GENERALISATION OF HERMITE-HADAMARD TYPE INEQUALITIES FOR $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$CONVEX FUNCTIONS IN MIXED KIND WITH APPLICATIONS

FARAZ MEHMOOD[1,2]

ABSTRACT. In this article, we present the 1st time the generalised notion of $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex(concave) function in mixed kind, which is the generalization of functions: convex(concave), $P-$convex(concave), quasi$-$convex (concave), $s-$convex(concave) in 1st kind, $s-$convex(concave) in 2nd kind, $m-$convex(concave), $(m_1, m_2)-$convex(concave), $(m, s)-$convex(concave) in 1st kind, $(m, s)-$convex(concave) in 2nd kind, $(s, r)-$convex(concave) in mixed kind, (α, β)-convex(concave) in 1st kind, $(\alpha, \beta)-$convex(concave) in 2nd kind, $(m, s, r)-$convex(concave) in mixed kind, (m, α, β) $-$convex(concave) in 1st kind, $(m, \alpha, \beta)-$convex(concave) in 2nd kind, $(m_1, m_2, s)-$convex(concave) function in 1st kind, $(m_1, m_2, s)-$convex(concave) function in 2nd kind, $(m_1, m_2, s, r)-$convex(concave) in mixed kind, $(m_1, m_2, \alpha, \beta)-$convex(concave) in 1st kind, $(m_1, m_2, \alpha, \beta)-$convex(concave) in 2nd kind, $(\alpha, \beta, \gamma, \mu)-$convex(concave) in mixed kind, $(m, \alpha, \beta, \gamma, \mu)-$convex(concave) in mixed kind. Our aim is to establish generalised inequalities of Hermite-Hadamard type for functions whose modulus of the derivatives are $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex functions by using different techniques including Hölder's inequality and power mean inequality and we would also give applications for probability theory and numerical integration. Moreover, we would obtain various results with respect to the convexity of function as special cases on several choices of $m_1, m_2, \alpha, \beta, \gamma, \mu$. Various established results of different authors of different articles would also be recaptured as special cases.

1. Introduction

About the features of convex functions, we code some lines from [21] "Convex functions appear in many problems in pure and applied mathematics. They play an extremely important role in the study of both linear and non-linear programming problems. The theory of convex functions is part of the general subject of convexity, since a convex function is one whose epigraph is a convex set. Nonetheless it is an important theory which touches almost all branches of mathematics. Graphical analysis is one of the first topics in mathematics which requires the concept of convexity. Calculus gives us a powerful tool in recognizing convexity, the second-derivative test".

The importance of convex functions for the generalization of integral inequalities due to the variety of their nature the notion have been established. Integral inequalities are satisfied by many convex functions. Among these, the well known is

Ostrowski inequality. To generalize the Ostrowski's inequality, we need to generalize the concept of convax functions, in this way we can easily see the generalizations and its particular cases. From the literature, we remind few definitions for several convex functions [2].

Definition 1.1. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as convex(concave), if

$$g(\tau y + (1 - \tau)z) \le (\ge)\tau g(y) + (1 - \tau)g(z), \tag{1.1}$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Here we remind defination of $P-$convex(concave) function see [4].

Definition 1.2. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as $P-$convex(concave), if function g is a non-negative, then we have

$$g(\tau y + (1 - \tau)z) \le (\ge)g(y) + g(z), \tag{1.2}$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

The definition of quasi-convex function is extracted from [7].

Definition 1.3. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is called a quasi-convex(concave), if

$$g(\tau y + (1 - \tau)z) \le (\ge)\max\{g(y), g(z)\} \tag{1.3}$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

We present defination of $s-$convax(concave) functions in the 1st kind as follows (see [20]).

Definition 1.4. Suppose $s \in (0, 1]$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $s-$convex(concave) in the 1st kind, if

$$g(\tau y + (1 - \tau)z) \le (\ge)\tau^s g(y) + (1 - \tau^s)g(z), \tag{1.4}$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Remark 1.5. "Note that in this definition we also included $s = 0$. Further if we put $s = 0$, we get quasi-convexity (see Definition 1.3)".

We also present definition of $s-$convex(concave) functons in the second kind from [20].

Definition 1.6. Suppose $s \in (0, 1]$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $s-$convex(concave) in the 2nd kind, if

$$g(\tau y + (1 - \tau)z) \le (\ge)\tau^s g(y) + (1 - \tau)^s g(z), \tag{1.5}$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Remark 1.7. "In the similar manner, we have slightly improved definition of 2nd kind convexity by including $s = 0$. Further if we put $s = 0$, we easily get $P-$convexity (see Definition 1.2)".

The following definition of $m-$convex(concave) function is extracted from [9]

Definition 1.8. Suppose $m \in [0, 1]$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as $m-$convex (concave), if

$$g(\tau y + m(1 - \tau)z) \le (\ge)\tau g(y) + m(1 - \tau)g(z) \tag{1.6}$$

$\forall\ y, z \in [0, \infty),\ \tau \in [0, 1]$.

Remark 1.9. "For $m = 1$ the above definition recaptures the concept of standard convex(concave) functions in the interval K and for $m = 0$ the concept star-shaped functions".

Following definition is extracted from [9]

Definition 1.10. Let $(m_1, m_2) \in (0, 1]^2$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as (m_1, m_2)−convex(concave), if

$$g(m_1\tau y + m_2(1 - \tau)z) \leq (\geq)m_1\tau g(y) + m_2(1 - \tau)g(z), \tag{1.7}$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

In [18], Mihesan stated (α, m)−convaxity as in the following:

Definition 1.11. Suppose $(\alpha, m) \in [0, 1]^2$. A function $g : [0, \infty) \to \mathbb{R}$ is known as (α, m)−convex(concave), if

$$g(\tau y + m(1 - \tau)z) \leq (\geq)\tau^\alpha g(y) + m(1 - \tau^\alpha)g(z) \tag{1.8}$$

$\forall\ y, z \in [0, \infty)$, $\tau \in [0, 1]$. Above function can also be written as (m, s)−convex (concave) function in the 1st kind.

Firstly, we introduce a new class of (m, s)−convax(concave) function in the 2nd kind that is given below:

Definition 1.12. Let $(m, s) \in (0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m, s)−convex(concave) in the 2nd kind, if

$$g(\tau y + m(1 - \tau)z) \leq (\geq)\tau^s g(y) + m(1 - \tau)^s g(z) \tag{1.9}$$

$\forall\ y, z \in K$, $\tau \in [0, 1]$.

A new class of (s, r)−convax(concave) functions in the mixed kind is extracted from [8].

Definition 1.13. Suppose $(s, r) \in [0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (s, r)−convex(concave) in the mixed kind, if

$$g(\tau y + (1 - \tau)z) \leq (\geq)\tau^{rs} g(y) + (1 - \tau^r)^s g(z), \tag{1.10}$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Definition 1.14. [6] Suppose $(\alpha, \beta) \in [0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (α, β)−convax(concave) in the 1st kind, if

$$g(\tau y + (1 - \tau)z) \leq (\geq)\tau^\alpha g(y) + (1 - \tau^\beta)g(z), \tag{1.11}$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Definition 1.15. [6] Suppose $(\alpha, \beta) \in [0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (α, β)−convax(concave) in the 2nd kind, if

$$g(\tau y + (1 - \tau)z) \leq (\geq)\tau^\alpha g(y) + (1 - \tau)^\beta g(z), \tag{1.12}$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Secondly, we introduce a new class of (m, s, r)−convax(concave) functions in mixed kind which is given below:

Definition 1.16. Let $(m, s, r) \in [0, 1]^3$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(m, s, r)-$convex(concave) in the mixed kind, if

$$g(\tau y + m(1 - \tau)z) \leq (\geq)\tau^{rs}g(y) + m(1 - \tau^r)^s g(z), \qquad (1.13)$$

$\forall \ y, z \in K, \tau \in [0, 1]$.

