2, \{ \}_2)$ is a k -linear map $\phi : L_1 \rightarrow L_2$ satisfying

$$\phi([x, y]_1) = [\phi(x), \phi(y)]_2 \quad \text{and} \quad \phi(\{x, y\}_1) = \{\phi(x), \phi(y)\}_2.$$

Definition 3.3. Let $(L, [], \{ \})$ be a compatible Leibniz algebra. A compatible L -bimodule is a vector space M together with four L -actions

$$\begin{aligned}
m_l^1 : L \otimes M &\rightarrow M, & m_r^1 : M \otimes L &\rightarrow M \\
m_l^2 : L \otimes M &\rightarrow M, & m_r^2 : M \otimes L &\rightarrow M
\end{aligned}$$

such that

- (M, m_l^1, m_r^1) is a bimodule over $(L, [])$.
- (M, m_l^2, m_r^2) is a bimodule over $(L, \{ \})$.
- the following compatibilities hold for all $x, y \in L, m \in M$

$$\begin{aligned}
LLM : & \quad m_l^1(x, m_r^2(y, m)) + m_l^2(x, m_l^1(y, m)) \\
= & \quad m_l^1(\{x, y\}, m) + m_l^2([x, y], m) + m_l^1(y, m_r^2(x, m)) + m_l^2(y, m_l^1(x, m))
\end{aligned}$$

$$\begin{aligned}
LML : & \quad m_l^1(x, m_r^2(m, y)) + m_l^2(x, m_r^1(m, y)) \\
= & \quad m_r^1(m_l^2(x, m), y) + m_r^2(m_l^1(x, m), y) + m_r^1(m, \{x, y\}) + m_r^2(m, [x, y])
\end{aligned}$$

$$\begin{aligned}
MLL : & \quad m_r^1(m, \{x, y\}) + m_r^2(m, [x, y]) \\
= & \quad m_r^1(m_r^2(m, x), y) + m_r^2(m_r^1(m, x), y) + m_l^1(x, m_r^2(m, y)) + m_l^2(x, m_r^1(m, y))
\end{aligned}$$

Note: Any compatible Leibniz algebra $(L, [\], \{ \})$ is a compatible L -bimodule in which $m_1^1 = m_r^1 = [\]$ and $m_1^2 = m_r^2 = \{ \}$. Equivalently in terms of endomorphisms, we can define a compatible L -bimodule to be a vector space M together with maps

$$\rho_1^L, \rho_1^R, \rho_2^L, \rho_2^R : L \rightarrow \text{End}(M), \text{ such that}$$

1. (L, ρ_1^L, ρ_1^R) is an L -bimodule over $(L, [\])$.
2. (L, ρ_2^L, ρ_2^R) is an L -bimodule over $(L, \{ \})$.
3. the following compatibilities hold for all $x, y \in L$

$$\begin{aligned} \rho_1^L\{x, y\} + \rho_2^L[x, y] &= [\rho_1^L(x), \rho_2^L(y)] + [\rho_2^L(x), \rho_1^L(y)] \\ \rho_1^R\{x, y\} + \rho_2^R[x, y] &= [\rho_1^R(x), \rho_2^R(y)] + [\rho_2^R(x), \rho_1^R(y)] \\ \rho_1^R(y)\rho_2^L(x) + \rho_2^R(y)\rho_1^L(x) &= -\rho_1^R(y)\rho_2^R(x) - \rho_2^R(y)\rho_1^R(x). \end{aligned}$$

3.1 Maurer-Cartan characterisation of Compatible Leibniz algebra

Definition 3.4. [1] Let $(\mathfrak{g}, [\], \delta_1)$ and $(\mathfrak{g}, [\], \delta_2)$ be two differential graded Lie algebras. We call $(\mathfrak{g}, [\], \delta_1, \delta_2)$ a bi-differential graded Lie algebra (b - $dgLa$) if δ_1 and δ_2 satisfy

$$\delta_1\delta_2 + \delta_2\delta_1 = 0.$$

Proposition 3.2. [1] Let $(\mathfrak{g}, [\], \delta_1)$ and $(\mathfrak{g}, [\], \delta_2)$ be two differential graded Lie algebras. Then $(\mathfrak{g}, [\], \delta_1, \delta_2)$ is a bi-differential graded Lie algebra iff for any k_1 and $k_2 \in K$, $(\mathfrak{g}, [\], \delta_{k_1k_2})$ is a differential graded Lie algebra where $\delta_{k_1k_2} = k_1\delta_1 + k_2\delta_2$.

Definition 3.5. Let $(\mathfrak{g}, [\], \delta_1, \delta_2)$ be a b - $dgLa$. A pair $(\pi_1, \pi_2) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1$ is called a Maurer-Cartan element of the b - $dgLa$ $(\mathfrak{g}, [\], \delta_1, \delta_2)$ if π_1 and π_2 are Maurer-Cartan elements of the $dgLa$ s $(\mathfrak{g}, [\], \delta_1)$ and $(\mathfrak{g}, [\], \delta_2)$ respectively, and

$$\delta_2\pi_1 + \delta_1\pi_2 + [\pi_1, \pi_2] = 0.$$

Proposition 3.3. A pair $(\pi_1, \pi_2) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1$ is a Maurer-Cartan element of the b - $dgLa$ $(\mathfrak{g}, [\], \delta_1, \delta_2)$ iff for any $k_1, k_2 \in K$, $k_1\pi_1 + k_2\pi_2$ is a Maurer-Cartan element of the $dgLa$ $(\mathfrak{g}, [\], \delta_{k_1k_2})$.

Theorem 3.1. Let L be a vector space and $\pi_1, \pi_2 \in C^2(L, L)$. Then (L, π_1, π_2) is a compatible Leibniz algebra iff (π_1, π_2) is a Maurer-Cartan element of the b - $dgLa$ $(\mathbb{C}^*(L, L), [\]_B, \delta_1 = 0, \delta_2 = 0)$.

Proof. (L, π_1, π_2) is a compatible Leibniz algebra gives (L, π_1) and (L, π_2) are Leibniz algebras. Hence we get $[\pi_1, \pi_1]_B = [\pi_2, \pi_2]_B = 0$. Further $\forall x, y, z \in L$ we have the compatibility condition,

$$\begin{aligned} & \pi_1(x, \pi_2(y, z)) + \pi_2(x, \pi_1(y, z)) \\ = & \pi_1(\pi_2(x, y), z) + \pi_2(\pi_1(x, y), z) + \pi_1(y, \pi_2(x, z)) + \pi_2(y, \pi_1(x, z)) \quad (3) \end{aligned}$$

We note that $[\pi_1, \pi_2]_B = \pi_1 \circ \pi_2 + \pi_2 \circ \pi_1$, where

$$\begin{aligned} \pi_1 \circ \pi_2(x, y, z) &= (\pi_1 \circ_1 \pi_2 - \pi_1 \circ_2 \pi_2)(x, y, z) \\ &= \pi_1(\pi_2(x, y), z) - \pi_1(x, \pi_2(y, z)) + \pi_1(y, \pi_2(x, z)) \end{aligned}$$

and

$$\begin{aligned} \pi_2 \circ \pi_1(x, y, z) &= (\pi_2 \circ_1 \pi_1 - \pi_2 \circ_2 \pi_1)(x, y, z) \\ &= \pi_2(\pi_1(x, y), z) - \pi_2(x, \pi_1(y, z)) + \pi_2(y, \pi_1(x, z)). \end{aligned}$$

i.e.,

$$\begin{aligned} [\pi_1, \pi_2]_B(x, y, z) &= \pi_1(\pi_2(x, y), z) - \pi_1(x, \pi_2(y, z)) + \pi_1(y, \pi_2(x, z)) \\ &\quad + \pi_2(\pi_1(x, y), z) - \pi_2(x, \pi_1(y, z)) + \pi_2(y, \pi_1(x, z)). \end{aligned}$$

We thus have that $[\pi_1, \pi_2]_B = 0$ is equivalent to the compatibility condition 3. \square

Theorem 3.2. [1] Let (π_1, π_2) be a Maurer-Cartan element of the b -dgLa $(\mathfrak{g}, [\cdot, \cdot], \delta_1, \delta_2)$.

