Multiple weak solutions for Leray-Lions type operators including singular terms

Zahirulhaq Musbah[1] and Abdolrahman Razani[2*]

Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Postal Code 3414896818, Qazvin, Iran.
E-mails: [1]Z.musbah@edu.ikiu.ac.ir and [2]razani@sci.ikiu.ac.ir

Abstract

In this paper, we are concerned with the Leray-Lions type operator along with the singular terms in variable exponent spaces. The existence of multiple solutions is established using variational methods.

Keywords: Leray-Lions type operator, variable exponent Sobolev space, singular terms.
2020 Mathematics Subject Classification: 35J60; 35J20; 35J60; 46E35

1 Introduction

The Leray-Lions type operators are elliptic operators which are appropriate for the study of the variable exponent problems of higher order. In general, they are

$$\begin{cases} -div\big(a(x,\nabla u)\big) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$, $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and λ is a positive parameter. The operator $div\,(a\,(x,\nabla u))$ generalizes a degenerate $p(x)$-Laplacian $div(w(x)|\nabla u|^{p(x)-2}\nabla u)$, where $p \in C(\overline{\Omega},(1,\infty))$ and w is a measurable positive function on Ω.

In 2013, Boureanu et al. [6] considered the problem

$$\begin{cases} -div\big(a(x,\nabla u)\big) + |u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = \text{constant} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

*Corresponding Author

and studied the existence and multiplicity of solutions to (1.2) in the case of both $(p(\cdot)-1)$-sublinear at infinity and $(p(\cdot)-1)$-superlinear at infinity of nonlinearity. Moreover, they used several three solutions type theorems to obtain at least three distinct solutions to (1.2) for the case of $(p(\cdot)-1)$-sublinear at infinity.

In 2015, Kim et al. [17] considered problem (1.1) with $a(x,\xi)=\phi(x,|\xi|)\xi)$ which is of type $|\xi|^{p(x)-2}\xi$ (non-degenerate cases) and proved the existence and multiplicity of solutions using the Mountain Pass Theorem and Fountain Theorem.

In 2017, Ky et al. [19] studied the existence and multiplicity of solutions to degenerate $p(x)$-Laplace equations with Leray-Lions type operators (1.1) using direct methods and critical point theories in Calculus of Variations

In 2021, Hai et al. [9] proved the existence of infinitely many solutions for a generalized $p(\cdot)$-Laplace equation involving Leray-Lions operators.

Heidari et al. [12] proved the existence of three solutions in Orlicz-Sobolev spaces for

$$\begin{cases} -M_i(\int_\Omega \Phi_i(\nabla u_i)dx)(div(\alpha_i(\nabla u_i)\nabla u_i) = \lambda F_{u_i}(x,u_1,\cdots,u_n) & in\ \Omega, \\ u_i = 0 & on\ \partial\Omega, \end{cases}$$

for $1\leq i\leq n$, where Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 3$, with smooth boundary $\partial\Omega$ and λ is a positive parameter, $F:\Omega\times\mathbb{R}^n\to\mathbb{R}$ is a measurable function with respect to $x\in\Omega$ for every $(t_1,\cdots,t_n)\in\mathbb{R}^n$ and is C^1 with respect to $(t_1,t_2\cdots,t_n)\in\mathbb{R}^n$ for a.e. $x\in\Omega$.

On the other hand, problems with the singular terms have been intensively studied. For instance, in [20] the existence and multiplicity of solutions for the following problem has been established

$$\begin{cases} -\Delta_{p(x)}u + \frac{u|u|^{s-2}}{|x|^s} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1<s<p(x)<\infty$. The existence of the solutions to the following weighted $(p(x),q(x))$-Laplace problem consisting of a singular term

$$\begin{cases} -a(x)\Delta_{p(x)}u - b(x)\Delta_{q(x)}u + \frac{u|u|^{s-2}}{|x|^s} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

have been proved in [21], where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary, $a,b\in L^\infty(\Omega)$ are positive functions with $a(x)\geq 1$ a.e. on Ω, $\lambda>0$ is a real parameter, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying the following growth condition

$$|f(x,t)|\leq \alpha+\beta|t|^{h(x)-1}$$

for all $(x,t)\in\Omega\times\mathbb{R}$. See also [2, 3, 5, 6, 14, 13, 17] and references therein.

