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СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

ISSN 1813-3304 Институт математики им. С.Л. Соболева СО РАН

Sobolev Institute of Mathematics SB RAS

Том 21, № 1, стр. 144–145 (2024) УДК 519.651

https://doi.org/10.33048/semi.2023.20.??? MSC 65D15

APPROXIMATION OF A FUNCTION BY POLYNOMIALS IN THE PRESENCE OF A REGION OF LARGE GRADIENTS

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COMMUNICATED

Abstract: The issue of approximating functions of one and two variables by polynomials in the presence of a region of large gradients is considered. The problem is that when applying Taylor's formula, the residual term can be significant if the function has large gradients. It is assumed that the function has a decomposition in the form of a sum of regular and boundary layer components. The boundary layer components are known up to a factor. This decomposition is valid for the solution of a singularly perturbed boundary value problem. The derivatives of the regular component are bounded to a certain order, and the boundary layer components have large gradients. Formulas for approximating a function by polynomials of an arbitrarily specified degree are constructed based on the fact that these formulas are exact for the boundary layer components. This approach has not been previously explored. Error estimates that are uniform in the boundary layer components are obtained.

Zadorin, A.I., Approximation of a function by polynomials in the presence of a region of large gradients.

This work was carried out as part of a state assignment for the Institute of Mathematics, Siberian Branch, Russian Academy of Sciences (project no. FWNF-2022-0016).

Received January 17, 2024, published December 31, 2024.

Keywords: function of one or two variables, large gradients, boundary layer components, approximation by polynomials, error estimation.

1 Introduction

The method of approximating a function by polynomials based on Taylor series expansion is well known [1]. However, if the function has large gradients, then the residual term of such an approximation can be significant.

In particular, the solution of the singularly perturbed problem has large gradients in the boundary layer region [2]. For the solution of a singularly perturbed problem a decomposition is valid in the form of a sum of regular and boundary layer components. In the case of a singularly perturbed second order ordinary differential equation, such a decomposition was constructed in [3]. In the case of a singularly perturbed problem for elliptic equation such the decomposition was constructed in [4]. In these works, the boundary layer components are known up to a factor.

It is of interest how to apply the Taylor series, so that the estimate of the error in approximating a function by polynomials depends only on the regular component and its derivatives.

This issue was considered in [5], where formulas were constructed for approximating functions with large gradients by polynomials on the basis that these formulas were exact for the boundary layer components. In the case of a function of one variable, approximation is carried out by polynomials of an arbitrarily specified degree. In the case of a function of two variables, formulas were constructed for approximation by polynomials of zero and first degree. Error estimates that are uniform in the boundary layer components and their derivatives are obtained.

This work develops the results of the work of [5]. The formula from [5] has been modified in the case of a function of one variable, while fewer restrictions are imposed on the boundary layer component. In the case of a function of two variables with boundary layer components, an approximation formula by polynomials of an arbitrarily specified degree has been developed.

Notations. By C and C_j we mean positive constants that are independent of h and of the functions p, Φ, Θ and their derivatives. We will limit the various quantities to one constant C_j, if this is clear from the text.

2 Approximation of a function of one variable

Let $u(x)$ be a sufficiently smooth function on the interval $[0, 1]$. The formula for approximating a function by polynomials based on the Taylor series expansion is known [1]:

$$u(x) = \sum_{j=0}^{k} \frac{u^{(j)}(x_0)}{j!}(x - x_0)^j + R_k(u, x), \tag{1}$$

where $x_0, x \in [0,1],$ k is degree of polynomial, $R_k(u,x)$ is remainder term, representable as

$$R_k(u,x) = \frac{1}{k!}\int\limits_{x_0}^{x} u^{(k+1)}(t)(x-t)^k\,dt. \tag{2}$$

Let's set $h = |x - x_0|$. We will assume that $x > x_0$. In the case of $x < x_0$, the interval $[x_0, x]$ is replaced by the interval $[x, x_0]$.

From (2) the estimate follows:

$$|R_k(u,x)| \le \frac{h^{k+1}}{(k+1)!}\max_{s\in[x_0,x]}\left|u^{(k+1)}(s)\right|. \tag{3}$$

According to (3), for some constant C $|R_k(u,x)| \le Ch^{k+1}$ if the derivative $u^{(k+1)}(x)$ is bounded. However, there may be an error significant if this derivative is not bounded.