Thirdly, we introduce a new class of $(m, \alpha, \beta)-$convax(concave) functions in the 1st kind which is given below:

Definition 1.17. Let $(m, \alpha, \beta) \in [0, 1]^3$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(m, \alpha, \beta)-$convax(concave) in the 1st kind, if

$$g(\tau y + m(1 - \tau)z) \leq (\geq)\tau^{\alpha}g(y) + m(1 - \tau^{\beta})g(z), \qquad (1.14)$$

$\forall \ y, z \in K, \tau \in [0, 1]$.

Fourthly, we introduce a new class of $(m, \alpha, \beta)-$convax(concave) functions in the 2nd kind which is given below:

Definition 1.18. Let $(m, \alpha, \beta) \in [0, 1]^3$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(m, \alpha, \beta)-$convax(concave) in the 2nd kind, if

$$g(\tau y + m(1 - \tau)z) \leq (\geq)\tau^{\alpha}g(y) + m(1 - \tau)^{\beta} g(z), \qquad (1.15)$$

$\forall \ y, z \in K, \tau \in [0, 1]$.

Following definition is extracted from [9]

Definition 1.19. Let $(\alpha, m_1, m_2) \in (0, 1]^3$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as $(\alpha, m_1, m_2)-$convax(concave), if

$$g(m_1\tau y + m_2(1 - \tau)z) \leq (\geq)m_1\tau^{\alpha}g(y) + m_2(1 - \tau^{\alpha})g(z), \qquad (1.16)$$

$\forall \ y, z \in K, \tau \in [0, 1]$. Above function can also be written as $(m_1, m_2, s)-$convex (concave) function in the 1st kind.

Fifthly, we introduce a new class of $(m_1, m_2, s)-$convax(concave) functions in the 2nd kind which is given below:

Definition 1.20. Let $(m_1, m_2, s) \in (0, 1]^3$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as $(m_1, m_2, s)-$convax(concave) in the 2nd kind, if

$$g(m_1\tau y + m_2(1 - \tau)z) \leq (\geq)m_1\tau^s g(y) + m_2(1 - \tau)^s g(z), \qquad (1.17)$$

$\forall \ y, z \in K, \tau \in [0, 1]$.

Sixthly, we introduce a new class of $(m_1, m_2, s, r)-$convax(concave) functions in mixed kind which is given below:

Definition 1.21. Let $(m_1, m_2, s, r) \in (0, 1]^4$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(m_1, m_2, s, r)-$convax(concave) in the mixed kind, if

$$g(m_1\tau y + m_2(1 - \tau)z) \leq (\geq)m_1\tau^{rs}g(y) + m_2(1 - \tau^r)^s g(z), \qquad (1.18)$$

$\forall \ y, z \in K, \tau \in [0, 1]$.

Seventhly, we introduce a new class of $(m_1, m_2, \alpha, \beta)-$convax(concave) functions in the 1st kind which is given below:

Definition 1.22. Let $(m_1, m_2, \alpha, \beta) \in (0,1]^4$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(m_1, m_2, \alpha, \beta)-$convax(concave) in the 1st kind, if

$$g(m_1\tau y + m_2(1 - \tau)z) \leq (\geq)m_1\tau^\alpha g(y) + m_2(1 - \tau^\beta)g(z), \qquad (1.19)$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Eighthly, we introduce a new class of $(m_1, m_2, \alpha, \beta)-$convex(concave) functions in the 2nd kind which is given below:

Definition 1.23. Suppose $(m_1, m_2, \alpha, \beta) \in (0,1]^4$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(m_1, m_2, \alpha, \beta)-$convax(concave) in the 2nd kind, if

$$g(m_1\tau y + m_2(1 - \tau)z) \leq (\geq)m_1\tau^\alpha g(y) + m_2(1 - \tau)^\beta g(z), \qquad (1.20)$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Upcoming definition is $(\alpha, \beta, \gamma, \mu)-$convex(concave) function which is extracted from [8].

Definition 1.24. Let $(\alpha, \beta, \gamma, \mu) \in [0,1]^4$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(\alpha, \beta, \gamma, \mu)-$convax(concave) in the mixed kind, if

$$g(\tau y + (1 - \tau)z) \leq (\geq)\tau^{\alpha\gamma} g(y) + (1 - \tau^\beta)^\mu g(z), \qquad (1.21)$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Ninthly, we introduce a new class of $(m, \alpha, \beta, \gamma, \mu)-$convax(concave) functons in mixed kind that is given below:

Definition 1.25. Let $(m, \alpha, \beta, \gamma, \mu) \in [0,1]^5$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(m, \alpha, \beta, \gamma, \mu)-$convax(concave) in the mixed kind, if

$$g(\tau y + m(1 - \tau)z) \leq (\geq)\tau^{\alpha\gamma} g(y) + m(1 - \tau^\beta)^\mu g(z), \qquad (1.22)$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Tenthly and Finally we introduce a new class of functon which would be called class of $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convax(concave) functon in mixed kind and containing all above classes of functions. This definition is used sequentially in this paper.

Definition 1.26. Let $(m_1, m_2, \alpha, \beta, \gamma, \mu) \in (0,1]^6$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convax(concave) in the mixed kind, if

$$g(m_1\tau y + m_2(1 - \tau)z) \leq (\geq)m_1\tau^{\alpha\gamma} g(y) + m_2(1 - \tau^\beta)^\mu g(z), \qquad (1.23)$$

$\forall\ y, z \in K, \tau \in [0, 1]$.

Remark 1.27. "In Definition 1.26, we have the following cases.
   (i) If we choose $m_1 = 1$, $m_2 = m$ in (1.23), we get $(m, \alpha, \beta, \gamma, \mu)-$convex (concave) function in the mixed kind.
  (ii) If we choose $m_1 = m_2 = 1$ in (1.23), we get $(\alpha, \beta, \gamma, \mu)-$convex(concave) function in the mixed kind.
 (iii) If we choose $\beta = \gamma = 1$ and $\mu = \beta$ in (1.23), we get $(m_1, m_2, \alpha, \beta)-$convex (concave) function in the 2nd kind.
 (iv) If we choose $\gamma = \mu = 1$ in (1.23), we get $(m_1, m_2, \alpha, \beta)-$convex(concave) function in the 1st kind.
  (v) If we choose $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in (1.23), we get (m_1, m_2, s, r) $-$convex(concave) function in mixed kind.

(vi) If we choose $\alpha = \mu = s$ and $\beta = \gamma = 1$ in (1.23), we get $(m_1, m_2, s)-$convex (concave) function in the 2nd kind.

(vii) If we choose $\gamma = s$ and $\alpha = \beta = \mu = 1$ in (1.23), we get $(m_1, m_2, s)-$convex (concave) function in the 1st kind.