Define $d_1 := \delta_1 + [\pi_1, \cdot]$ and $d_2 := \delta_2 + [\pi_2, \cdot]$. Then $(\mathfrak{g}, [\cdot, \cdot], d_1, d_2)$ is a b -dgLa. Further for any $\tilde{\pi}_1, \tilde{\pi}_2 \in \mathfrak{g}_1$, $(\pi_1 + \tilde{\pi}_1, \pi_2 + \tilde{\pi}_2)$ is a Maurer-Cartan element of the b -dgLa $(\mathfrak{g}, [\cdot, \cdot], \delta_1, \delta_2)$ iff $(\tilde{\pi}_1, \tilde{\pi}_2)$ is a Maurer-Cartan element of the b -dgLa $(\mathfrak{g}, [\cdot, \cdot], d_1, d_2)$.

Let (L, π_1, π_2) be a compatible Leibniz algebra. From theorems 3.1 and 3.2, we conclude the following important results:

Theorem 3.3. $(\mathbb{C}^*(L, L), [\cdot, \cdot], d_1, d_2)$ is a b -dgLa where d_1 and d_2 are given by $d_1 := [\pi_1, \cdot]_B$ and $d_2 := [\pi_2, \cdot]_B$.

Theorem 3.4. For any $\tilde{\pi}_1, \tilde{\pi}_2 \in C^2(L, L)$, $(L, \pi_1 + \tilde{\pi}_1, \pi_2 + \tilde{\pi}_2)$ is a compatible Leibniz algebra iff $(\pi_1 + \tilde{\pi}_1, \pi_2 + \tilde{\pi}_2)$ is a Maurer-Cartan element of the b -dgLa $(\mathbb{C}^*(L, L), [\cdot, \cdot]_B, d_1, d_2)$.

3.2 Cohomology of compatible Leibniz algebra

Let $(L, [\cdot, \cdot], \{ \cdot, \cdot \})$ be a compatible Leibniz algebra with $\pi_1(x, y) = [x, y]$ and $\pi_2(x, y) = \{x, y\}$. By theorem 3.1, (π_1, π_2) is a Maurer-Cartan element of the b -dgLa $(\mathbb{C}^*(L, L), [\cdot, \cdot]_B, 0, 0)$.

We define the cochains as follows: For $n \geq 1$,

$$LC^n(L, L) := \underbrace{\mathbb{C}^n(L, L) \oplus \mathbb{C}^n(L, L) \dots \oplus \mathbb{C}^n(L, L)}_{n \text{ times}}$$

and $d^n : LC^n(L, L) \rightarrow LC^{n+1}(L, L)$ by

$$d^1 f = ([\pi_1, f]_B, [\pi_2, f]_B), \quad \forall f \in LC^1(L, L)$$

$$d^n(f_1, f_2, \dots, f_n) = (-1)^{n-1}([\pi_1, f_1]_B, \dots, [\pi_2, f_{i-1}]_B + [\pi_1, f_i]_B, \dots, [\pi_2, f_n]_B),$$

where $(f_1, f_2, \dots, f_n) \in LC^n(L, L)$ and $2 \leq i \leq n$.

d defined as above gives the following theorem.

Theorem 3.5. *We have $d^{n+1} \circ d^n = 0$.*

Proof. We first note that since (π_1, π_2) is a Maurer-Cartan element of the b-dgLa $(\mathbb{C}^*(L, L), [\]_B, 0, 0)$ we have $[\pi_1, \pi_1]_B = 0$, $[\pi_1, \pi_2]_B = 0$, $[\pi_2, \pi_2]_B = 0$. For any $(f_1, f_2, \dots, f_n) \in LC^n(L, L)$, $2 \leq i \leq n$ we have

$$\begin{aligned} & d^{n+1} d^n(f_1, f_2, \dots, f_n) \\ &= (-1)^{n-1} d^{n+1}([\pi_1, f_1]_B, \dots, [\pi_2, f_{i-1}]_B + [\pi_1, f_i]_B, \dots, [\pi_2, f_n]_B) \\ &= -([\pi_1, [\pi_1, f_1]_B]_B, [\pi_2, [\pi_1, f_1]_B]_B + [\pi_1, [\pi_2, f_1]_B]_B + [\pi_1, [\pi_1, f_2]_B]_B, \dots \\ & \quad [\pi_2, [\pi_2, f_{i-2}]_B]_B + [\pi_2, [\pi_1, f_{i-1}]_B]_B + [\pi_1, [\pi_2, f_{i-1}]_B]_B + [\pi_1, [\pi_1, f_i]_B]_B, \dots, \\ & \quad [\pi_2, [\pi_2, f_{n-1}]_B]_B + [\pi_2, [\pi_1, f_n]_B]_B + [\pi_1, [\pi_2, f_n]_B]_B, [\pi_2, [\pi_2, f_n]_B]_B) \quad (3 \leq i \leq n-1) \\ &= -\left(\frac{1}{2}[[\pi_1, \pi_1]_B, f_1]_B, [[\pi_1, \pi_2]_B, f_1]_B + \frac{1}{2}[[\pi_1, \pi_1]_B, f_2]_B, \dots \right. \\ & \quad \left. \frac{1}{2}[[\pi_2, \pi_2]_B, f_{i-2}]_B + [[\pi_1, \pi_2]_B, f_{i-1}]_B + \frac{1}{2}[[\pi_1, \pi_1]_B, f_i]_B, \dots, \right. \\ & \quad \left. \frac{1}{2}[[\pi_2, \pi_2]_B, f_{n-1}]_B + [[\pi_1, \pi_2]_B, f_n]_B, \frac{1}{2}[[\pi_2, \pi_2]_B, f_n]_B\right) \\ &= (0, 0, \dots, 0). \end{aligned}$$

□

Hence we have that $LC^*(L, L) = (\oplus_{n \in \mathbb{N}} LC^n(L, L), d^*)$ is a cochain complex.

Definition 3.6. *Let $(L, [\], \{ \})$ be a compatible Leibniz algebra. The cohomology of the cochain complex $(LC^*(L, L), d^*)$ is called the cohomology of $(L, [\], \{ \})$. We denote the cohomology group by $H^n(L, L)$.*

4 Infinitesimal deformations of compatible Leibniz algebras

Definition 4.1. *Let $(L, [\], \{ \})$ be a compatible Leibniz algebra. A formal one-parameter deformation of L is a pair of $k[[t]]$ -linear maps*

$$\mu_t : L[[t]] \otimes L[[t]] \rightarrow L[[t]] \quad \text{and}$$

$$m_t : L[[t]] \otimes L[[t]] \rightarrow L[[t]] \quad \text{such that :}$$

(a) $\mu_t(a, b) = \sum_{i=0}^{\infty} \mu_i(a, b)t^i$, $m_t(a, b) = \sum_{i=0}^{\infty} m_i(a, b)t^i$
for all $a, b \in L$, where $\mu_i, m_i : L \otimes L \rightarrow L$ are k -linear and $\mu_0(a, b) = [a, b]$
and $m_0(a, b) = \{a, b\}$.