For example, consider the formula (1) for $k = 1$:

$$u(x) \approx u(x_0) + (x - x_0)u'(x_0). \tag{4}$$

Let $u(x) = e^{-x/\varepsilon}$, where $\varepsilon \in (0,1], x \in [0,1]$. This function has large gradients for small values of the parameter ε and corresponds to the presence of an exponential boundary layer [2].

In the case $x_0 = 0$, $x = \varepsilon$ we have

$$R_1(u,x) = u(\varepsilon) - u(0) - \varepsilon u'(0) = e^{-1}.$$

Thus, in this case, the error of the formula (4) does not decrease with decreasing h, if $\varepsilon = h$. So, the question of approximating functions with large gradients by polynomials is of interest.

Let the decomposition be valid for a sufficiently smooth function $u(x)$:

$$u(x) = p(x) + \gamma\Phi(x),\ x \in [0,1], \tag{5}$$

where $p(x)$ is a regular component with bounded derivatives up to a certain order, $\Phi(x)$ is a boundary layer component, which is a function of general form and is responsible for large gradients of the function $u(x)$. Function $\Phi(x)$ is assumed to be known, $p(x)$ and γ are not specified.

Let us note the cases of functions representable in the form (5).

Let $u(x)$ be the solution of a singularly perturbed boundary value problem:

$$\varepsilon u''(x) + a_1(x)u'(x) - a_2(x)u(x) = f(x),\quad u(0) = A,\ u(1) = B,$$

where $a_1(x) \ge \beta > 0, a_2(x) \ge 0,$ $\varepsilon \in (0,1]$, functions a_1, a_2, f are smooth enough. According to [3], decomposition (5) is valid when given

$$\Phi(x) = e^{-\alpha x/\varepsilon},\ \alpha = a_1(0) > 0. \tag{6}$$

In the presence of a power-law boundary layer [6] we have

$$\Phi(x) = (x+\varepsilon)^{\alpha}, 0 < \alpha < 1,\ x \in [0,1], \varepsilon \in (0,1].$$

In the presence of a logarithmic singularity

$$\Phi(x) = \ln x, x \in [\varepsilon, 1], 1 \gg \varepsilon > 0.$$

Decomposition of a function with such a feature was used in [7], where a method for solving an elliptic equation in a region with a small hole of radius ε was studied.

In [8] decomposition (5) is used to construct an interpolation formula that is exact on the component $\Phi(x)$. The formula contains an arbitrarily specified number of interpolation nodes, in accordance with [8] the error of this formula is uniform over the component $\Phi(x)$.

If $\Phi(x)$ corresponds to (6), then the estimate holds $|\Phi^{(n)}(x)| \le C/\varepsilon^n$, therefore, in accordance with (3) the error $|R_k(u,x)|$ can be significant for small values of ε.

To approximate the function $u(x)$ of the form (5), we correct Taylor's formula (1) so that the formula becomes exact on the component $\Phi(x)$.

In [5], for approximating the function $u(x)$ of the form (5) by polynomials, a formula is constructed:

$$u(x) \approx \tilde{G}_k(u,x) = \sum_{j=0}^{k} \frac{u^{(j)}(x_0)}{j!}(x-x_0)^j +$$

$$+\Big[\Phi(x) - \sum_{j=0}^{k} \frac{\Phi^{(j)}(x_0)}{j!}(x-x_0)^j\Big] \frac{u^{(k+1)}(x_0)}{\Phi^{(k+1)}(x_0)}, \ \Phi^{(k+1)}(x_0) \neq 0. \qquad (7)$$

It has been proven that if for some constant C_1

$$\max_{s\in[x_0,x]} \Big|\Phi^{(k+1)}(s)\Big| \le C_1 \Big|\Phi^{(k+1)}(x_0)\Big|, \qquad (8)$$

then for some constant C_2 the following error estimate is valid

$$\Big|u(x) - \tilde{G}_k(u,x)\Big| \le C_2 h^{k+1} \max_{s\in[x_0,x]} \Big|p^{(k+1)}(s)\Big|, \ x_0, x \in [0,1].$$

The error estimate is uniform over the boundary layer component and its derivatives. However, the condition (8) is satisfied in the case of $\Phi(x) = e^{-\alpha x/\varepsilon}$ and is not satisfied, for example, in the case of $\Phi(x) = e^{\alpha(x-1)/\varepsilon}$, where $\alpha > 0$, $\varepsilon \in (0,1]$, $x \in [0,1], x_0 < x$. The fulfillment of the condition (8) depends on the type of the function $\Phi(x)$.