(viii) If we choose $m_1 = 1$, $m_2 = m$, $\beta = \gamma = 1$ and $\mu = \beta$ in (1.23), we get $(m, \alpha, \beta)-$convex(concave) function in the 2nd kind.

(ix) If we choose $m_1 = 1$, $m_2 = m$ and $\gamma = \mu = 1$ in (1.23), we get (m, α, β) $-$convex(concave) function in the 1st kind.

(x) If we choose $m_1 = 1$, $m_2 = m$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in (1.23), we get $(m, s, r)-$convex(concave) function in the mixed kind.

(xi) If we choose $m_1 = m_2 = 1$, $\beta = \gamma = 1$ and $\mu = \beta$ in (1.23), we get $(\alpha, \beta)-$convex(concave) function in the 2nd kind.

(xii) If we choose $m_1 = m_2 = 1$ and $\gamma = \mu = 1$ in (1.23), we get $(\alpha, \beta)-$convex (concave) function in the 1st kind.

(xiii) If we choose $m_1 = m_2 = 1$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in (1.23), we get $(s, r)-$convex(concave) function in the mixed kind.

(xiv) If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in (1.23), we get $(m, s)-$convex(concave) function in the 2nd kind.

(xv) If we choose $m_1 = 1$, $m_2 = m$, $\gamma = s$ and $\alpha = \beta = \mu = 1$ in (1.23), we get $(m, s)-$convex(concave) function in the 1st kind.

(xvi) If we choose $\alpha = \beta = \gamma = \mu = 1$ in (1.23), we get $(m_1, m_2)-$convex(concave) functnion.

(xvii) If we choose $m_1 = 1$, $m_2 = m$ and $\alpha = \beta = \gamma = \mu = 1$ in (1.23), we get $m-$convex(concave) function.

(xviii) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in (1.23), we get $s-$convex(concave) function in the 2nd kind.

(xix) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = s$ and $\gamma = \mu = 1$ in (1.23), we get $s-$convex (concave) function in the 1st kind.

(xx) If we choose $m_1 = m_2 = 1$, $\gamma = s$ and $\alpha = \beta = \mu = 1$ in (1.23), we get $s-$convex(concave) function in the 1st kind.

(xxi) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = 0$, and $\gamma = \mu = 1$ in (1.23), we get quasi-convex(concave) function.

(xxii) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = 0$ and $\beta = \gamma = 1$ in (1.23), we get $P-$convex(concave) function.

(xxiii) If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in (1.23), gives us ordinary convex(concave) function.

In almost every field of science, inequalities play an important role. Although it is very vast discipline but our focus is mainly on Hermite–Hadamard type inequalities.

The convexity theory has close relationship with theory of inequalities. Many inequalities known in the literature are direct consequences of the applications of convex functions. An important inequality for convex functions which has been extensively studied in recent decades is Hermite-Hadamard's inequality, which was obtained by Hermite and Hadamard independently and stated as: A function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function, $j, k \in K$ with $j < k$, if and only if,

$$g\left(\frac{j+k}{2}\right) \leq \frac{1}{k-j} \int_j^k g(y)dy \leq \frac{g(j)+g(k)}{2}, \tag{1.24}$$

which is known as Hermite-Hadamard inequality. This inequality (1.24) has become an important cornerstone in probability and optimization. An account on the history of this inequality can be found in [19]. Moreover, the inequality (1.24) has been refined or generalised for convex, s-convex, and quasi-convex functions and other kinds of functions by a number of mathematicians.

In [5], the following result which was obtained by Dragomir and Agarwal contains the Hermite-Hadamard type integral inequality.

Proposition 1.28. *Suppose $g : K^0 \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping in the K^0, $j, k \in K^0$ with $j < k$. If $|g'|$ is convex in the $[j, k]$, then the following inequality holds:*

$$\left| \frac{g(j) + g(k)}{2} - \frac{1}{k-j} \int_j^k g(u)du \right| \leq \frac{(k-j)(|g'(j)| + |g'(k)|)}{8}. \qquad (1.25)$$

For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, see [1, 3, 10, 11, 13, 15, 19, 24] and the references cited therein.

In 2011, Kavurmaci et. al. [10] gave some new inequalities of hermite-hadamard type for convex functions with applications by using Hölder inequality and Power-mean inequality. In 2016, Liu et. al. [13] established some Hermite-Hadamard type inequalities for $MT-$convex functions via classical integrals and Riemann-Liouville fractional integrals, respectively. In [14], [16] and [17], Mehmood et. al. established generalised Hermite-Hadamard type inequalities for $(s, r)-$convex functions, $(\alpha, \beta, \gamma, \mu)-$convex functions and $(m, \alpha, \beta, \gamma, \mu)-$convex functions respectively with applications by using Hölder inequality and Power-mean inequality".

The main aim of this article is to generalise some Hermite-Hadamard type inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex functions in mixed kind through classical integrals using Hölder inequality and Power-mean inequality and applications are also provided for probability theory and numerical integration. We would recapture several results of different articles [5, 10, 14, 16, 17] and also provide special cases of class of $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex function on different choices of $m_1, m_2, \alpha, \beta, \gamma, \mu$ as remarks.

2. Generalisation of Hermite-Hadamard Type Inequalities

Derivation of our main theorems, we require folowing lemma which is extracted from [10]:

Lemma 2.1. *Suppose $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping in the $K^0 \subset \mathbb{R}$, where $m_1, m_2 j, m_2 k \in K$ with $m_2 j < m_2 k$. If $g' \in L[m_2 j, m_2 k]$, then*

$$\frac{(m_2 k - m_1 y)g(m_2 k) + (m_1 y - m_2 j)g(m_2 j)}{k - j} - \frac{1}{k-j} \int_{m_2 j}^{m_2 k} g(u)du$$

$$= \frac{(m_1 y - m_2 j)^2}{k-j} \int_0^1 (\tau - 1)g'(m_1 \tau y + m_2(1-\tau)j)d\tau$$

$$+ \frac{(m_2 k - m_1 y)^2}{k-j} \int_0^1 (1 - \tau)g'(m_1 \tau y + m_2(1-\tau)k)d\tau.$$

Proof. We obtain the required result by using similar techniques of proof of Lemma 1 of [10]. □

Remark 2.2. If we choose $m_1 = 1$ and $m_2 = m$ we recapture Lemma 2.1 of [17].

Remark 2.3. If we choose $m_1 = m_2 = 1$ we recapture Lemma 1 of [10].

The following results can be obtained by using Lemma 2.1.

Theorem 2.4. *Suppose $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping in the $K^0 \subset \mathbb{R}$ provided that $g' \in L[m_2 j, m_2 k]$, where $m_1, m_2 j, m_2 k \in K$ with $m_2 j < m_2 k$. If $|g'|$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex in the $[m_2 j, m_2 k]$, then*

$$\left| \frac{(m_2 k - m_1 y)g(m_2 k) + (m_1 y - m_2 j)g(m_2 j)}{k - j} - \frac{1}{k - j} \int_{m_2 j}^{m_2 k} g(u)du \right|$$

$$\leq \frac{(m_1 y - m_2 j)^2}{k - j} \left[\frac{m_1 |g'(y)|}{(\alpha\gamma + 1)(\alpha\gamma + 2)} + \frac{m_2 |g'(j)|}{\beta} \left(B\left(\frac{1}{\beta}, \mu + 1\right) - B\left(\frac{2}{\beta}, \mu + 1\right) \right) \right]$$

$$+ \frac{(m_2 k - m_1 y)^2}{k - j} \left[\frac{m_1 |g'(y)|}{(\alpha\gamma + 1)(\alpha\gamma + 2)} + \frac{m_2 |g'(k)|}{\beta} \left(B\left(\frac{1}{\beta}, \mu + 1\right) - B\left(\frac{2}{\beta}, \mu + 1\right) \right) \right]$$

for every $g \in [m_2 j, m_2 k]$ and $\beta > 0$.