(b) For any t , $(L[[t]], \mu_t, m_t)$ is a compatible Leibniz algebra.

Definition 4.2. Let $(L, [\], \{ \ })$ be a compatible Leibniz algebra. Let $\mu_1, m_1 \in C^2(L, L)$. Define

$$\mu_t(x, y) = [x, y] + t\mu_1(x, y), \quad m_t(x, y) = \{x, y\} + tm_1(x, y), \quad \forall x, y \in L.$$

If for any t , (L, μ_t, m_t) is a compatible Leibniz algebra, we say that (L, μ_t, m_t) defines an infinitesimal deformation of $(L, [\], \{ \ })$.

We also say that (μ_1, m_1) generates an infinitesimal deformation of $(L, [\], \{ \ })$. For convenience we write $[x, y] = \mu_0(x, y)$ and $\{x, y\} = m_0(x, y)$.

By 3.1 we have that (L, μ_t, m_t) is a compatible Leibniz algebra if and only if (μ_t, m_t) is a Maurer-Cartan element of $(C^*(L, L), [\]_B, 0, 0)$. “ (μ_t, m_t) is a Maurer-Cartan element ” is equivalent to following condition

$$[\mu_t, \mu_t]_B = 0 \quad [m_t, m_t]_B = 0 \quad [\mu_t, m_t]_B = 0 \quad (4)$$

Condition 4 is equivalent to following conditions

$$\begin{aligned} [\mu_0, \mu_0]_B = 0, \quad [\mu_0, \mu_1]_B = 0, \quad [\mu_1, \mu_1]_B = 0 \\ [m_0, m_0]_B = 0, \quad [m_0, m_1]_B = 0, \quad [m_1, m_1]_B = 0 \\ [\mu_0, m_0]_B = 0, \quad [\mu_0, m_1]_B + [\mu_1, m_0]_B = 0, \quad [\mu_1, m_1]_B = 0 \end{aligned}$$

Reordering the terms and excluding the trivial equations we get that (L, μ_t, m_t) defines an infinitesimal deformation of $(L, [\], \{ \ })$ iff

$$\begin{aligned} [\mu_0, \mu_1]_B = 0, \quad [m_0, m_1]_B = 0, \quad [\mu_0, m_1]_B + [\mu_1, m_0]_B = 0 \\ [\mu_1, \mu_1]_B = 0, \quad [m_1, m_1]_B = 0, \quad [\mu_1, m_1]_B = 0. \end{aligned}$$

Note that the first line above implies $d^2(\mu_1, m_1) = 0$ i.e (μ_1, m_1) is a 2-cocycle and the second line implies that (L, μ_1, m_1) is a compatible Leibniz algebra.

Hence we have the following theorem.

Theorem 4.1. Let $(L, [\], \{ \ })$ be a compatible Leibniz algebra. If $(\mu_1, m_1) \in LC^2(L, L)$ generates an infinitesimal deformation then (μ_1, m_1) is a cocycle.

Definition 4.3. Two infinitesimal deformations (L, μ_t, m_t) and (L, μ'_t, m'_t) are said to be equivalent if there exists a linear bijection $N : L \rightarrow L$ such that

$$Id + tN : (L, \mu_t, m_t) \rightarrow (L, \mu'_t, m'_t)$$

is a compatible Leibniz algebra homomorphism.

$Id + tN$ being a compatible Leibniz algebra homomorphism implies

1. $[x, y] = [x, y]'$
2. $\mu_1(x, y) - \mu'_1(x, y) = [x, N(y)] + [N(x), y] - N[x, y]$
3. $N\mu_1(x, y) = \mu'_1(x, N(y)) + \mu'_1(N(x), y) + [N(x), N(y)]$
4. $\mu'_1(N(x), N(y)) = 0$
5. $\{x, y\} = \{x, y\}'$
6. $m_1(x, y) - m'_1(x, y) = \{x, N(y)\} + \{N(x), y\} - N\{x, y\}$
7. $Nm_1(x, y) = m'_1(x, N(y)) + m'_1(N(x), y) + \{N(x), N(y)\}$
8. $m'_1(N(x), N(y)) = 0$

2 and 6 gives

$$\begin{aligned}
(\mu_1 - \mu'_1, m_1 - m'_1)(x, y) &= ([x, N(y)] + [N(x), y] - N[x, y], \{x, N(y)\} \\
&\quad + \{N(x), y\} - N\{x, y\}) \\
&= ([\mu_0, N]_B, [m_0, N]_B)(x, y) \\
&= d^1N(x, y).
\end{aligned}$$

Thus we have the following theorem

Theorem 4.2. *If two infinitesimal deformations (L, μ_t, m_t) and (L, μ'_t, m'_t) of a compatible Leibniz algebra (L, μ_0, m_0) are equivalent then, (μ_1, m_1) and (μ'_1, m'_1) are in the same cohomology class.*

Definition 4.4. *Let $(L, [\])$ be a Leibniz algebra. A linear map $N : L \rightarrow L$ is said to be a Nijenhuis operator on L if*

$$N([x, N(y)] + [N(x), y] - N[x, y]) = [N(x), N(y)] \quad \forall x, y \in L.$$

We define linear $[\]_N : L \otimes L \rightarrow L$ as

$$[x, y]_N = [x, N(y)] + [N(x), y] - N[x, y]$$

$T_{[\]_N} : L \otimes L \rightarrow L$ denotes the Nijenhuis torsion of N defined as

$$T_{[\]_N}(x, y) = N([x, y]_N) - [N(x), N(y)], \quad \forall x, y \in L.$$

When N is a Nijenhuis operator we get that $T_{[\]_N} = 0$.

Proposition 4.1. *If $N : L \rightarrow L$ is a Nijenhuis operator on Leibniz algebra $(L, [\])$, then $(L, [\]_N)$ is also a Leibniz algebra. Further N is a Leibniz algebra homomorphism from $(L, [\]_N)$ to $(L, [\])$. Furthermore $(L, [\], [\]_N)$ forms a compatible Leibniz algebra.*

Proof.

$$[x, [y, z]_N]_N = [[x, y]_N, z]_N + [y, [x, z]_N]_N, \quad \forall x, y, z \in L$$

Further, $N([x, y]_N) = [N(x), N(y)]$ follows from the definition of Nijenhuis operator and $[\]_N$.

To show $(L, [\], [\]_N)$ is a compatible Leibniz algebra we first note that $\pi_N = [\pi, N]_B$. For any k_1 and $k_2 \in K$,

$$\begin{aligned} [k_1\pi + k_2\pi_N, k_1\pi + k_2\pi_N]_B &= k_1k_2([\pi, \pi_N]_B + [\pi_N, \pi]_B) \\ &= 2k_1k_2[\pi, \pi_N]_B \\ &= 2k_1k_2[\pi, [\pi, N]_B]_B \\ &= 0. \end{aligned}$$

□

Definition 4.5. Let $(L, [\], \{ \})$ be a compatible Leibniz algebra. A linear map $N : L \rightarrow L$ is said to be a Nijenhuis operator on $(L, [\], \{ \})$ if N is a Nijenhuis operator on the Leibniz algebras $(L, [\])$ and $(L, \{ \})$.