For this reason, we modify the formula (7).

For given k we define $a \in [0,1]$ such that

$$|\Phi^{(k+1)}(a)| = \max_{s\in[0,1]} |\Phi^{(k+1)}(s)|. \qquad (9)$$

Then we define the approximation:

$$u(x) \approx G_k(u,x) = \sum_{j=0}^{k} \frac{u^{(j)}(x_0)}{j!}(x-x_0)^j +$$

$$+\Big[\Phi(x)-\sum_{j=0}^{k}\frac{\Phi^{(j)}(x_0)}{j!}(x-x_0)^j\Big]\frac{u^{(k+1)}(a)}{\Phi^{(k+1)}(a)}. \tag{10}$$

Lemma 1. *Let the function $u(x)$ has decomposition (5). Then the following error estimate holds*

$$|u(x)-G_k(u,x)|\le\frac{2h^{k+1}}{(k+1)!}\max_{s\in[0,1]}|p^{(k+1)}(s)|, \tag{11}$$

where $x_0, x\in[0,1]$, $G_k(u,x)$ corresponds to (10).

Proof. From (10) it follows that $G_k(\gamma\Phi,x)=\gamma\Phi(x)$. Taking into account (5), we get

$$u(x)-G_k(u,x)=p(x)-G_k(p,x). \tag{12}$$

Taking into account (12), from (10) we obtain

$$u(x)-G_k(u,x)=p(x)-\sum_{j=0}^{k}\frac{p^{(j)}(x_0)}{j!}(x-x_0)^j-$$

$$\Big[\Phi(x)-\sum_{j=0}^{k}\frac{\Phi^{(j)}(x_0)}{j!}(x-x_0)^j\Big]\frac{p^{(k+1)}(a)}{\Phi^{(k+1)}(a)}. \tag{13}$$

According to (1), the relation (13) can be written as:

$$u(x)-G_k(u,x)=R_k(p,x)-\frac{R_k(\Phi,x)}{\Phi^{(k+1)}(a)}p^{(k+1)}(a). \tag{14}$$

From (3), (14) it follows

$$|u(x)-G_k(u,x)|\le\frac{h^{k+1}}{(k+1)!}\Big[1+\frac{\max_s|\Phi^{(k+1)}(s)|}{|\Phi^{(k+1)}(a)|}\Big]\max_s|p^{(k+1)}(s)|, \tag{15}$$

where $s\in[0,1]$. From the estimates (15) and (9) we get the estimate (11). The lemma is proven.

In contrast to the estimate (3) for formula (1), the error estimate (11) for formula (10) is uniform in the boundary layer component $\Phi(x)$ and its derivatives.

3 Approximation of a function of two variables

Let us consider the issue of approximating a function of two variables with large gradients by polynomials.

Let $u(x,y)$ be a sufficiently smooth function, where $(x,y)\in[0,1]^2$. The formula for expanding the function $u(x,y)$ into a Taylor series near the point (x_0,y_0) in accordance with [1] has the form:

$$u(x,y)=G_k(u,x,y)+R_k(u,x,y), \tag{16}$$

where

$$G_k(u,x,y)=\sum_{j=0}^{k}\frac{1}{j!}\Big((x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\Big)^j u(x_0,y_0), \tag{17}$$

the remainder term has the form:

$$R_k(u,x,y)=\frac{1}{(k+1)!}\Big((x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\Big)^{k+1} u(s_1,s_2) \tag{18}$$

for some $s_1\in[x_0,x], s_2\in[y_0,y]$.

Here

$$\Big((x-x_0)\frac{\partial}{\partial x}\Big)^{k+1}=(x-x_0)^{k+1}\frac{\partial^{k+1}}{\partial x^{k+1}}.$$

We assume that $x>x_0, y>y_0$, other cases are considered similarly with replacing the interval. For example, for $x<x_0$ the interval $[x_0,x]$ is replaced by the interval $[x,x_0]$. Let $h=\sqrt{(x-x_0)^2+(y-y_0)^2}$.