Proof. By Lemma 2.1 and using the modulus, we have

$$\left| \frac{(m_2 k - m_1 y)g(m_2 k) + (m_1 y - m_2 j)g(m_2 j)}{k - j} - \frac{1}{k - j} \int_{m_2 j}^{m_2 k} g(u)du \right|$$

$$\leq \frac{(m_1 y - m_2 j)^2}{k - j} \int_0^1 (1 - \tau)|g'(m_1 \tau y + m_2 (1 - \tau)j)|d\tau$$

$$+ \frac{(m_2 k - m_1 y)^2}{k - j} \int_0^1 (1 - \tau)|g'(m_1 \tau y + m_2 (1 - \tau)k)|d\tau$$

Since $|f'|$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex in mixed kind, then we get

$$\left| \frac{(m_2 k - m_1 y)g(m_2 k) + (m_1 y - m_2 j)g(m_2 j)}{k - j} - \frac{1}{k - j} \int_{m_2 j}^{m_2 k} g(u)du \right|$$

$$\leq \frac{(m_1 y - m_2 j)^2}{k - j} \int_0^1 (1 - \tau)\left[m_1 \tau^{\alpha\gamma}|g'(y)| + m_2 (1 - \tau^\beta)^\mu |g'(j)| \right]d\tau$$

$$+ \frac{(m_2 k - m_1 y)^2}{k - j} \int_0^1 (1 - \tau)\left[m_1 \tau^{\alpha\gamma}|g'(y)| + m_2 (1 - \tau^\beta)^\mu |g'(k)| \right]d\tau$$

$$= \frac{(m_1 y - m_2 j)^2}{k - j} \left[\frac{m_1 |g'(y)|}{(\alpha\gamma + 1)(\alpha\gamma + 2)} + \frac{m_2 |g'(j)|}{\beta} \left(B\left(\frac{1}{\beta}, \mu + 1\right) - B\left(\frac{2}{\beta}, \mu + 1\right) \right) \right]$$

$$+ \frac{(m_2 k - m_1 y)^2}{k - j} \left[\frac{m_1 |g'(y)|}{(\alpha\gamma + 1)(\alpha\gamma + 2)} + \frac{m_2 |g'(k)|}{\beta} \left(B\left(\frac{1}{\beta}, \mu + 1\right) - B\left(\frac{2}{\beta}, \mu + 1\right) \right) \right]$$

which completes the proof.

Note: Where B is Beta function and it is stated as $B(l, m) = \int_0^1 \tau^{l-1}(1 - \tau)^{m-1}d\tau = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l + m)}$. Since $\Gamma(l) = \int_0^\infty e^{-u}u^{l-1}du$. $\square$

Remark 2.5. "Some remarks about Theorem 2.4 are following as special cases:

    (i) If we choose $\beta = \gamma = 1$ and $\mu = \beta$ in Theorem 2.4, we can get inequality for $(m_1, m_2, \alpha, \beta)-$convex function in the 2nd kind.

(ii) If we choose $\gamma = \mu = 1$ in Theorem 2.4, we can get inequality for $(m_1, m_2, \alpha, \beta)-$ convex function in the 1st kind.

(iii) If we choose $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in Theorem 2.4, we can get inequality for (m_1, m_2, s, r) $-$convex function in mixed kind.

(iv) If we choose $\alpha = \mu = s$ and $\beta = \gamma = 1$ in Theorem 2.4, we can get inequality for $(m_1, m_2, s)-$convex function in the 2nd kind.

(v) If we choose $\gamma = s$ and $\alpha = \beta = \mu = 1$ in Theorem 2.4, we can get inequality for $(m_1, m_2, s)-$convex function in the 1st kind.

(vi) If we choose $m_1 = 1$, $m_2 = m$, $\beta = \gamma = 1$ and $\mu = \beta$ in Theorem 2.4, we can get inequality for $(m, \alpha, \beta)-$convex function in the 2nd kind.

(vii) If we choose $m_1 = 1$, $m_2 = m$ and $\gamma = \mu = 1$ in Theorem 2.4, we can get inequality for (m, α, β) $-$convex function in the 1st kind.

(viii) If we choose $m_1 = 1$, $m_2 = m$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in Theorem 2.4, we can get inequality for $(m, s, r)-$convex function in the mixed kind.

(ix) If we choose $m_1 = m_2 = 1$, $\beta = \gamma = 1$ and $\mu = \beta$ in Theorem 2.4, we can get inequality for $(\alpha, \beta)-$convex function in the 2nd kind.

(x) If we choose $m_1 = m_2 = 1$ and $\gamma = \mu = 1$ in Theorem 2.4, we can get inequality for $(\alpha, \beta)-$convex function in the 1st kind.

(xi) If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in Theorem 2.4, we can get inequality for $(m, s)-$convex function in the 2nd kind.

(xii) If we choose $\alpha = \beta = \gamma = \mu = 1$ in Theorem 2.4, we can get inequality for $(m_1, m_2)-$convex function.

(xiii) If we choose $m_1 = 1$, $m_2 = m$ and $\alpha = \beta = \gamma = \mu = 1$ in Theorem 2.4, we can get inequality for $m-$convex function.

(xiv) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in Theorem 2.4, we can get inequality for $s-$convex function in the 2nd kind.

(xv) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = s$ and $\gamma = \mu = 1$ in Theorem 2.4, we can get inequality for $s-$convex function in the 1st kind.

(xvi) If we choose $m_1 = m_2 = 1$, $\gamma = s$ and $\alpha = \beta = \mu = 1$ in Theorem 2.4, we can get inequality for $s-$convex function in the 1st kind.

(xvii) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = 0$, and $\gamma = \mu = 1$ in Theorem 2.4, we can get inequality for quasi-convex function.

(xviii) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = 0$ and $\beta = \gamma = 1$ in Theorem 2.4, we can get inequality for $P-$convex function.

Remark 2.6. If we choose $m_1 = 1$ and $m_2 = m$ in Theorem 2.4, we recapture the main Theorem 2.3 of [17].

Remark 2.7. If we choose $m_1 = m_2 = 1$ in Theorem 2.4, we recapture the main Theorem 2.2 of [16].

Remark 2.8. If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$, $\beta = 1$ and $\gamma = r$, where $s, r \in (0, 1]$ in Theorem 2.4, we recapture the main Theorem 2.2 of [14].

Remark 2.9. If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in Theorem 2.4, we recapture the main Theorem 4 of [10]".