Proposition 4.2. Let $(L, [\], \{ \})$ be a compatible Leibniz algebra. A linear map $N : L \rightarrow L$ is a Nijenhuis operator on $(L, [\], \{ \})$ iff for any $k_1, k_2 \in K$, N is a Nijenhuis operator on the Leibniz algebra $(L, [\])$, where $[[x, y]] = k_1[x, y] + k_2\{x, y\}$, $\forall x, y \in L$.

Proof. We have

$$\begin{aligned} T_{[[\]_N}(x, y) &= N([[x, y]]_N) - [[N(x), N(y)]] \\ &= N(k_1[x, Ny] + k_2\{x, Ny\} + k_1[Nx, y] + k_2\{Nx, y\}) \\ &\quad - k_1[N(x), N(y)] - k_2\{N(x), N(y)\} \\ &= k_1(N([x, y]_N) - [N(x), N(y)]) + k_2(N(\{x, y\}_N) - \{N(x), N(y)\}) \\ &= k_1T_{[\]_N}(x, y) + k_2T_{\{ \}_N}(x, y) \end{aligned} \tag{5}$$

Hence we have,

$$T_{[[\]_N} = 0 \quad \text{iff} \quad T_{[\]_N} = T_{\{ \}_N} = 0.$$

Proposition 4.3. Let $(L, [\], \{ \})$ be a compatible Leibniz algebra and $N : L \rightarrow L$ be a Nijenhuis operator on $(L, [\], \{ \})$. Then $(L, [\]_N, \{ \}_N)$ is also a compatible Leibniz algebra and N is a compatible Leibniz algebra homomorphism from $(L, [\]_N, \{ \}_N)$ to $(L, [\], \{ \})$.

Proof. Let $N : L \rightarrow L$ be a Nijenhuis operator on $(L, [\], \{ \})$. then by the previous theorem N is a Nijenhuis operator on the Leibniz algebra $(L, [\])$ for any k_1, k_2 in K .

Using result 4.1 we get that $(L, [[\]_N)$ is a Leibniz algebra and N is a Leibniz algebra homomorphism from $(L, [[\]_N)$ to $(L, [[\])$.

Hence we have that $(L, [\]_N, \{ \}_N)$ is a compatible Leibniz algebra. And we also get that N is a compatible Leibniz algebra homomorphism from $(L, [\]_N, \{ \}_N)$ to $(L, [\], \{ \})$.

Definition 4.6. An infinitesimal deformation (L, μ_t, m_t) of compatible Leibniz algebra (L, μ_0, m_0) generated by (μ_1, m_1) is trivial if there exists linear $N : L \rightarrow L$ such that $Id + tN : (L, \mu_t, m_t) \rightarrow (L, \mu_0, m_0)$ is a compatible Leibniz algebra homomorphism.

$Id + tN$ is a compatible Leibniz algebra homomorphism iff

1. $\mu_1(x, y) = [x, N(y)] + [N(x), y] - N[x, y]$
2. $m_1(x, y) = \{x, N(y)\} + \{N(x), y\} - N\{x, y\}$
3. $N\mu_1(x, y) = [N(x), N(y)]$
4. $Nm_1(x, y) = \{N(x), N(y)\}$

1 and 3 gives that N is a Nijenhuis operator on (L, μ_0) . 2 and 4 gives that N is a Nijenhuis operator on (L, m_0) .

Thus we have the following theorem.

Theorem 4.3. A trivial infinitesimal deformation of a compatible Leibniz algebra gives rise to a Nijenhuis operator.

Theorem 4.4. A Nijenhuis operator on a compatible Leibniz algebra $(L, [\], \{ \ })$ gives rise to a trivial deformation.

Proof. Let N be a Nijenhuis operator on a compatible Leibniz algebra $(L, [\], \{ \ })$. Take

$$\begin{aligned}\mu_1(x, y) &= [x, N(y)] + [N(x), y] - N[x, y] \\ m_1(x, y) &= \{x, N(y)\} + \{N(x), y\} - N\{x, y\}\end{aligned}$$

for any $x, y \in L$. Then

$$\begin{aligned}d^1 N(x, y) &= ([\mu_0, N]_B, [m_0, N]_B)(x, y) \\ &= ([x, N(y)] + [N(x), y] - N[x, y], \{x, N(y)\} + \{N(x), y\} - N\{x, y\}) \\ &= (\mu_1(x, y), m_1(x, y)).\end{aligned}$$

i.e., (μ_1, m_1) is a 2-cocycle.

Further since N is a Nijenhuis operator on $(L, [\], \{ \ })$, and $\mu_1 = [\]_N$ and $m_1 = \{ \ }_N$, by proposition (4.3) we get that $(L, [\]_N, \{ \ }_N)$ is a compatible Leibniz algebra.

These two statements implies that (μ_1, m_1) give rise to an infinitesimal deformation of L . Showing the deformation is trivial is straightforward.

5 Cohomologies of compatible Leibniz algebras with coefficients in arbitrary representation

For vector spaces g_1 and g_2 , we define $g^{l,k}$ to be the direct sum of tensor products of g_1 and g_2 , where g_1 is repeated l times and g_2 is repeated k times. For example $g^{1,1} = (g_1 \otimes g_2) \oplus (g_2 \otimes g_1)$ and $g^{2,1} = (g_1 \otimes g_1 \otimes g_2) \oplus (g_1 \otimes g_2 \otimes g_1) \oplus (g_2 \otimes g_1 \otimes g_1)$.

Thus $\otimes^n(g_1 \oplus g_2) \equiv \oplus_{l+k=n} g^{l,k}$.

For any linear map $f : g_{i_1} \otimes g_{i_2} \cdots \otimes g_{i_n} \rightarrow g_j$, where $i_1, i_2, \dots, i_n, j \in \{1, 2\}$ we define $\hat{f} \in C^n(g_1 \oplus g_2, g_1 \oplus g_2)$ as

$$\hat{f} = \begin{cases} f, & \text{on } g_{i_1} \otimes g_{i_2} \cdots \otimes g_{i_n} \\ 0, & \text{otherwise} \end{cases}$$

\hat{f} is called a lift of f .

In particular, for the linear maps we encountered in the previous sections:

$$\pi : L \otimes L \rightarrow L, \quad m_l : L \otimes M \rightarrow M, \quad m_r : M \otimes L \rightarrow M$$

we get lifts

$$\hat{\pi} : (L \oplus M)^2 \rightarrow L \oplus M, \text{ defined as } \hat{\pi}((x_1, v_1), (x_2, v_2)) = (\pi(x_1, x_2), 0)$$

$$\hat{m}_l : (L \oplus M)^2 \rightarrow L \oplus M, \text{ defined as } \hat{m}_l((x_1, v_1), (x_2, v_2)) = (0, m_l(x_1, v_2))$$

$$\hat{m}_r : (L \oplus M)^2 \rightarrow L \oplus M, \text{ defined as } \hat{m}_r((x_1, v_1), (x_2, v_2)) = (0, m_r(v_1, x_2))$$

By property of the Hom-functor we get

$$C^n(g_1 \oplus g_2, g_1 \oplus g_2) \equiv \sum_{l+k=n} C^n(g^{l,k}, g_1) \oplus \sum_{l+k=n} C^n(g^{l,k}, g_2).$$

Definition 5.1. A linear map $f \in \text{Hom}(\otimes^n(g_1 \oplus g_2), (g_1 \oplus g_2))$ has bidegree $l|k$ if

1. $l + k + 1 = n$
2. if $X \in g^{l+1,k}$ then $f(X) \in g_1$
3. if $X \in g^{l,k+1}$ then $f(X) \in g_2$
4. $f(X) = 0$ in all other cases.

We use notation $\|f\| = l|k$. We say that f is homogeneous if f has a bidegree.