Considering the above results, we shall consider the following degenerate $p(x)$-Laplace equations with Leray-Lions type operators

$$\begin{cases} -div\big(a_i(x,\nabla u_i)\big) + \dfrac{|u_i|^{s_i-2}u_i}{|x|^{s_i}} = \lambda F_{u_i}(x,u_1,\cdots,u_n) & in\ \Omega, \\ u_i = 0 & on\ \partial\Omega, \end{cases} \tag{1.3}$$

for $i = 1,\cdots,n$, where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$, with smooth boundary, $\lambda \in (0,+\infty)$ is a parameter.

For $i = 1,\cdots,n$, we assume that $1 < s_i < N$ and $p_i \in C(\overline{\Omega})$ with

$$N < \inf_{x\in\Omega} p_i(x) \leq p_i(x) \leq \sup_{x\in\Omega} p_i(x) < \infty;$$

For each $i = 1,\cdots,n$, the potential

$$a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$$

is a Carathéodory function satisfying some suitable supplementary conditions:

(A1) a_i is a Carathéodory function such that $a_i(x,0) = 0$, for a.e. $x \in \Omega$.

(A2) There exist $C_i > 0$ such that

$$|a_i(x,t)| \leq C_i(1 + |t|^{p_i(x)-1}),$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

(A3) For all $s,t \in \mathbb{R}$, the inequality

$$(a_i(x,t) - a_i(x,s))(t-s) \geq 0$$

holds, for a.e. $x \in \Omega$.

(A4) There exists $c_i \geq 1$ such that

$$c_i|t|^{p_i(x)} \leq \min\{a_i(x,t)t, p_i(x)A_i(x,t)\},$$

for a.e. $x \in \Omega$ and all $s,t \in \mathbb{R}$, where

$$A_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$$

represents the antiderivative of a_i; that is

$$A_i(x,t) := \int_0^t a_i(x,s)ds.$$

The function $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function with respect to $x \in \Omega$ for each $(t_1,\cdots,t_n) \in \mathbb{R}^n$ and is C^1 with respect to $(t_1,\cdots,t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$.

The main result of the paper is as follows:

Theorem 1.1. *Assume that the conditions* $(A1)-(A4)$ *are held and* $F:\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}$ *satisfies the following conditions*

$(F1)$ $F(x,0,\cdots,0)=0$, *for a.e.* $x\in\Omega$;

$(F2)$ *There exist* $\eta\in L^1(\Omega)$ *and* n *positive constants* γ_i, $1\le i\le n$, *with* $\gamma_i(x)<p_i(x)$ *a.e in* Ω *such that*

$$0\le F(x,u_1,\cdots,u_n)\le \eta(x)(1+\sum_{i=1}^{n}|u_i|^{\gamma_i(x)})$$

$(F3)$ *There exist* $r>0$, $\delta>0$ *and* $1\le i_*\le n$ *such that*

$$\frac{c_{i_*}}{p_{i_*}^+}(\frac{2\delta}{D})^{\check{p_{i_*}}}m(D^N-(\frac{D}{2})^N)>r,$$

where $m:=\frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$ *is the measure of unit ball of* $\mathbb{R}^N$ *and* Γ *is the Gamma function.*

Such that

$$A_r<B_\delta, \tag{1.4}$$

where

$$A_r:=\frac{\|\eta\|_1}{r}\big(1+M\sum_{i=1}^{n}(\frac{p_i^+}{c_i}r)^{\frac{\hat{\gamma_i}}{\check{p_i}}}\big),$$

and

$$B_\delta:=\frac{\inf\limits_{x\in\Omega}F(x,\delta,\cdots,\delta)}{\sum\limits_{i=1}^{n}\Big(\hat{C}(\frac{2\delta}{D})^{\hat{p_i}}+\frac{\kappa}{H}(\frac{2\delta}{D})^{s_i}\Big)(2^N-1)}.$$

Then for each $\lambda\in\Lambda_{r,\delta}:=(\frac{1}{B_\delta},\frac{1}{A_r})$, *problem* (1.3) *possess at least three distinct weak solutions in* X.