Corollary 2.10. *If we select* $y = \frac{m_2 j + m_2 k}{2}$ *in Theorem 2.4, then*

$$\left| \frac{(2m_2 k - m_1 m_2 j - m_1 m_2 k)g(m_2 k) + (m_1 m_2 j + m_1 m_2 k - 2m_2 j)g(m_2 j)}{2(k-j)} \right.$$

$$\left. - \frac{1}{k-j} \int_{m_2 j}^{m_2 k} g(u) du \right|$$

$$\leq \frac{(m_1 m_2 j + m_1 m_2 k - 2m_2 j)g(m_2 j)}{2(k-j)}$$

$$\times \left[\frac{m_1 |g'(\frac{m_2 j + m_2 k}{2})|}{(\alpha\gamma + 1)(\alpha\gamma + 2)} + \frac{m_2 |g'(j)|}{\beta} \left(B\left(\frac{1}{\beta}, \mu + 1\right) - B\left(\frac{2}{\beta}, \mu + 1\right) \right) \right]$$

$$+ \frac{(2m_2 k - m_1 m_2 j - m_1 m_2 k)g(m_2 k)}{2(k-j)}$$

$$\times \left[\frac{m_1 |g'(\frac{m_2 j + m_2 k}{2})|}{(\alpha\gamma + 1)(\alpha\gamma + 2)} + \frac{m_2 |g'(k)|}{\beta} \left(B\left(\frac{1}{\beta}, \mu + 1\right) - B\left(\frac{2}{\beta}, \mu + 1\right) \right) \right]$$

Remark 2.11. "Some remarks about Corollary 2.10 are following as special cases:

(i) In Corollary 2.10, using the convexity of $|g'|$, then we can get established inequality (1.25) (recapture Theorem 2.2 of [5]).

(ii) If we choose $\beta = \gamma = 1$ and $\mu = \beta$ in Corollary 2.10, we can get inequality for $(m_1, m_2, \alpha, \beta)-$convex function in the 2nd kind.

(iii) If we choose $\gamma = \mu = 1$ in Corollary 2.10, we can get inequality for $(m_1, m_2, \alpha, \beta)-$ convex function in the 1st kind.

(iv) If we choose $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in Corollary 2.10, we can get inequality for $(m_1, m_2, s, r)-$convex function in mixed kind.

(v) If we choose $\alpha = \mu = s$ and $\beta = \gamma = 1$ in Corollary 2.10, we can get inequality for $(m_1, m_2, s)-$convex function in the 2nd kind.

(vi) If we choose $\gamma = s$ and $\alpha = \beta = \mu = 1$ in Corollary 2.10, we can get inequality for $(m_1, m_2, s)-$convex function in the 1st kind.

(vii) If we choose $m_1 = 1$, $m_2 = m$, $\beta = \gamma = 1$ and $\mu = \beta$ in Corollary 2.10, we can get inequality for $(m, \alpha, \beta)-$convex function in the 2nd kind.

(viii) If we choose $m_1 = 1$, $m_2 = m$ and $\gamma = \mu = 1$ in Corollary 2.10, we can get inequality for $(m, \alpha, \beta)-$convex function in the 1st kind.

(ix) If we choose $m_1 = 1$, $m_2 = m$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in Corollary 2.10, we can get inequality for $(m, s, r)-$convex function in the mixed kind.

(x) If we choose $m_1 = m_2 = 1$, $\beta = \gamma = 1$ and $\mu = \beta$ in Corollary 2.10, we can get inequality for $(\alpha, \beta)-$convex function in the 2nd kind.

(xi) If we choose $m_1 = m_2 = 1$ and $\gamma = \mu = 1$ in Corollary 2.10, we can get inequality for $(\alpha, \beta)-$convex function in the 1st kind.

(xii) If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in Corollary 2.10, we can get inequality for $(m, s)-$convex function in the 2nd kind.

(xiii) If we choose $\alpha = \beta = \gamma = \mu = 1$ in Corollary 2.10, we can get inequality for $(m_1, m_2)-$convex function.

(xiv) If we choose $m_1 = 1$, $m_2 = m$ and $\alpha = \beta = \gamma = \mu = 1$ in Corollary 2.10, we can get inequality for $m-$convex function.

(xv) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in Corollary 2.10, we can get inequality for $s-$convex function in the 2nd kind.

(xvi) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = s$ and $\gamma = \mu = 1$ in Corollary 2.10, we can get inequality for $s-$convex function in the 1st kind.

(xvii) If we choose $m_1 = m_2 = 1$, $\gamma = s$ and $\alpha = \beta = \mu = 1$ in Corollary 2.10, we can get inequality for $s-$convex function in the 1st kind.

(xviii) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = 0$, and $\gamma = \mu = 1$ in Corollary 2.10,we can get inequality for quasi-convex function.

(xix) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = 0$ and $\beta = \gamma = 1$ in Corollary 2.10, we can get inequality for $P-$convex function.

Remark 2.12. If we choose $m_1 = 1$ and $m_2 = m$ in Corollary 2.10, we recapture the Corollary 2.8 of [17].

Remark 2.13. If we choose $m_1 = m_2 = 1$ in Corollary 2.10, we recapture the Corollary 2.6 of [16].

Remark 2.14. If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$, $\beta = 1$ and $\gamma = r$, where $s, r \in (0, 1]$ in Corollary 2.10, we recapture the Corollary 2.5 of [14].

Remark 2.15. If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in Corollary 2.10, we recapture the Corollary 2 of [10]".

Theorem 2.16. *Suppose $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping in the $K^0 \subset \mathbb{R}$ provided that $g' \in L[m_2j, m_2k]$, where $m_1, m_2j, m_2k \in K$ with $m_2j < m_2k$. If $|g'|^{\frac{p}{p-1}}$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex in the $[m_2j, m_2k]$ and for some fixed $q > 1$, then*

$$\left| \frac{(m_2k - m_1y)g(m_2k) + (m_1y - m_2j)g(m_2j)}{k - j} - \frac{1}{k - j} \int_{m_2j}^{m_2k} g(u)du \right|$$

$$\leq \frac{1}{k - j} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[(m_1y - m_2j)^2 \left(\frac{m_1|g'(y)|^q}{\alpha\gamma + 1} + \frac{m_2|g'(j)|^q}{\beta} B\left(\frac{1}{\beta}, \mu + 1 \right) \right)^{\frac{1}{q}} \right.$$

$$\left. + (m_2k - m_1y)^2 \left(\frac{m_1|g'(y)|^q}{\alpha\gamma + 1} + \frac{m_2|g'(k)|^q}{\beta} B\left(\frac{1}{\beta}, \mu + 1 \right) \right)^{\frac{1}{q}} \right]$$

for every $g \in [m_2j, m_2k]$ and $\beta > 0$.