Considering examples above, we have $\|\hat{\pi}\| = \|\hat{m}_l\| = \|\hat{m}_r\| = 1|0$.

In the next three lemmas, we consider a few standard results regarding bidegrees [12].

Lemma 5.1. If $f_1, f_2, \dots, f_k \in C^n(g_1 \oplus g_2, g_1 \oplus g_2)$ be homogeneous linear maps and the bidegrees of f_i are different. Then $f_1 + f_2 + \dots + f_k = 0$ iff $f_1 = f_2 = \dots = f_k = 0$.

Lemma 5.2. If $\|f\| = -1|l$ ($l - 1$) and $\|g\| = -1|k$ ($k - 1$) then $[f, g]_B = 0$.

Lemma 5.3. $f \in C^n(g_1 \oplus g_2, g_1 \oplus g_2)$ and $g \in C^m(g_1 \oplus g_2, g_1 \oplus g_2)$ be homogeneous linear maps with bidegrees $l_f|k_f$ and $l_g|k_g$ respectively. Then $[f, g]_B$ is a linear map of bidegree $l_f + l_g|k_f + k_g$.

Theorem 5.1. [3] Let $(L, \pi = [\])$ be a Leibniz algebra. (V, m_l, m_r) is a representation of L iff $\hat{m}_l + \hat{m}_r$ is a Maurer-Cartan element of the dgLA $(C^*(L \oplus V, L \oplus V), [\]_B, \partial_{\hat{\pi}} = [\hat{\pi}, \cdot]_B)$.

Corollary 5.1. If (V, m_l, m_r) is a representation of (L, π) , then $[\hat{\pi} + \hat{m}_l + \hat{m}_r, \hat{\pi} + \hat{m}_l + \hat{m}_r]_B = 0$.

Let (V, m_l, m_r) be a representation of the Leibniz algebra (L, π) . We denote the set of all homogeneous elements of degree $p|q$ of $C^{p+q+1}(L \oplus V, L \oplus V)$ by $C^{p|q}(L \oplus V, L \oplus V)$. We define the set of n -cochains as

$$C^n(L, V) := C^{n|^{-1}}(L \oplus V, L \oplus V) \cong \text{Hom}(\otimes^n L, V) \text{ using the lift map}$$

and coboundary operator $d_{\hat{\pi} + \hat{m}_l + \hat{m}_r}^n : C^n(L, V) \rightarrow C^{n+1}(L, V)$ as

$$d_{\hat{\pi} + \hat{m}_l + \hat{m}_r}^n f := (-1)^{n-1} [\hat{\pi} + \hat{m}_l + \hat{m}_r, \hat{f}]_B, \quad \forall f \in C^n(L, V).$$

Note that since $\hat{\pi} + \hat{m}_l + \hat{m}_r \in C^{1|0}$ and $\hat{f} \in C^{n|^{-1}}$, Lemma 5.3 gives us that $[\hat{\pi} + \hat{m}_l + \hat{m}_r, \hat{f}]_B \in C^{n+1|^{-1}}$.

Further note that $d^{n+1} d^n f = -[\hat{\pi} + \hat{m}_l + \hat{m}_r, [\hat{\pi} + \hat{m}_l + \hat{m}_r, \hat{f}]_B]_B = 0$ by the graded Jacobi identity.

Thus we have a well defined cochain complex $(C^*(L, V), d_{\hat{\pi} + \hat{m}_l + \hat{m}_r}^*)$.

Theorem 5.2. Let $(L, \pi_1 = [\], \pi_2 = \{ \ })$ be a compatible Leibniz algebra and $(M, m_l^1, m_r^1, m_l^2, m_r^2)$ a representation of L . Then $(\hat{\pi}_1 + \hat{m}_l^1 + \hat{m}_r^1, \hat{\pi}_2 + \hat{m}_l^2 + \hat{m}_r^2)$ is a Maurer-Cartan element of the bi-differential graded Lie Algebra $(C^*(L \oplus M, L \oplus M), [\]_B, 0, 0)$ i.e

$$[\hat{\pi}_1 + \hat{m}_l^1 + \hat{m}_r^1, \hat{\pi}_1 + \hat{m}_l^1 + \hat{m}_r^1]_B = 0, \quad (6)$$

$$[\hat{\pi}_1 + \hat{m}_l^1 + \hat{m}_r^1, \hat{\pi}_2 + \hat{m}_l^2 + \hat{m}_r^2]_B = 0, \quad (7)$$

$$[\hat{\pi}_2 + \hat{m}_l^2 + \hat{m}_r^2, \hat{\pi}_2 + \hat{m}_l^2 + \hat{m}_r^2]_B = 0 \quad (8)$$

Proof. Since (M, m_l^1, m_r^1) is a representation of the Leibniz algebra $(L, \pi_1 = [\])$, by corollary 5.1 equation 6 holds. Likewise (M, m_l^2, m_r^2) is a representation of the Leibniz algebra $(L, \pi_2 = \{ \ })$, by corollary 5.1 equation 8 holds.

For $x_1, x_2, x_3 \in L, v_1, v_2, v_3 \in V$

$$\begin{aligned} & [\hat{\pi}_1 + \hat{m}_l^1 + \hat{m}_r^1, \hat{\pi}_2 + \hat{m}_l^2 + \hat{m}_r^2]_B(x_1, v_1), (x_2, v_2), (x_3, v_3) \\ &= (\pi_1(\pi_2(x_1, x_2), x_3), m_l^1(\pi_2(x_1, x_2), v_3) + m_r^1(m_l^2(x_1, v_2) + (m_r^2(u_1, x_2), x_3))) \\ &+ (-\pi_1(x_1, \pi_2(x_2, x_3)), -m_l^1(x_1, m_l^2(x_2, v_3) + m_r^2(u_2, x_3)) - m_r^1(u_1, \pi_2(x_2, x_3))) \\ &+ (\pi_1(x_2, \pi_2(x_1, x_3)), m_l^1(x_2, m_l^2(x_1, v_3) + m_r^2(u_1, x_3)) + m_r^1(u_2, \pi_2(x_1, x_3))) \\ &+ (\pi_2(\pi_1(x_1, x_2), x_3), m_l^2(\pi_1(x_1, x_2), v_3) + m_r^2(m_l^1(x_1, v_2) + m_r^1(u_1, x_2), x_3))) \\ &+ (-\pi_2(x_1, \pi_1(x_2, x_3)), -m_l^2(x_1, m_l^1(x_2, v_3) + m_r^1(u_2, x_3)) - m_r^2(u_1, \pi_1(x_2, x_3))) \\ &+ ((\pi_2(x_2, \pi_1(x_1, x_3)), m_l^2(x_2, m_l^1(x_1, v_3) + m_r^1(u_1, x_3)) + m_r^2(u_2, \pi_1(x_1, x_3))) \\ &= 0. \end{aligned} \quad (9)$$

We get the above by the compatibility conditions 1, *LLM*, *LML* and *MLL*.