Remark 1.1. *The purpose of the main result is to establish multiplicity of weak solutions for the system* (1.3) *by using three critical points theorem due to Bonanno. The novelty of this paper is to consider the nonlinear functions in the right-hand side of each equation which depend on all the functions* $u_1,\cdots,u_n$, *not only on a certain* u_i. *Moreover, we consider the singular terms in* (1.3). *Up to our knowledge, this is the first time that* (1.3) *is considered and the multiplicity of weak solutions is studied.*

Remark 1.2. *One can study the system* (1.3) *under the Steklov boundary conditions [16] or on Heisenberg Sobolev spaces or on Orlicz Sobolev spaces. Interested reader can see details of these spaces in [11, 12, 23, 22, 25, 27, 28, 29] and references therein.*

The rest of the paper is organized as follows. In Section 2, we present a brief survey of notions and results related to our problem. We also prove some remarks which we shall need later. In Section 3, we sketch the main result of the paper and its proof.

2 Preliminaries

Variable exponent Lebesgue spaces appeared in the literature for the first time by W. Orlicz 1931 and it began to be actively studied in 1990s. The notion variable exponent Lebesgue and Sobolev spaces are directly related to the classical Lebesgue and Sobolev spaces where the constant p is replaced with the function $p(\cdot)$ which may depend on a variable. These kinds of spaces of functions provide a useful tool for the description of non-linear phenomena in elastic and fluid mechanics, and in image restoration, among other fields. In this section we recall some facts which are necessary in the sequel (see [7, 8, 10] and references therein).

Here Ω is a bounded domain in $\mathbb{R}^N(N \geq 2)$, with smooth boundary. We suppose that $1 < s_i < N$ and $p_i \in C(\overline{\Omega})$, $i = 1, \cdots, n$, satisfy the following condition

$$N < p_i^- := \inf_{x\in\Omega} p_i(x) \leqslant p(x) \leqslant p_i^+ := \sup_{x\in\Omega} p_i(x) < +\infty. \tag{2.1}$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : \; u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

with the Luxemburg norm

$$\|u\|_{p(x)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx < 1 \right\}.$$

Following [26], for any $\vartheta > 0$, we put

$$\vartheta^{\check{r}} := \begin{cases} \vartheta^{r^+} & \vartheta < 1, \\ \vartheta^{r^-} & \vartheta \geq 1; \end{cases}$$

and

$$\vartheta^{\hat{r}} := \begin{cases} \vartheta^{r^-} & \vartheta < 1, \\ \vartheta^{r^+} & \vartheta \geq 1; \end{cases}$$

for $r \in \{p_i : i = 1, \cdots, n\}$. Then the well-known [15, Proposition 2.7] will be rewritten as follows.

Proposition 2.1. *For each $u \in L^{p(x)}(\Omega)$, we have*

$$\|u\|_{p(x)}^{\check{p}} \leq \int_\Omega |u(x)|^{p(x)} dx \leq \|u\|_{p(x)}^{\hat{p}}.$$

In the sequel, $W^{1,p_i(x)}(\Omega)$, $i = 1, \cdots, n$, denote the variable exponent Sobolev spaces.

Remark 2.1. *As a consequence of [18, Theorem 3.8] we have*

$$W_0^{1,q(x)}(\Omega) \hookrightarrow W_0^{1,p(x)}(\Omega), \tag{2.2}$$

if $p(x) \leq q(x)$ *a.e.* $x \in \Omega$. *In a special case, for* $p_i, i = 1, \cdots, n$ *with the condition* (2.1),

$$W^{1,p_i(x)}(\Omega) \hookrightarrow W^{1,p_i^-}(\Omega)$$

is embedded continuously and since $p_i^- > N$, $W^{1,p_i^-}(\Omega)$ *is compactly embedded in* $C^0(\overline{\Omega})$. *Then*

$$W^{1,p_i(x)}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega}).$$

So, in particular, there exist positive constants $m_i > 0$, $i = 1, \cdots, n$, *such that*

$$|u|_\infty \leqslant m_i \|\nabla u\|_{p_i(x)},$$

for each $u \in W^{1,p_i(x)}(\Omega)$ *where* $|u|_\infty := \sup_{x\in\Omega} |u(x)|$.