Proof. By Lemma 2.1 and $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convexity of $|g'|$ and then applying the wel-known Hölder inequality, we have

$$\left| \frac{(m_2k - m_1y)g(m_2k) + (m_1y - m_2j)g(m_2j)}{k - j} - \frac{1}{k - j} \int_{m_2j}^{m_2k} g(u)du \right|$$

$$\leq \frac{(m_1y - m_2j)^2}{k - j} \int_0^1 (1 - \tau)|g'(m_1\tau y + m_2(1 - \tau)j)|d\tau$$

$$+ \frac{(m_2k - m_1y)^2}{k - j} \int_0^1 (1 - \tau)|g'(m_1\tau y + m_2(1 - \tau)k)|d\tau$$

$$\leq \frac{(m_1y - m_2j)^2}{k - j} \int_0^1 (1 - \tau) \left[m_1\tau^{\alpha\gamma}|g'(y)| + m_2(1 - \tau^\beta)^\mu|g'(j)| \right] d\tau$$

$$+ \frac{(m_2k - m_1y)^2}{k - j} \int_0^1 (1 - \tau) \left[m_1\tau^{\alpha\gamma}|g'(y)| + m_2(1 - \tau^\beta)^\mu|g'(k)| \right] d\tau$$

$$\leq \frac{(m_1 y - m_2 j)^2}{k-j}\left(\int_0^1 (1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[\int_0^1 \left(m_1 \tau^{\alpha\gamma}|g'(y)| + m_2(1-\tau^\beta)^\mu|g'(j)|\right)^q d\tau\right]^{\frac{1}{p}}$$

$$+ \frac{(m_2 k - m_1 y)^2}{k-j}\left(\int_0^1 (1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[\int_0^1 \left(m_1 \tau^{\alpha\gamma}|g'(y)| + m_2(1-\tau^\beta)^\mu|g'(k)|\right)^q d\tau\right]^{\frac{1}{p}}$$

$$\leq \frac{(m_1 y - m_2 j)^2}{k-j}\left(\int_0^1 (1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[m_1 \int_0^1 \tau^{\alpha\gamma}|g'(y)|^q d\tau + m_2 \int_0^1 (1-\tau^\beta)^\mu|g'(j)|^q d\tau\right]^{\frac{1}{p}}$$

$$+ \frac{(m_2 k - m_1 y)^2}{k-j}\left(\int_0^1 (1-\tau)^p d\tau\right)^{\frac{1}{p}}\left[m_1 \int_0^1 \tau^{\alpha\gamma}|g'(y)|^q d\tau + m_2 \int_0^1 (1-\tau^\beta)^\mu|g'(k)|^q d\tau\right]^{\frac{1}{p}}$$

$$\leq \frac{1}{k-j}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(m_1 y - m_2 j)^2\left(\frac{m_1|g'(y)|^q}{\alpha\gamma+1} + \frac{m_2|g'(j)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}\right.$$

$$\left. + (m_2 k - m_1 y)^2\left(\frac{m_1|g'(y)|^q}{\alpha\gamma+1} + \frac{m_2|g'(k)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}\right]$$

$\square$

Remark 2.17. "All remarks hold for Theorem 2.16 as we have given remarks (i) to (xviii) for Theorem 2.4.

Remark 2.18. If we choose $m_1 = 1$ and $m_2 = m$ in Theorem 2.16, we recapture the main Theorem 2.13 of [17].

Remark 2.19. If we choose $m_1 = m_2 = 1$ in Theorem 2.16, we recapture the main Theorem 2.10 of [16].

Remark 2.20. If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$, $\beta = 1$ and $\gamma = r$, where $s, r \in (0,1]$ in Theorem 2.16, we recapture the Theorem 2.8 of [14].

Remark 2.21. If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in Theorem 2.16, we recapture the Theorem 5 of [10]".

Corollary 2.22. *If we select* $y = \frac{m_2 j + m_2 k}{2}$ *in Theorem* 2.16, *we get*

$$\left|\frac{(2m_2 k - m_1 m_2 j - m_1 m_2 k)g(m_2 k) + (m_1 m_2 j + m_1 m_2 k - 2m_2 j)g(m_2 j)}{2(k-j)}\right.$$

$$\left. - \frac{1}{k-j}\int_{m_2 j}^{m_2 k} g(u)du\right|$$

$$\leq \frac{1}{4(k-j)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(m_1 m_2 j + m_1 m_2 k - 2m_2 j)^2\left(\frac{m_1|g'(\frac{m_2 j + m_2 k}{2})|^q}{\alpha\gamma+1}\right.\right.$$

$$\left.+ \frac{m_2|g'(j)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}} + (2m_2 k - m_1 m_2 j - m_1 m_2 k)^2\left(\frac{m_1|g'(\frac{m_2 j + m_2 k}{2})|^q}{\alpha\gamma+1}\right.$$

$$\left.\left.+ \frac{m_2|g'(k)|^q}{\beta}B\left(\frac{1}{\beta},\mu+1\right)\right)^{\frac{1}{q}}\right]$$

Remark 2.23. "All remarks hold for Corollary 2.22 as we have given remarks (ii) to (xix) for Corollary 2.10.

Remark 2.24. If we choose $m_1 = 1$ and $m_2 = m$ in Corollary 2.22, we recapture the Corollary 2.18 of [17].

Remark 2.25. If we choose $m_1 = m_2 = 1$ in Corollary 2.22, we recapture the Corollary 2.14 of [16].

Remark 2.26. If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$, $\beta = 1$ and $\gamma = r$, where $s, r \in (0, 1]$ in Corollary 2.22, we recapture the Corollary 2.11 of [14].

Remark 2.27. If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in Corollary 2.22, we recapture the Corollary 3 of [10]".

Theorem 2.28. *Suppose $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping in the $K^0 \subset \mathbb{R}$ provided that $g' \in L[m_2 j, m_2 k]$, where $m_1, m_2 j, m_2 k \in K$ with $m_2 j < m_2 k$. If $|g'|^q$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex in the $[m_2 j, m_2 k]$ and for some fixed $q \geq 1$, then*

$$\left| \frac{(m_2 k - m_1 y) g(m_2 k) + (m_1 y - m_2 j) g(m_2 j)}{k - j} - \frac{1}{k - j} \int_{m_2 j}^{m_2 k} g(u) du \right|$$

$$\leq \frac{1}{2^{1-\frac{1}{q}} (k - j)} \left[(m_1 y - m_2 j)^2 \left(\frac{m_1 |g'(y)|^q}{(\alpha\gamma + 1)(\alpha\gamma + 2)} \right. \right.$$

$$\left. + \frac{m_2 |g'(j)|^q}{\beta} \left(B\left(\frac{1}{\beta}, \mu + 1 \right) - B\left(\frac{2}{\beta}, \mu + 1 \right) \right) \right)^{\frac{1}{q}} + (m_2 k - m_1 y)^2$$

$$\left. \times \left(\frac{m_1 |g'(y)|^q}{(\alpha\gamma + 1)(\alpha\gamma + 2)} + \frac{m_2 |g'(k)|^q}{\beta} \left(B\left(\frac{1}{\beta}, \mu + 1 \right) - B\left(\frac{2}{\beta}, \mu + 1 \right) \right) \right)^{\frac{1}{q}} \right]$$

for every $g \in [m_2 j, m_2 k]$, $q = \frac{p}{p-1}$ and $\beta > 0$.