Note that the coboundary operator for (L, π_1) with coefficients in (M, m_l^1, m_r^1) and for (L, π_2) with coefficients in (M, m_l^2, m_r^2) are respectively given by

$$d_{\pi_1+m_l^1+m_r^1}^n f := (-1)^{n-1}[\hat{\pi}_1 + \hat{m}_l^1 + \hat{m}_r^1, \hat{f}]_B, \text{ and}$$

$$d_{\pi_2+m_l^2+m_r^2}^n f := (-1)^{n-1}[\hat{\pi}_2 + \hat{m}_l^2 + \hat{m}_r^2, \hat{f}]_B, \forall f \in C^n(L, M)$$

By the graded Jacobi identity it can be shown that the three conditions 6 , 7, 8 implies

$$\begin{aligned} d_{\pi_1+\hat{m}_l^1+\hat{m}_r^1}^{n+1} d_{\pi_1+\hat{m}_l^1+\hat{m}_r^1}^n &= 0 \\ d_{\pi_2+\hat{m}_l^2+\hat{m}_r^2}^{n+1} d_{\pi_2+\hat{m}_l^2+\hat{m}_r^2}^n &= 0 \\ d_{\pi_1+\hat{m}_l^1+\hat{m}_r^1}^{n+1} d_{\pi_2+\hat{m}_l^2+\hat{m}_r^2}^n + d_{\pi_2+\hat{m}_l^2+\hat{m}_r^2}^{n+1} d_{\pi_1+\hat{m}_l^1+\hat{m}_r^1}^n &= 0 \end{aligned} \quad (10)$$

For $n \geq 1$ we define the space of n-cochains $LC^n(L, M)$ as

$$LC^n(L, M) = C^n(L, M) \oplus C^n(L, M) \oplus \cdots \oplus C^n(L, M) \text{ } \}n\text{-copies}$$

and coboundary for $n \geq 1$, $\partial^n : LC^n \rightarrow LC^{n+1}$ as

$$\partial^1 f = (d_{\hat{\pi}_1+\hat{m}_l^1+\hat{m}_r^1} f, d_{\hat{\pi}_2+\hat{m}_l^2+\hat{m}_r^2} f) \quad \forall f \in Hom(L, M)$$

and for $2 \leq i \leq n$ and $(f_1, f_2, \dots, f_n) \in LC^n(L, V)$,

$$\begin{aligned} \partial^n(f_1, f_2, \dots, f_n) \\ = (d_{\pi_1+\hat{m}_l^1+\hat{m}_r^1}^n f_1, \dots, d_{\pi_2+\hat{m}_l^2+\hat{m}_r^2}^n f_{i-1} + d_{\pi_1+\hat{m}_l^1+\hat{m}_r^1}^n f_i, \dots, d_{\pi_2+\hat{m}_l^2+\hat{m}_r^2}^n f_n). \end{aligned}$$

Using 10 or just like in Theorem 3.5 it can be shown that $\partial^2 = 0$.

Definition 5.2. Let $(M, m_l^1, m_r^1, m_l^2, m_r^2)$ be a representation of a compatible Leibniz algebra (L, π_1, π_2) . The cohomology of the cochain complex $(\oplus_{n=1}^{\infty} LC^n(L, L), \partial)$ is called the cohomology of (L, π_1, π_2) with coefficient in the representation $(M, m_l^1, m_r^1, m_l^2, m_r^2)$. The corresponding n^{th} cohomology group is denoted by $\mathbb{H}^n(L, M)$.

6 Abelian extension of compatible Leibniz algebras

Definition 6.1. Let (L, π_1, π_2) and (V, μ_1, μ_2) be two compatible Leibniz algebras. An extension of (L, π_1, π_2) by (V, μ_1, μ_2) is a short exact sequence of compatible Leibniz algebra morphisms

$$0 \rightarrow V \xrightarrow{i} \tilde{L} \xrightarrow{\rho} L \rightarrow 0, \quad (11)$$

where $(\tilde{L}, [\]_{\tilde{L}}, \{ \}_{\tilde{L}})$ is a compatible Leibniz algebra.

A section of the extension $(\tilde{L}, \tilde{\pi}_1, \tilde{\pi}_2)$ of (L, π_1, π_2) by (V, μ_1, μ_2) is a linear map $s : L \rightarrow \tilde{L}$ such that $\rho \circ s = Id_L$ where Id_L is the identity morphism on L . \tilde{L} is called an abelian extension of L by V , if the compatible Leibniz algebra structure on V is trivial i.e., $\mu_1(x, y) = \mu_2(x, y) = 0, \forall x, y \in V$.

Definition 6.2. Two abelian extensions, $(L_1, [\]_1, \{ \}_1)$ and $(L_2, [\]_2, \{ \}_2)$ of (L, π_1, π_2) by $(V, 0, 0)$ given by exact sequences $0 \rightarrow V \xrightarrow{i} L_1 \xrightarrow{\rho} L \rightarrow 0$ and $0 \rightarrow V \xrightarrow{i'} L_2 \xrightarrow{\rho'} L \rightarrow 0$ are equivalent if there exists an isomorphism $F : L_1 \rightarrow L_2$ such that the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & L_1 & \xrightarrow{\rho} & L \longrightarrow 0 \\ & & \downarrow Id_V & & \downarrow F & & \downarrow Id_L \\ 0 & \longrightarrow & V & \xrightarrow{i'} & L_2 & \xrightarrow{\rho'} & L \longrightarrow 0. \end{array} \quad (12)$$

Consider an abelian extension \tilde{L} of L by V and section $s : L \rightarrow \tilde{L}$ For $x, y \in L, u \in V$ define linear maps $\theta : L \otimes L \rightarrow V, \tilde{\theta} : L \otimes L \rightarrow V, m_l^2 : L \otimes V \rightarrow V$ and $m_r^1, m_r^2 : V \otimes L \rightarrow V$ as

$$\begin{aligned} \theta(x, y) &= [s(x), s(y)]_{\tilde{L}} - s[x, y] \\ \tilde{\theta}(x, y) &= \{s(x), s(y)\}_{\tilde{L}} - s\{x, y\} \\ m_l^1(x, u) &= [s(x), u]_{\tilde{L}}, \quad m_r^1(u, x) = [u, s(x)]_{\tilde{L}} \\ m_l^2(x, u) &= \{s(x), u\}_{\tilde{L}}, \quad m_r^2(u, x) = \{u, s(x)\}_{\tilde{L}}. \end{aligned}$$

Theorem 6.1. With the notations introduced above, $(V, m_l^1, m_r^1, m_l^2, m_r^2)$ is a representation of the compatible Leibniz algebra (L, π_1, π_2) . Further this representation is independent of the choice of sections s .

Proof. For $x, y \in L$ and $u \in V$, since $\theta(x, y) \in V$, we have

$$\begin{aligned} 0 &= [\theta(x, y), u]_{\tilde{L}} \\ &= [[s(x), s(y)]_{\tilde{L}}, u]_{\tilde{L}} - [s[x, y], u]_{\tilde{L}} \\ &= [s(x), [s(y), u]_{\tilde{L}}]_{\tilde{L}} - [s(y), [s(x), u]_{\tilde{L}}]_{\tilde{L}} - [s[x, y], u]_{\tilde{L}} \end{aligned} \quad (13)$$

i.e., we have $m_l^1([x, y], u) = m_l^1(x, m_l^1(y, u)) - m_l^1(y, m_l^1(x, u))$.

Similarly, we can show

$$\begin{aligned} m_r^1(u, [x, y]) &= m_r^1(x, m_r^1(u, y)) - m_r^1(m_r^1(x, u), y) \text{ and} \\ m_r^1(u, [x, y]) &= m_r^1(m_r^1(u, x), y) + m_r^1(x, m_r^1(u, y)). \end{aligned}$$

Thus (V, m_l^1, m_r^1) is a representation of $(L, [\])$.

Similarly we can prove that (V, m_l^2, m_r^2) is a representation of $(L, \{ \})$ and the

compatibility conditions LLM , LML and MLL hold.