Remark 2.2. *Let* $1 < s_i < N$ *and* $p_i \in C(\Omega)$ *be as in relation* (2.1), *for* $i = 1, \cdots, n$. *Then there exists* κ *such that*

$$\int_\Omega \frac{|u(x)|^{s_i}}{|x|^{s_i}} dx \leq \frac{\kappa}{H} \|\nabla u\|_{p_i(x)}^{s_i},$$

for $u \in W_0^{1,p_i(x)}(\Omega)$, *where* H *is given by the classical Hardy's inequality in [1].*

Next we define the suitable function space

$$X := \prod_{i=1}^n W_0^{1,p_i(x)}(\Omega),$$

endowed with the norm

$$\|u\| = \|(u_1, \cdots, u_n)\| = \sum_{i=1}^n \|\nabla u_i\|_{p_i(x)},$$

for $u = (u_1, \cdots, u_n) \in X$.

Remark 2.1 implies the embedding

$$X \hookrightarrow C^0(\overline{\Omega}) \times \cdots \times C^0(\overline{\Omega})$$

is compact and if $M := \max_{1\leq i\leq n} m_i$, then $M > 0$ and one has

$$|u|_\infty \leqslant M\|\nabla u\|_{p_i(x)} \qquad i = 1, \cdots, n. \tag{2.3}$$

We set

$$\delta(x) = sup\,\{\delta > 0 : B(x, \delta) \subseteq \Omega\} \quad \text{and} \quad D := sup_{x\in\Omega}\delta(x).$$

Obviously, there exists $x_0 \in \Omega$ such that

$$B(x_0, D) \subseteq \Omega.$$

In the next section we study the existence of weak solution for system (1.3).

3 Multiple weak solutions

The definition of weak solution for system (1.3) is as follows.

Definition 3.1. *It is said that* $(u_1, \cdots, u_n) \in X \setminus \{0\}$ *is a weak solution of problem* (1.3) *if*

$$u_i = 0 \quad \text{on} \;\; \partial\Omega \qquad 1 \leq i \leq n,$$

and

$$\sum_{i=1}^{n} \int_\Omega a_i(x, \nabla u_i)\nabla v_i dx + \sum_{i=1}^{n} \int_\Omega \frac{|u_i|^{s_i-2}u_i v_i}{|x|^{s_i}} dx \\ - \lambda \sum_{i=1}^{n} \int_\Omega F_{u_i}(x, u_1, \cdots, u_n) v_i dx = 0,$$

for each $v = (v_1, \cdots, v_n) \in X$.

Applying a variational theorem from [4] or [24, Theorem 1.1], we can prove Theorem 1.1. To this end, we need to bring some auxiliary remarks.

Remark 3.1. *Assume conditions* $(A1) - (A4)$ *are held, then for* $i = 1, \cdots, n$, *we have*

(I) $A_i(x, t)$ *is a* C^1*-Carathéodory function; i.e. for every* $t \in \mathbb{R}$,

$$A_i(\cdot, t) : \Omega \to \mathbb{R}$$

is measurable and for a.e. $x \in \Omega$, $A_i(x, \cdot)$ *is* $C^1(\mathbb{R})$.

(II) There exist constants C_i', $i = 1, \cdots, n$, *such that*

$$\frac{c_i}{p_i(x)}|t|^{p_i(x)} \leq |A_i(x, t)| \leq C_i'(|t| + |t|^{p_i(x)}),$$

for a.e. $x \in \Omega$ *and all* $t \in \mathbb{R}$, *where for* $i = 1, \cdots, n$ *constants* c_i *are as in condition* $(A4)$.

We define the functional $\Upsilon : X \longrightarrow \mathbb{R}$ as follows

$$\Upsilon(u_1, \cdots, u_n) := \sum_{i=1}^{n} \int_\Omega A_i(x, \nabla u_i) dx + \sum_{i=1}^{n} \int_\Omega \frac{|u_i|^{s_i}}{s_i |x|^{s_i}} dx.$$

Remark 3.2. *There exists positive constant* $\hat{C}$ *such that*

$$\frac{c_i}{p_i^+} \|\nabla u_i\|_{p_i(x)}^{\check{p}_i} \leq \Upsilon(u_1, \cdots, u_n) \leq \hat{C} \sum_{i=1}^{n} \|\nabla u_i\|_{p_i(x)}^{\hat{p}_i} + \frac{\kappa}{H} \sum_{i=1}^{n} \|\nabla u_i\|_{p_i(x)}^{s_i},$$

for each $1 \leq i \leq n$ *and* $u = (u_1, \cdots, u_n) \in X$.