Proof. Consider $q \geq 1$ and from Lemma 2.1 and applying the wel-known power-mean inequality, we have

$$\left| \frac{(m_2 k - m_1 y) g(m_2 k) + (m_1 y - m_2 j) g(m_2 j)}{k - j} - \frac{1}{k - j} \int_{m_2 j}^{m_2 k} g(u) du \right|$$

$$\leq \frac{(m_1 y - m_2 j)^2}{k - j} \int_0^1 (1 - \tau) |g'(m_1 \tau y + m_2 (1 - \tau) j)| d\tau$$

$$+ \frac{(m_2 k - m_1 y)^2}{k - j} \int_0^1 (1 - \tau) |g'(m_1 \tau y + m_2 (1 - \tau) k)| d\tau$$

$$\leq \frac{(m_1 y - m_2 j)^2}{k - j} \left(\int_0^1 (1 - \tau) d\tau \right)^{1-\frac{1}{q}} \left(\int_0^1 (1 - \tau) |g'(m_1 \tau y + m_2 (1 - \tau) j)|^q d\tau \right)^{\frac{1}{q}}$$

$$+ \frac{(m_2 k - m_1 y)^2}{k - j} \left(\int_0^1 (1 - \tau) d\tau \right)^{1-\frac{1}{q}} \left(\int_0^1 (1 - \tau) |g'(m_1 \tau y + m_2 (1 - \tau) k)|^q d\tau \right)^{\frac{1}{q}}$$

Since $|f'|$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex in mixed kind, then we consider

$$\int_0^1 (1-\tau)|g'(m_1\tau y + m_2(1-\tau)j)|^q d\tau$$

$$\leq \int_0^1 (1-\tau)\left[m_1\tau^{\alpha\gamma}|g'(y)|^q + m_2(1-\tau^\beta)^\mu|g'(j)|^q\right]d\tau$$

$$= \frac{m_1|g'(y)|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m_2|g'(j)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)$$

Analogously,

$$\int_0^1 (1-\tau)|g'(m_1\tau y + m_2(1-\tau)k)|^q d\tau$$

$$\leq \frac{m_1|g'(y)|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m_2|g'(k)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right)\right)$$

We get the required result by collecting all above inequalities. $\square$

Remark 2.29. "All remarks hold for Theorem 2.28 as we have given remarks (i) to (xviii) for Theorem 2.4.

Remark 2.30. If we choose $m_1 = 1$ and $m_2 = m$ in Theorem 2.28, we recapture the main Theorem 2.23 of [17].

Remark 2.31. If we choose $m_1 = m_2 = 1$ in Theorem 2.28, we recapture the main Theorem 2.18 of [16].

Remark 2.32. If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$, $\beta = 1$ and $\gamma = r$, where $s, r \in (0, 1]$ in Theorem 2.28, we recapture the Theorem 2.14 of [14].

Remark 2.33. If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in Theorem 2.28, we recapture the Theorem 7 of [10]".

Corollary 2.34. *If we select $y = \frac{m_2 j + m_2 k}{2}$ in Theorem 2.28, we get*

$$\left| \frac{(2m_2 k - m_1 m_2 j - m_1 m_2 k)g(m_2 k) + (m_1 m_2 j + m_1 m_2 k - 2m_2 j)g(m_2 j)}{2(k-j)} \right.$$

$$\left. - \frac{1}{k-j}\int_{m_2 j}^{m_2 k} g(u)du \right|$$

$$\leq \frac{2^{\frac{1}{q}-3}}{(k-j)}\left[(m_1 m_2 j + m_1 m_2 k - 2m_2 j)^2 \left(\frac{m_1|g'(\frac{m_2 j + m_2 k}{2})|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} \right. \right.$$

$$\left. + \frac{m_2|g'(j)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right) \right) \right)^{\frac{1}{q}}$$

$$+ (2m_2 k - m_1 m_2 j - m_1 m_2 k)^2 \left(\frac{m_1|g'(\frac{m_2 j + m_2 k}{2})|^q}{(\alpha\gamma+1)(\alpha\gamma+2)} \right.$$

$$\left. \left. + \frac{m_2|g'(k)|^q}{\beta}\left(B\left(\frac{1}{\beta},\mu+1\right) - B\left(\frac{2}{\beta},\mu+1\right) \right) \right)^{\frac{1}{q}} \right]$$

Remark 2.35. "All remarks hold for Corollary 2.34 as we have given remarks (ii) to (xix) for Corollary 2.10.

Remark 2.36. If we choose $m_1 = 1$ and $m_2 = m$ in Corollary 2.34, we recapture the Corollary 2.28 of [17].

Remark 2.37. If we choose $m_1 = m_2 = 1$ in Corollary 2.34, we recapture the Corollary 2.22 of [16].

Remark 2.38. If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$, $\beta = 1$ and $\gamma = r$, where $s, r \in (0, 1]$ in Corollary 2.34, we recapture the Corollary 2.17 of [14].

Remark 2.39. If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in Corollary 2.34, we recapture the Corollary 4 of [10]".

3. Application to Numerical Integration

3.1. **The Trapezoidal Formula.** Suppose $d : m_2j = m_2\theta_0 < m_2\theta_1 < \cdots < m_2\theta_n = m_2k$ is division of $[m_2j, m_2k]$ and $h_i = \theta_{i+1} - \theta_i$, $(i = 0, 1, \cdots, n-1)$ and consider the quadrature formula

$$\int_{m_2j}^{m_2k} g(u)du = Q(g, d) + R(g, d), \tag{3.1}$$

where

$$Q(g, d) = \sum_{i=0}^{n-1} \left((m_2\theta_{i+1} - m_1y)g(m_2\theta_{i+1}) + (m_1y - m_2\theta_i)g(m_2\theta_i) \right)$$

for trapezoidal version and $R(g, d)$ represents the associated approximation error.

Theorem 3.1. *Suppose the assumptions of Theorem 2.4 is true, for each division $'d'$ of $[m_2j, m_2k]$. Then in (3.1), the trapezoidal error estimate satisfies:*

$$\left| R(g, d) \right| \leq \sum_{i=0}^{n-1} (m_1y - m_2\theta_i)^2 \left[\frac{m_1|g'(y)|}{(\alpha\gamma + 1)(\alpha\gamma + 2)} \right.$$

$$+ \frac{m_2|g'(\theta_i)|}{\beta} \left(B\left(\frac{1}{\beta}, \mu + 1\right) - B\left(\frac{2}{\beta}, \mu + 1\right) \right) \right]$$

$$+ \sum_{i=0}^{n-1} (m_2\theta_{i+1} - m_1y)^2 \left[\frac{m_1|g'(y)|}{(\alpha\gamma + 1)(\alpha\gamma + 2)} \right.$$

$$+ \frac{m_2|g'(\theta_{i+1})|}{\beta} \left(B\left(\frac{1}{\beta}, \mu + 1\right) - B\left(\frac{2}{\beta}, \mu + 1\right) \right) \right]$$

for every $g \in [m_2j, m_2k]$ and $\beta > 0$.

Proof. Using Theorem 2.4 on the sub-interval $[m_2\theta_i, m_2\theta_{i+1}]$ $(i = 0, 1, ..., n-1)$, then

$$\left| \frac{(m_2\theta_{i+1} - m_1y)g(m_2\theta_{i+1}) + (m_1y - m_2\theta_i)g(m_2\theta_i)}{h_i} - \frac{1}{h_i}\int_{m_2\theta_i}^{m_2\theta_{i+1}} g(u)du \right|$$

$$\leq \frac{(m_1y - m_2\theta_i)^2}{h_i}\left[\frac{m_1|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m_2|g'(\theta_i)|}{\beta}\left(B\left(\frac{1}{\beta}, \mu+1\right) - B\left(\frac{2}{\beta}, \mu+1\right) \right) \right]$$

$$+ \frac{(m_2\theta_{i+1} - m_1y)^2}{h_i}\left[\frac{m_1|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m_2|g'(\theta_{i+1})|}{\beta}\left(B\left(\frac{1}{\beta}, \mu+1\right) - B\left(\frac{2}{\beta}, \mu+1\right) \right) \right]$$