Suppose there is another section $s' : L \rightarrow \tilde{L}$ w.r.t which $(V, m_l^1, m_r^1, m_l^2, m_r^2)$ is the representation of (L, π_1, π_2) . Then for any $x \in L$ and $u \in V$

$$\begin{aligned} m_l^1(x, u) - m_l^1(x, u) &= [s(x), u]_{\tilde{L}} - [s'(x), u]_{\tilde{L}} \\ &= [s(x) - s'(x), u]_{\tilde{L}} \\ &= 0, \quad (\text{since } s(x) - s'(x) \in V) \end{aligned}$$

Hence $m_l^1 = m_l^1$.

Similarly, it can be shown that $m_r^1 = m_r^1$, $m_l^2 = m_l^2$, $m_r^2 = m_r^2$.

Theorem 6.2. *Two equivalent abelian extensions, $(L_1, []_1, \{ \}_1)$ and $(L_2, []_2, \{ \}_2)$ of (L, π_1, π_2) by $(V, 0, 0)$ give rise to the same representation of (L, π_1, π_2) as in the previous theorem.*

Proof. Let s_1 and s_2 be sections of $(L_1, []_1, \{ \}_1)$ and $(L_2, []_2, \{ \}_2)$ respectively and $(V, m_l^1, m_r^1, m_l^2, m_r^2)$ and $(V, m_l^1, m_r^1, m_l^2, m_r^2)$ be their corresponding representations given by Theorem 6.1.

If F is the isomorphism from L_1 to L_2 as given in the commutative diagram 14 then define $s'_1 : L \rightarrow L_1$ by $s'_1 = F^{-1} \circ s_2$. Then

$$\rho \circ s'_1 = \rho \circ F^{-1} \circ s_2 = \rho' \circ F \circ F^{-1} \circ s_2 = \rho' \circ s_2 = Id_L.$$

Thus s'_1 is a section of L_1 that gives the representation $(V, m_l^1, m_r^1, m_l^2, m_r^2)$. For all $x \in L, u \in V$ and $F|_V = Id_V$ we have

$$\begin{aligned} m_l^1(x, u) &= [s'_1(x), u]_1 \\ &= [F^{-1} \circ s_2(x), u]_1 \\ &= F^{-1}[s_2(x), u]_2 \\ &= F^{-1}m_l^1(x, u) = m_l^1(x, u). \end{aligned}$$

Similarly, we can show $m_r^1 = m_r^1$, $m_l^2 = m_l^2$, $m_r^2 = m_r^2$.

Theorem 6.3. *With $\theta, \tilde{\theta}, m_l^1, m_r^1, m_l^2, m_r^2$ defined above, $L \oplus V$ with the following bilinear operations is a compatible Leibniz algebra.*

$$\begin{aligned} [x + u, y + v]_{\oplus} &= [x, y] + \theta(x, y) + m_l^1(x, v) + m_r^1(u, y) \\ \{x + u, y + v\}_{\oplus} &= \{x, y\} + \tilde{\theta}(x, y) + m_l^2(x, v) + m_r^2(u, y), \end{aligned}$$

$\forall x, y \in L, u, v \in V$.

Proof. The proof of the theorem is straightforward.

Theorem 6.4. *$(\theta, \tilde{\theta})$ is a 2-cocycle of (L, π_1, π_2) with coefficients in the representation $(V, m_l^1, m_r^1, m_l^2, m_r^2)$.*

Proof. For any $x, y, z \in L$

$$\begin{aligned}
& d_{\pi_1+m_1^1+m_1^1}^2 \theta(x, y, z) \\
&= -[\hat{\pi}_1 + \hat{m}_1^1 + \hat{m}_1^1, \hat{\theta}]_B \\
&= m_r(\theta(x, y, z) - m_l(x, \theta(y, z)) + m_l(y, \theta(x, z)) + \theta([x, y], z) - \theta(x, [y, z]) + \theta(y, [x, z]) \\
&= [\theta(x, y), s(z)] - [s(x), \theta(y, z)] + [s(y), \theta(x, z)] + [s[x, y], s(z)] - \\
&\quad s[[x, y], z] - [s(x), s[y, z]] + s[x, [y, z]] + [s(y), s[x, z]] - s[y, [x, z]] \\
&= [[s(x), s(y)], s(z)] - \cancel{[s[x, y], s(z)]} - [s(x), [s(y), s(z)]] + \cancel{[s(x), s[y, z]]} + [s(y), [s(x), s(z)]] - \\
&\quad \cancel{[s(y), s[x, z]]} + \cancel{[s[x, y], s(z)]} - s[[x, y], z] - \cancel{[s(x), s[y, z]]} + s[x, [y, z]] + \cancel{[s(y), s[x, z]]} - s[y, [x, z]] \\
&= [[s(x), s(y)], s(z)] - [s(x), [s(y), s(z)]] + [s(y), [s(x), s(z)]] - s[[x, y], z] + s[x, [y, z]] - s[y, [x, z]] \\
&= 0.
\end{aligned}$$

Similarly we can show $d_{\pi_2+m_1^2+m_2^2}^2 \tilde{\theta}(x, y, z) = 0$.

$$\begin{aligned}
& d_{\pi_2+m_1^2+m_2^2}^2 \theta(x, y, z) + d_{\pi_1+m_1^1+m_1^1}^2 \tilde{\theta}(x, y, z) \\
&= \{\theta(x, y), s(z)\} - \{s(x), \theta(y, z)\} + \{s(y), \theta(x, z)\} + \{s[x, y], s(z)\} - s\{\{x, y\}, z\} - [s(x), s\{y, z\}] + \\
&\quad s\{x, \{y, z\}\} + [s(y), s\{x, z\}] - s\{y, \{x, z\}\} + \\
&\quad [\tilde{\theta}(x, y), s(z)] - [s(x), \tilde{\theta}(y, z)] + [s(y), \tilde{\theta}(x, z)] + \{s[x, y], s(z)\} - s\{\{x, y\}, z\} - \{s(x), s\{y, z\}\} + \\
&\quad s\{x, \{y, z\}\} + \{s(y), s\{x, z\}\} - s\{y, \{x, z\}\} \\
&= \{[s(x), s(y)], s(z)\} - \cancel{\{s[x, y], s(z)\}} - \{s(x), [s(y), s(z)]\} + \cancel{\{s(x), s[y, z]\}} + \{s(y), [s(x), s(z)]\} \\
&\quad - \cancel{\{s(y), s[x, z]\}} + \cancel{\{s[x, y], s(z)\}} - s\{\{x, y\}, z\} - \cancel{[s(x), s\{y, z\}]} + s\{x, \{y, z\}\} + \cancel{[s(y), s\{x, z\}]} \\
&\quad - s\{y, \{x, z\}\} + \{s(x), s(y)\}, s(z) - \cancel{[s\{x, y\}, s(z)]} - [s(x), \{s(y), s(z)\}] + \cancel{[s(x), s\{y, z\}]} \\
&\quad + [s(y), \{s(x), s(z)\}] - \cancel{[s(y), s\{x, z\}]} + \cancel{[s[x, y], s(z)]} - s\{\{x, y\}, z\} - \cancel{\{s(x), s\{y, z\}\}} \\
&\quad + s\{x, \{y, z\}\} + \cancel{\{s(y), s\{x, z\}\}} - s\{y, \{x, z\}\} \\
&= \{[s(x), s(y)], s(z)\} - \{s(x), [s(y), s(z)]\} + \{s(y), [s(x), s(z)]\} - s\{\{x, y\}, z\} + s\{x, \{y, z\}\} \\
&\quad - s\{y, \{x, z\}\} + \{s(x), s(y)\}, s(z) - [s(x), \{s(y), s(z)\}] + [s(y), \{s(x), s(z)\}] \\
&\quad - s\{\{x, y\}, z\} + s\{x, \{y, z\}\} - s\{y, \{x, z\}\} \\
&= 0, \quad \text{by the compatibility condition (1)}
\end{aligned}$$