Proof. From (2.2) and Remark 3.1, for every $1 \le i \le n$, one has the following estimate

$$
\begin{aligned}
\frac{c_i}{p_i^+}\|\nabla u_i\|_{p_i(x)}^{\check{p_i}} &\le \frac{c_i}{p_i^+}\int_\Omega |\nabla u_i|^{p_i(x)}dx \\
&\le \sum_{i=1}^n \frac{c_i}{p_i^+}\int_\Omega |\nabla u_i|^{p_i(x)}dx \\
&\le \Upsilon(u_1,\cdots,u_n) = \sum_{i=1}^n \int_\Omega A_i(x,\nabla u_i)dx + \sum_{i=1}^n \int_\Omega \frac{|u_i|^{s_i}}{s_i|x|^{s_i}}dx \\
&\le \sum_{i=1}^n C_i' \int_\Omega (|\nabla u_i| + |\nabla u_i|^{p_i(x)})dx + \frac{\kappa}{H}\sum_{i=1}^n \|\nabla u_i\|_{p_i(x)}^{s_i} \\
&\le \sum_{i=1}^n C_i'(K_i+1)\|\nabla u_i\|_{p_i(x)}^{\hat{p_i}} + \frac{\kappa}{H}\sum_{i=1}^n \|\nabla u_i\|_{p_i(x)}^{s_i}.
\end{aligned}
$$

It is enough to set $\hat{C} = \max\{C_i'(K_1+1) : 1 \le i \le n\}$. □

Remark 3.3. *Remark 3.2 ensures that Υ is coercive.*

Proof. Let $u = (u_1,\cdots,u_n) \in X$ and $\|u\| \to \infty$. Thanks to definition of $\|\cdot\|$, there exists $1 \le i_0 \le n$ such that $\|\nabla u_{i_0}\|_{p_{i_0}(x)} \to \infty$. Then Remark 3.2 deduce that $\Upsilon(u) \to \infty$. □

Notice that Υ is sequentially weakly lower semicontinuous and it is known that Υ is continuously Gâteaux differentiable functional; moreover,

$$\Upsilon'(u_1,\cdots,u_n)(v_1,\cdots,v_n) = \sum_{i=1}^n \int_\Omega \Big(a_i(x,\nabla u_i)\nabla v_i + \frac{|u_i|^{s_i-2}u_i v_i}{|x|^{s_i}}\Big)dx,$$

for each $(v_1,\cdots,v_n) \in X$. And we define $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with

$$\Lambda(u_1,\cdots,u_n) := \int_\Omega F(x,u_1,\cdots,u_n)dx.$$

The functional Λ is well defined, continuously Gâteaux differentiable with compact derivative, whose Gâteaux derivative at point $u \in X$ is as follows

$$\Lambda'(u_1,\cdots,u_n)(v_1,\cdots,v_n) = \sum_{i=1}^n \int_\Omega F_{u_i}(x,u_1(x),\cdots,u_n(x))v_i(x)dx,$$

for every $(v_1,\cdots,v_n) \in X$. The energy functional corresponding to the problem is

$$\mathcal{G}_\lambda(u) = \Upsilon(u) - \lambda\Lambda(u),$$

for each $u = (u_1,\cdots,u_n)$; or equivalently, weak solutions of (1.3) are exactly the critical points of $\mathcal{G}_\lambda$.

Therefore, the functional $\Upsilon, \Lambda : \mathbb{R}^N \to \mathbb{R}$ defined as above satisfy the regularity assumptions of [24, Theorem 1.1]. From definition of Υ, Λ and condition $(F1)$, it is clear that

$$\inf_{x\in X} \Upsilon = \Upsilon(0) = \Lambda(0) = 0.$$

For $\delta > 0$ and D defined as above, we denote by w, a function of $W^{1,p_i(x)}(\Omega), 1 \leq i \leq n$, defined by