Let's estimate $R_k(u,x,y)$. We write (18) in terms of Newton's binomial:

$$R_k(u,x,y)=\frac{1}{(k+1)!}\sum_{j=0}^{k+1}C_{k+1}^j\Big((x-x_0)\frac{\partial}{\partial x}\Big)^j\Big((y-y_0)\frac{\partial}{\partial y}\Big)^{k+1-j}u(s_1,s_2),$$

$$C_{k+1}^j=\frac{(k+1)!}{(k+1-j)!\,j!}.$$

Hence,

$$R_k(u,x,y)=$$

$$=\frac{1}{(k+1)!}\sum_{j=0}^{k+1}C_{k+1}^j(x-x_0)^j(y-y_0)^{k+1-j}\frac{\partial^{k+1}}{\partial x^j\partial y^{k+1-j}}u(s_1,s_2). \tag{19}$$

Considering the known relation

$$\sum_{j=0}^{k+1}C_{k+1}^j=2^{k+1},$$

from (19) we get

$$|R_k(u,x,y)|\le\frac{2^{k+1}}{(k+1)!}\max_{j,s_1,s_2}\Big|\frac{\partial^{k+1}}{\partial x^j\partial y^{k+1-j}}u(s_1,s_2)\Big|h^{k+1}, \tag{20}$$

where

$$j\in\{0,1,\dots,k+1\}, s_1\in[0,1], s_2\in[0,1].$$

If the partial derivatives of the function $u(x,y)$ up to order $(k+1)$ are bounded, then from (20) it follows that for some constant C

$$|R_k(u,x,y)|\le Ch^{k+1}.$$

However, if the function $u(x,y)$ has a region of large gradients, for example, in the presence of exponential boundary layers [2], then, as in the one-dimensional case, the error $|R_k(u,x,y)|$ can be significant.

In [5] is investigated the issue of approximation by polynomials of a function of two variables with large gradients due to the presence of a boundary layer component on each variable. Such a function corresponds to the solution of the singularly perturbed elliptic problem [4]. To approximate such a function by polynomials, the formula (17) was modified in the cases of $k = 0$ and $k = 1$. The formulas in [5] were constructed so that they were exact for the boundary layer components. For the constructed formulas, error estimates obtained of the order of $O(h)$ for $k = 0$ and of $O(h^2)$ for $k = 1$. Restrictions are imposed under which the obtained error estimates are uniform in the boundary layer components and their derivatives.

In this work, we modify the formula (17) based on fit to the boundary layer components in the case of an arbitrary value of k and estimate its error.

So, let the following decomposition be valid for a sufficiently smooth function $u(x, y)$:

$$u(x, y) = p(x, y) + \gamma_1\Phi(x) + \gamma_2\Theta(y), (x, y) \in [0, 1]^2. \tag{21}$$

We assume that the regular component $p(x, y)$ is not specified and has derivatives bounded to a certain order, coefficients γ_1, γ_2 are not specified, $\Phi(x)$ and $\Theta(y)$ are known boundary layer components with large gradients.

Decomposition (21) is possessed, in particular, by the solution of a singularly perturbed problem for an elliptic equation:

$$\varepsilon u_{xx} + \varepsilon u_{yy} + a_1(x)u_x + a_2(y)u_y - c(x, y)u = f(x, y),\ (x, y) \in \Omega = (0, 1)^2,$$
$$u(x, y) = g(x, y),\ (x, y) \in \Gamma, \tag{22}$$

where $\Gamma = \overline{\Omega}\setminus\Omega$, functions a_1, a_2, c, f, g are smooth enough,

$$a_1(x) \geq \beta_1 > 0,\quad a_2(y) \geq \beta_2 > 0,\quad c(x, y) \geq 0,\ \varepsilon \in (0, 1].$$

It is known that the solution of the problem (22) for small values of the parameter ε has large gradients at the boundaries $x = 0, y = 0$ [2].

In accordance with [4], the decomposition (21) can be valid for the solution of the problem (22), given

$$\Phi(x) = e^{-\alpha x/\varepsilon},\ \Theta(y) = e^{-\beta y/\varepsilon},$$

where $\alpha = a_1(0), \beta = a_2(0)$, coefficients γ_1, γ_2 and the function $p(x, y)$ are not specified explicitly.

Let's modify the formula (17) so that the formula becomes exact for the boundary layer components.