Thus

$$\partial^2(\theta, \tilde{\theta}) = (d_{\pi_1+m_1^1+m_1^1}^2 \theta, d_{\pi_2+m_1^2+m_2^2}^2 \theta + d_{\pi_1+m_1^1+m_1^1}^2 \tilde{\theta}, d_{\pi_2+m_1^2+m_2^2}^2 \tilde{\theta}) = 0.$$

Lemma 6.1. *The cohomology class of the cocycle $(\theta, \tilde{\theta})$ does not depend on the choice of sections.*

Proof. Let $(\theta, \tilde{\theta})$ and $(\theta', \tilde{\theta}')$ be the cocycles corresponding to sections s and s' . Define $\psi : L \rightarrow V$ as $\psi(x) = s(x) - s'(x)$

Then

$$\begin{aligned}
\theta(x, y) &= [s(x), s(y)]_{\bar{L}} - s[x, y] \\
&= [\psi(x) + s'(x), \psi(y) + s'(y)]_{\bar{L}} - s'[x, y] - \psi[x, y] \\
&= [s'(x), \psi(y)]_{\bar{L}} + [\psi(x), s'(y)]_{\bar{L}} + [s'(x), s'(y)]_{\bar{L}} - s'[x, y] - \psi[x, y] \\
&= m_l^1(x, \psi(y)) + m_r^1(y, \psi(x)) + \tilde{\theta}'(x, y) - \psi[x, y] \\
&= \partial_{\pi_1+m_1^1+m_1^1} \psi(x, y) + \tilde{\theta}'(x, y)
\end{aligned}$$

Hence we have $\theta - \theta' = \partial_{\pi_1+m_1^1+m_2^2} \psi$. Similarly it can be shown $\tilde{\theta} - \tilde{\theta}' = \partial_{\pi_2+m_1^2+m_2^2} \psi$.

Thus $(\theta - \theta', \tilde{\theta} - \tilde{\theta}') = \partial\psi$ i.e. $(\theta, \tilde{\theta})$ and $(\theta', \tilde{\theta}')$ are in the same cohomology class.

Theorem 6.5. *Equivalent abelian extensions give rise to the same cocycle $(\theta_1, \tilde{\theta}_1)$.*

Proof. Let L_1 and L_2 be equivalent abelian extensions of the compatible Leibniz algebra (L, π_1, π_2) by $(V, 0, 0)$ as detailed by the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & V & \xrightarrow{i_1} & L_1 & \xrightarrow{\rho_1} & L \longrightarrow 0 \\ & & \downarrow id_V & & \downarrow \phi & & \downarrow id_L \\ 0 & \longrightarrow & V & \xrightarrow{i_2} & L_2 & \xrightarrow{\rho_2} & L \longrightarrow 0. \end{array} \quad (14)$$

and sections $s_1 : L \rightarrow L_1$ and $s_2 : L \rightarrow L_2$ and cocycles $(\theta_1, \tilde{\theta}_1)$ and $(\theta_2, \tilde{\theta}_2)$ respectively.

We note that

$$\rho_2(\phi \circ s_1) = (\rho_2 \circ \phi)s_1 = \rho_1 \circ s_1 = Id_L.$$

Hence $\phi \circ s_1$ is a section of L_2 .

From Theorem 6.1 we know that the cohomological class of the cocycle $(\theta, \tilde{\theta})$ does not depend on the choice of sections. Hence we can take $s_2 = \phi \circ s_1$.

Thus $\forall x, y \in L$

$$\begin{aligned} \theta_2(x, y) &= [s_2(x), s_2(y)]_2 - s_2[x, y] \\ &= [\phi \circ s_1(x), \phi \circ s_1(y)]_2 - \phi \circ s_1[x, y] \\ &= \phi([s_1(x), s_1(y)]_1 - s_1[x, y]) \\ &= [s_1(x), s_1(y)]_1 - s_1[x, y] \\ &= \theta_1(x, y). \end{aligned}$$

Likewise we can show $\tilde{\theta}_1 = \tilde{\theta}_2$.

Theorem 6.6. *Cohomologous cocycles $(\theta_1, \tilde{\theta}_1)$ and $(\theta_2, \tilde{\theta}_2)$ give rise to equivalent abelian extensions.*

Proof. Since $(\theta_1, \tilde{\theta}_1)$ and $(\theta_2, \tilde{\theta}_2)$ are cohomologous there exists $\phi : L \rightarrow V$ such that

$$\begin{aligned} \theta_1 &= \theta_2 + \partial_{\pi_1 + m_l^1 + m_r^1} \phi \\ \tilde{\theta}_1 &= \tilde{\theta}_2 + \partial_{\pi_2 + m_l^2 + m_r^2} \phi. \end{aligned}$$

Suppose $(L \oplus_1 V, []_1, \{ \}_1)$ and $(L \oplus_2 V, []_2, \{ \}_2)$ are abelian extensions of $(L, [], \{ \})$ by $(V, 0, 0)$ with respect to (θ_1, θ_1) and $(\theta_2, \tilde{\theta}_2)$ respectively.

Define $\psi : L \oplus_1 V \rightarrow L \oplus_2 V$ by $\psi(x + u) = x + u + \phi(x)$, $\forall x \in L, u \in V$.
For $x, y \in L, u, v \in V$,

$$\begin{aligned}
& \psi([x + u, y + v]_1) - [\psi(x + u), \psi(y + v)]_2 \\
= & \psi([x, y] + \theta_1(x, y) + m_l^1(x, v) + m_r^1(u, y)) - [x + u + \phi(x), y + v + \phi(y)]_2 \\
= & [x, y] + \theta_1(x, y) + m_l^1(x, v) + m_r^1(u, y) + \phi[x, y] - \{[x, y] + \theta_2(x, y) + \\
& m_l^1(x, v) + m_l^1(x, \phi(y)) + m_r^1(u, y) + m_r^1(\phi(x), y)\} \\
= & \theta_1(x, y) + \phi[x, y] - \theta_2(x, y) - m_l^1(x, \phi(y)) - m_r^1(\phi(x), y) \\
= & \theta_1(x, y) - \theta_2(x, y) - \partial_{\pi_1 + m_l^1 + m_r^1} \phi(x, u) \\
= & 0.
\end{aligned}$$

Similarly, we can show $\psi\{x + u, y + v\}_1 = \{\psi(x + u), \psi(y + v)\}_2$.
It is routine to verify that the following diagram commutes

$$\begin{array}{ccccccc}
0 & \longrightarrow & V & \xrightarrow{i_1} & L \oplus_1 V & \xrightarrow{\rho_1} & L \longrightarrow 0 \\
& & \downarrow Id_v & & \downarrow \phi & & \downarrow Id_L \\
0 & \longrightarrow & V & \xrightarrow{i_2} & L \oplus_2 V & \xrightarrow{\rho_2} & L \longrightarrow 0.
\end{array}$$

Remark 6.1. From the last two theorems we conclude that the abelian extensions of compatible Leibniz algebra (L, π_1, π_2) by $(V, 0, 0)$ are characterised by the 2nd cohomology group $H^2(L, V)$.

7 Declarations

Funding and/or Conflicts of interests/Competing interests.

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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