$$w(x) := \begin{cases} 0 & x \in \Omega \setminus B(x_0, D), \\ \delta & x \in B(x_0, \frac{D}{2}), \\ \frac{2\delta}{D}(D - |x - x_0|) & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}), \end{cases}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. By Remark 3.2, for $1 \leq i_* \leq n$, we have

$$\begin{aligned} \frac{c_{i_*}}{p^+_{i_*}}(\frac{2\delta}{D})^{\check{p}_{i_*}} m(D^N - (\frac{D}{2})^N) &\leq \Upsilon(w, \cdots, w) \\ &\leq \sum_{i=1}^{n} \Big(\hat{C}(\frac{2\delta}{D})^{\hat{p_i}} + \frac{\kappa}{H}(\frac{2\delta}{D})^{s_i}\Big) m(D^N - (\frac{D}{2})^N). \end{aligned}$$

Then by assumption $(F3)$, we gain $\Upsilon(w, \cdots, w) > r$. On the other hand, we have the following estimate

$$\begin{aligned} \Lambda(w, \cdots, w) &\geq \int_{B(x_0, \frac{D}{2})} F(x, w, \cdots, w)dx \\ &\geq \inf_{x\in\Omega} F(x, \delta, \cdots, \delta) m(\frac{D}{2})^N, \end{aligned}$$

where m is the measure of unit ball of $\mathbb{R}^N$ and so,

$$\begin{aligned} \frac{\Lambda(w, \cdots, w)}{\Upsilon(w, \cdots, w)} &\geq \frac{\inf_{x\in\Omega} F(x, \delta, \cdots, \delta) m(\frac{D}{2})^N}{\sum_{i=1}^{n} \Big(\hat{C}(\frac{2\delta}{D})^{\hat{p_i}} + \frac{\kappa}{H}(\frac{2\delta}{D})^{s_i}\Big) m(D^N - (\frac{D}{2})^N)} \\ &= \frac{\inf_{x\in\Omega} F(x, \delta, \cdots, \delta)}{\sum_{i=1}^{n} \Big(\hat{C}(\frac{2\delta}{D})^{\hat{p_i}} + \frac{\kappa}{H}(\frac{2\delta}{D})^{s_i}\Big)(2^N - 1)} = B_\delta. \end{aligned} \tag{3.1}$$

Now, let $u = (u_1, \cdots, u_n) \in \Upsilon^{-1}(-\infty, r)$, from Remark 3.2, we gain

$$\|\nabla u_i\|_{p_i(x)} \leq \Big(\frac{p_i^+}{c_i}\Upsilon(u_1, \cdots, u_n)\Big)^{\frac{1}{\hat{p_i}}} \leq \Big(\frac{p_i^+}{c_i} r\Big)^{\frac{1}{\hat{p_i}}}, \tag{3.2}$$

for each $i = 1, \cdots, n$. Then, for every $u = (u_1, \cdots, u_n) \in X$, using condition $(F2)$, Hölder inequality and (2.3) , we have

$$\begin{aligned}\int_\Omega F(x, u_1, \cdots, u_n)dx &\leq \int_\Omega sup_{u\in\Upsilon^{-1}(-\infty,r)} F(x, u_1, \cdots, u_n)dx \\ &\leq \int_\Omega \eta(x)(1 + \sum_{i=1}^{n} |u_i|^{\gamma_i(x)})dx \\ &\leq \|\eta\|_1(1 + \sum_{i=1}^{n} |u_i|_\infty^{\hat{\gamma_i}}) \\ &\leq \|\eta\|_1(1 + M\sum_{i=1}^{n} \|\nabla u_i\|_{p_i(x)}^{\hat{\gamma_i}}).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{r} sup_{u\in\Upsilon^{-1}(-\infty,r)} \Lambda(u) &= \frac{1}{r} sup_{u\in\Upsilon^{-1}(-\infty,r)} \int_\Omega F(x, u_1, \cdots, u_n)dx \\ &\leq \frac{\|\eta\|_1}{r}\Big(1 + M\sum_{i=1}^{n} (\frac{p_i^+}{c_i} r)^{\frac{\hat{\gamma_i}}{\check{p_i}}}\Big) = A_r.\end{aligned}$$

From assumption (1.4), relation (3.1) and the last inequality, one has

$$\frac{1}{r} sup_{u\in\Upsilon^{-1}(-\infty,r)} \Lambda(u_i) < \frac{\Lambda(w, \cdots, w)}{\Upsilon(w, \cdots, w)},$$