Let us set a formula that is exact for the components $\gamma_1\Phi(x), \gamma_2\Theta(y)$:

$$u(x, y) \approx D_k(u, x, y) =$$
$$= G_k(u, x, y) + \frac{1}{\Phi^{(k+1)}(x_0)}\Big[\Phi(x) - \sum_{j=0}^{k}\frac{\Phi^{(j)}(x_0)}{j!}(x - x_0)^j\Big]\frac{\partial^{k+1}}{\partial x^{k+1}}u(x_0, y_0)+$$
$$+\frac{1}{\Theta^{(k+1)}(y_0)}\Big[\Theta(y) - \sum_{j=0}^{k}\frac{\Theta^{(j)}(y_0)}{j!}(y - y_0)^j\Big]\frac{\partial^{k+1}}{\partial y^{k+1}}u(x_0, y_0), \tag{23}$$

where $G_k(u,x,y)$ corresponds to the classical formula (17).

From (23), (17) it follows

$$D_k(\gamma_1\Phi,x,y)=\gamma_1\Phi(x),\ \ D_k(\gamma_2\Theta,x,y)=\gamma_2\Theta(y),$$

therefore, in accordance with the decomposition (21) we have

$$u(x,y)-D_k(u,x,y)=p(x,y)-D_k(p,x,y). \tag{24}$$

Given (3), (24), from (23) we have

$$|u(x,y)-D_k(u,x,y)|=|p(x,y)-D_k(p,x,y)|\le|p(x,y)-G_k(p,x,y)|+$$
$$+\frac{h^{k+1}}{(k+1)!}\Big[\max_{s\in[x_0,x]}|\Phi^{(k+1)}(s)|\Big/|\Phi^{(k+1)}(x_0)|\times\Big|\frac{\partial^{k+1}}{\partial x^{k+1}}p(x_0,y_0)\Big|+$$
$$+\max_{s\in[y_0,y]}|\Theta^{(k+1)}(s)|\Big/|\Theta^{(k+1)}(y_0)|\times\Big|\frac{\partial^{k+1}}{\partial y^{k+1}}p(x_0,y_0)\Big|\Big]. \tag{25}$$

Applying (25) and estimating $|p(x,y)-G_k(p,x,y)|$ based on (16), (20) with $u(x,y)$ replaced by $p(x,y)$, we are convinced of the validity of the following lemma.

Lemma 2. *Let for some constant* C_1

$$\max_{s\in[x_0,x]}|\Phi^{(k+1)}(s)|\Big/|\Phi^{(k+1)}(x_0)|\le C_1,$$
$$\max_{s\in[y_0,y]}|\Theta^{(k+1)}(s)|\Big/|\Theta^{(k+1)}(y_0)|\le C_1. \tag{26}$$

Then for all $(x_0,y_0)\in[0,1]^2,(x,y)\in[0,1]^2$ *the following error estimate is valid:*

$$|u(x,y)-D_k(u,x,y)|\le\frac{2^{k+1}+2C_1}{(k+1)!}\max_{j,s_1,s_2}\Big|\frac{\partial^{k+1}}{\partial x^j\partial y^{k+1-j}}p(s_1,s_2)\Big|h^{k+1}, \tag{27}$$

where $D_k(u,x,y)$ *is given according to (23),*

$$j\in\{0,1,\dots,k+1\},s_1\in[0,1],s_2\in[0,1].$$

Let's compare the classical formula (17) and the constructed formula (23). According to assessment (20) the error of the formula (17) can be significant if the function $u(x,y)$ has a decomposition (21). In accordance with the obtained estimate (27), the error of the formula (23) is uniform in the boundary layer components and their derivatives.

3.1. Modified formula. Conditions (26) may be not fulfilled for some functions. For example, these conditions are satisfied in the case $\Phi(x)=e^{-x/\varepsilon}$ and are not satisfied in the case $\Phi(x)=e^{(x-1)/\varepsilon}$. For this reason, we modify the formula (23).