Now, we prove that, for each $\lambda > 0$, $\mathcal{G}_\lambda$ is coercive.
With the same arguments as used before, we have

$$\Lambda(u) = \int_\Omega F(x, u_1, \cdots, u_n)dx \leq \|\eta\|_1(1 + M\sum_{i=1}^{n} \|\nabla u_i\|_{p_i(x)}^{\hat{\gamma_i}}).$$

The last inequality and Remark 3.2 lead to

$$\mathcal{G}_\lambda(u) \geq \frac{c_i}{p_i^+} \|\nabla u_i\|_{p_i(x)}^{\check{p_i}} - \lambda\|\eta\|_1(1 + M\sum_{i=1}^{n} \|\nabla u_i\|_{p_i(x)}^{\hat{\gamma_i}}),$$

for each $i = 1, \cdots, n$. Now, imagine that $u \in X$ and $\|u\| \to \infty$. So, there exists $1 \leq i_0 \leq n$ such that $\|\nabla u_{i_0}\|_{p_{i_0}(x)} \to \infty$. Since according to our assumptions $\gamma_{i_0}(x) < p_{i_0}(x)$ a.e. in Ω, coercivity of $\mathcal{G}_\lambda$ is obtained.
Taking into account that

$$\Lambda_{\delta,r} := (\frac{1}{B_\delta}, \frac{1}{A_r}) \subseteq \Big(\frac{\Upsilon(w, \cdots, w)}{\Lambda(w, \cdots, w)}, \frac{r}{sup_{u\in\Upsilon^{-1}(-\infty,r)} \Lambda(u_i)}\Big),$$

Thus [24, Theorem 1.1] ensures that for each $\lambda \in \Lambda_{r,\delta}$, the functional $\mathcal{G}_\lambda$ admits at least three critical points in X that are weak solutions of the problem (1.3).

Declaration

Ethical Approval

Not applicable.

Competing interests

The authors have no relevant financial or non-financial interests to disclose.

Authors' contributions

All authors wrote and reviewed the manuscript.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Author Contributions Statement

All authors reviewed the manuscript.

References

[1] J.G. Azorero and I.P. Alonso, Hardy inequalities and some critical elliptic and parabolic problems, *J. Differential Equations* **144** (1998), 441-476.

[2] F. Behboudi, A. Razani and M. Oveisiha, Existence of a mountain pass solution for a nonlocal fractional (p, q)-Laplacian problem, *Bound. Value Probl.*, **2020** (2020), no. 1, 1-14. https://doi.org/10.1186/s13661-020-01446-w

[3] F. Behboudi and A. Razani, Two weak solutions for a singular (p, q)-Laplacian problem, *Filomat*, **33** (2019), No. 11, 3399–3407.

[4] G. Bonanno and S.A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.*, **89** (2010), 1-18.

[5] M.M. Boureanu and F. Preda, Infinitely many solutions for elliptic problems with variable exponent and nonlinear boundary conditions, *NoDEA Nonlinear Differ. Equ. Appl.*, **19** (2012), 235-251.

[6] M.M. Boureanu and D.N. Udrea, Existence and multiplicity results for elliptic problems with $p(\cdot)$-growth conditions, *Nonl. Anal.*, **14** (2013), 1829-1844.

[7] L. Diening, P. Harjulehto, P. Hästö and Michael Ruzicka, Lebesgue and Sobolev Spaces with Variable Exponents, (Lecture Notes in Mathematics, Vol. 2017). Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8

[8] P. Górka, N. Karak and D.J. Pons, Variable exponent Sobolev spaces and regularity of domains, *J. Geom. Anal.*, **31** (2021), 73047319. https://doi.org/10.1007/s12220-020-00499-8

[9] H. Hai Ha, K. Ho and I. Sim, Infinitely many solutions for a generalized $p(\cdot)$-Laplace equation involving Leray-Lions type operators, *Math. Methods Appl. Sci.*, (2021).

[10] P. Harjulehto and P. Hästö, Orlicz Spaces and Generalized Orlicz Spaces, Lecture Notes in Mathematics, vol. 2236, Springer, Cham, 2019.

[11] S. Heidari and A. Razani, Infinitely many solutions for nonlocal elliptic systems in Orlicz-Sobolev spaces, *Georgian Math. J.*, **29** (2021) no. 1, 45–54. https://doi.org/10.1515/gmj-2021-2110

[12] S. Heidari and A. Razani, Multiple solutions for a class of nonlocal quasilinear elliptic systems in Orlicz-Sobolev spaces, *Bound. Value Probl.*, (2021), no. 1, 1-15.

[13] K. Ho and I. Sim, Existence and some properties of solutions for degenerate elliptic equations with exponent variable, *Nonlinear Anal.*, **98** (2014), 146-164. Corrigendum to Existence and some properties of solutions for degenerate elliptic equations with exponent variable, *Nonlinear Anal*, **128** (2015), 423-426. https://doi.org/10.1016/j.na.2013.12.003

[14] K. Ho and I. Sim, Existence results for degenerate $p(x)$-Laplace equations with Leray-Lions type operators, *Sci. China Math.*, **60** (2017), 133-146.

[15] Y. Karagiorgos and N. Yannakaris, A Neumann problem involving the $p(x)$-Laplacian with $p = \infty$ in a subdomain, *Adv. Calc. Var.* **9** (2016), no. 1, 65-76.

[16] A. Khaleghi and A. Razani, Existence and multiplicity of solutions for $p(x)$-Laplacian problem with Steklov boundary condition, *Bound. Value Probl.*, **2022** (2022), no. 39, 11 pages, https://doi.org/10.1186/s13661-022-01624-y.

[17] I.H. Kim and Y.H. Kim, Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents, *Manuscripta Math.*, **147** (2015), 169-191.

[18] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslov. Math. J.* **41** (1991), no. 4, 592-618.

[19] H. Ky and S. Inbo, Existence results for degenerate $p(x)$-Laplace equations with Leray-Lions type operators, *Sci. China Math.*, **60** (2017), 133-146. https://doi.org/10.1007/s11425-015-0385-0

[20] R. Mahdavi and A. Razani, Existence of at least three weak solutions for a singular fourth-order elliptic problems ,*J. Math. Anal.* **8** (2017), 45–51.

[21] M. Mahshid and A. Razani, A weak solution for a $(p(x), q(x))$-Laplacian elliptic problem with a singular term, *Bound. Value Probl.*, **2021, 80** (2021). https://doi.org/10.1186/s13661-021-01557-y

[22] M. Makvand Chaharlang and A. Razani, Existence of infinitely many solutions for a class of nonlocal problems with Dirichlet boundary condition, *Commun. Korean Math. Soc.*, **34** (2019), no. 1, 155-167, https://doi.org/10.4134/CKMS.c170456.

[23] M. Makvand Chaharlang and A. Razani, Existence of weak solutions for a nonlocal singular elliptic problem, *Azerb. J. Math.*, **11** (2021), no. 1, 80-91.

[24] Z. Musbah and A. Razani, A class of biharmonic nonlocal quasilinear systems consisting of Leray-Lions type operators with Hardy potentials, *Bound. Value Probl.*, **2022:80** (2022). https://doi.org/10.1186/s13661-022-01666-2

[25] M.A. Ragusa and A. Razani, Weak solutions for a system of quasilinear elliptic equations, *Contrib. Math.* **1** (2020) 11-16.

[26] M.A. Ragusa, A. Razani and F. Safari, Existence of radial solutions for a $p(x)$-Laplacian Dirichlet problem, *Adv. Differential Equations*, (2021), no. 1, 1-14. https://doi.org/10.1186/s13662-021-03369-x

[27] F. Safari and A. Razani, Existence of positive radial solution for Neumann problem on the Heisenberg groups, *Bound. Value Probl.*, **2022:88** (2022). https://doi.org/10.1186/s13661-020-01386-5

[28] F. Safari and A. Razani, Existence of radial solutions of the Kohn-Laplacian problem, *Complex Var. Elliptic Equ.*, (2020), 1-15, https://doi.org/10.1080/17476933.2020.1818733

[29] F. Safari and A. Razani, Radial solutions for a general form of a p-Laplace equation involving nonlinearity terms, *Complex Var. Elliptic Equ.*, (2021), 1-11. https://doi.org/10.1080/17476933.2021.1991331.