Let us set $a,b\in[0,1]$ from the following conditions:

$$|\Phi^{(k+1)}(a)|=\max_{s\in[0,1]}|\Phi^{(k+1)}(s)|,|\Theta^{(k+1)}(b)|=\max_{s\in[0,1]}|\Theta^{(k+1)}(s)|. \tag{28}$$

Now, by analogy with (23), let's set the formula:

$$u(x,y) \approx \tilde{D}_k(u,x,y) =$$

$$= G_k(u,x,y) + \frac{1}{\Phi^{(k+1)}(a)}\Big[\Phi(x) - \sum_{j=0}^{k} \frac{\Phi^{(j)}(x_0)}{j!}(x-x_0)^j\Big]\frac{\partial^{k+1}}{\partial x^{k+1}}u(a,b)+$$

$$+\frac{1}{\Theta^{(k+1)}(b)}\Big[\Theta(y) - \sum_{j=0}^{k} \frac{\Theta^{(j)}(x_0)}{j!}(y-y_0)^j\Big]\frac{\partial^{k+1}}{\partial y^{k+1}}u(a,b), \qquad (29)$$

where $G_k(u,x,y)$ corresponds to the classical formula (17).

From (29), (17) it follows

$$\tilde{D}_k(\gamma_1\Phi,x,y) = \gamma_1\Phi(x),\ \ \tilde{D}_k(\gamma_2\Theta,x,y) = \gamma_2\Theta(y),$$

therefore, in accordance with the decomposition (21) we have

$$u(x,y) - \tilde{D}_k(u,x,y) = p(x,y) - \tilde{D}_k(p,x,y). \qquad (30)$$

Given (3), (30), from (29) we have

$$|u(x,y) - \tilde{D}_k(u,x,y)| = |p(x,y) - \tilde{D}_k(p,x,y)| \le |p(x,y) - G_k(p,x,y)|+$$

$$+\frac{h^{k+1}}{(k+1)!}\Big[\max_{s\in[x_0,x]}|\Phi^{(k+1)}(s)|\Big/|\Phi^{(k+1)}(a)| \times \Big|\frac{\partial^{k+1}}{\partial x^{k+1}}p(a,b)\Big|+$$

$$+\max_{s\in[y_0,y]}|\Theta^{(k+1)}(s)|\Big/|\Theta^{(k+1)}(b)| \times \Big|\frac{\partial^{k+1}}{\partial y^{k+1}}p(a,b)\Big|\Big]. \qquad (31)$$

Applying (31), (28) and estimating $|p(x,y) - G_k(p,x,y)|$ based on (16), (20), we are convinced of the validity of the following lemma.

Lemma 3. *For all $(x_0,y_0) \in [0,1]^2, (x,y) \in [0,1]^2$ the following error estimate is valid:*

$$|u(x,y) - \tilde{D}_k(u,x,y)| \le \frac{2^{k+1}+2}{(k+1)!}\max_{j,s_1,s_2}\Big|\frac{\partial^{k+1}}{\partial x^j \partial y^{k+1-j}}p(s_1,s_2)\Big|h^{k+1}, \qquad (32)$$

where $\tilde{D}_k(u,x,y)$ is given according to (29),

$$j \in \{0,1,\dots,k+1\}, s_1 \in [0,1], s_2 \in [0,1].$$

Remark. The condition (28), when specifying (a,b) can be weakened and replaced with the following condition

$$\max_{s\in[0,1]}|\Phi^{(k+1)}(s)| \le C_1|\Phi^{(k+1)}(a)|,\ \max_{s\in[0,1]}|\Theta^{(k+1)}(s)| \le C_1|\Theta^{(k+1)}(b)|. \qquad (33)$$

From (33) we have condition (28) if $C_1 = 1$. In accordance with (31), the estimate (32) remains valid taking into account the replacement of $(2^{k+1}+2)$ by $(2^{k+1}+2C_1)$.

Similarly, in Lemma 1, when specifying a, the constraint (9) can be replaced by a condition

$$\max_{s\in[0,1]}|\Phi^{(k+1)}(s)|\Big/|\Phi^{(k+1)}(a)| \le C_1.$$

4 Conclusion

The question of approximation by polynomials of functions with large gradients is studied. The problem is that the remainder term of Taylor's formula can be significant if the function has large gradients. Functions of one and two variables are considered, having a decomposition in the form of a sum of regular and boundary layer components. The boundary layer components are known up to a factor and responsible for large gradients of the function. The solution of a singularly perturbed boundary value problem has such a decomposition. Formulas for approximating a function by polynomials of arbitrarily specified degree are constructed. The formulas are constructed in such a way that they are exact for the boundary layer components. For the constructed formulas, error estimates are obtained that are uniform in the boundary layer components and their derivatives.

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