Hence in (3.1) we have to sum over i from 0 to $n-1$, then

$$\left| \int_{m_2j}^{m_2k} g(u)du - Q(g,d) \right| = \left| \sum_{i=0}^{n-1}\left[\int_{m_2\theta_i}^{m_2\theta_{i+1}} g(u)du - \left((m_2\theta_{i+1} - m_1y)g(m_2\theta_{i+1}) \right.\right.\right.$$

$$\left.\left.\left. + (m_1y - m_2\theta_i)g(m_2\theta_i) \right) \right] \right|$$

$$\leq \sum_{i=0}^{n-1}\left| \int_{m_2\theta_i}^{m_2\theta_{i+1}} g(u)du - \left((m_2\theta_{i+1} - m_1y)g(m_2\theta_{i+1}) + (m_1y - m_2\theta_i)g(m_2\theta_i) \right) \right|$$

$$\leq \sum_{i=0}^{n-1}(m_1y - m_2\theta_i)^2\left[\frac{m_1|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m_2|g'(\theta_i)|}{\beta}\left(B\left(\frac{1}{\beta}, \mu+1\right) - B\left(\frac{2}{\beta}, \mu+1\right) \right) \right]$$

$$+ \sum_{i=0}^{n-1}(m_2\theta_{i+1} - m_1y)^2\left[\frac{m_2|g'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m_2|g'(\theta_{i+1})|}{\beta}\left(B\left(\frac{1}{\beta}, \mu+1\right) - B\left(\frac{2}{\beta}, \mu+1\right) \right) \right]$$

Which completes the proof. $\square$

Remark 3.2. We can get above similar results for Corollary 2.10, Theorem 2.16, Corollary 2.22, Theorem 2.28 and Corollary 2.34.

Remark 3.3. All remarks of section 2 are also hold for Theorem 3.1 and for Remark 3.2.

4. Applications to Probability Theory

Suppose Y is a random variable choosing values in the finite interval $[m_2j, m_2k]$, with the probability density function $g : [m_2j, m_2k] \to [0,1]$ and with the cumulative distribution function $G(y) = P(Y \leq y) = \int_{m_2j}^{m_2k} g(u)du$.

Theorem 4.1. *Suppose the assumptions of Theorem* 2.4 *is true, then*

$$\left| \frac{(m_2k - m_1y)G(m_2k) + (m_1y - m_2j)G(m_2j)}{k-j} - \frac{m_2k - E(Y)}{k-j} \right|$$

$$\leq \frac{(m_1y - m_2j)^2}{k-j}\left[\frac{m_1|G'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m_2|G'(j)|}{\beta}\left(B\left(\frac{1}{\beta}, \mu+1\right) - B\left(\frac{2}{\beta}, \mu+1\right) \right) \right]$$

$$+ \frac{(m_2k - m_1y)^2}{k-j}\left[\frac{m_1|G'(y)|}{(\alpha\gamma+1)(\alpha\gamma+2)} + \frac{m_2|G'(k)|}{\beta}\left(B\left(\frac{1}{\beta}, \mu+1\right) - B\left(\frac{2}{\beta}, \mu+1\right) \right) \right]$$

$$(4.1)$$

for every $g \in [m_2j, m_2k]$ and $\beta > 0$. Where $E(Y)$ is the expectation of Y.

Proof. Select $g = G$, we obtain (4.1), by applying the identity

$$E(Y) = \int_{m_2j}^{m_2k} uG(u)du = m_2k - \int_{m_2j}^{m_2k} G(u)du.$$

Since $G(m_2j) = 0$ and $G(m_2k) = 1$.
We left the details to research scholars. $\qquad\square$

Remark 4.2. We can get above similar results for Corollary 2.10, Theorem 2.16, Corollary 2.22, Theorem 2.28 and Corollary 2.34.

Remark 4.3. All remarks of section 2 are also hold for Theorem 4.1 and for Remark 4.2.

5. **Conclusion**

In this article, we have generalised some results about famous Hermite-Hadamard type inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex functions in mixed kind via classical integrals using Hölder inequality and Power-mean inequality and applications are also provided for probability theory and numerical integration. We have re-captured various results of different articles [5, 10, 14, 16, 17] and also provided special cases of class of $(m_1, m_2, \alpha, \beta, \gamma, \mu)-$convex function on several choices of $m_1, m_2, \alpha, \beta, \gamma, \mu$ as remarks.

Declarations

Funding
Not applicable.

Data availability
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Contribution
Author has contributed the main idea of manuscript and wrote the whole manuscript and applications and also reviewed the complete manuscript.

References

[1] M. ALOMARI, M. DARUS AND S. S. DRAGOMIR, Inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are quasi-convex, *RGMIA Res. Rep. Coll.*, **12** (2009), 11 pages.

[2] E. F. BECKENBACH, Convex functions, *Bull. Amer. Math. Soc.*, **54** (1948), 439—460.

[3] S. S. DRAGOMIR, On some new inequalities of Hermite-Hadamard type for $m-$convex functions, *Tamkang J. Math.*, **33** (2002), 55–65.

[4] S. S. DRAGOMIR, J. PEČARIĆ AND L. PERSSON, Some inequalities of Hadamard type, *Soochow J. Math.*, **21**(3) (1995), 335—341.

[5] S. S. DRAGOMIR AND R. P. AGARWAL, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.*, **11**(5) (1998), 91—95.

[6] A. EKINCI, Klasik Eşitsizlikler Yoluyla Konveks Fonksiyonlar için Integral Eşitsizlikler, Ph.D. Thesis, Thesis ID: 361162 in tez2.yok.gov.tr Atatürk University, 2014.

[7] E. K. GODUNOVA AND V. I. LEVIN, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions, *Numerical Mathematics and Mathematical Physics*, (Russian), **166** (1985), 138–142.

[8] ALI HASSAN AND ASIF R. KHAN, Generalized Fractional Ostrowski Type Inequalities Via $(\alpha, \beta, \gamma, \delta)-$Convex Functions, *Fractional Differential Calculus*, **12**(1) (2022), 13–36.

[9] HURIYE KADAKAL, $(\alpha, m_1, m_2)-$Convexity and some Inequalities of Hermite-Hadamard Type, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, **68**(2) (2019), 2128–2142.

[10] HAVVA KAVURMACI, MERVE AVCI AND M. E. ÖZDEMIR, New inequalities of hermite-hadamard type for convex functions with applications, *Journal of Inequalities and Applications*, **2011**(86) (2011), 11 pages.

[11] U. S. KIRMACI, M. KLARIČIĆ BAKULA, M. E. ÖZDEMIR AND J. PEČARIĆ, Hadamard-type inequalities for s-convex functions, *Appl. Math. Comput.*, **193**(1) (2007), 26—35.

[12] U. S. KIRMACI, Improvement and further generalization of inequalities for differentiable mappings and applications, *Computers and Mathematics with Applications*, **55** (2008), 485—493.

[13] W. LIU, W. WEN AND J. PARK, Hermite-Hadamard type inequalities for $MT-$convex functions via classical integrals and fractional integrals, *J. Nonlinear Sci. Appl.*, **9** (2016), 766–777.

[14] FARAZ MEHMOOD, FAISAL NAWAZ AND AKHMADJON SOLEEV, Generalized Hermite-Hadamard Type Inequalities for $(s, r)-$convex functions in Mixed Kind with Applications, *J. Math. Computer Sci.*, *J. Math. Computer Sci.*, **30**(4) (2023), 372–380.

[15] FARAZ MEHMOOD, ALI HASSAN, ATIF IDREES AND FAISAL NAWAZ, Ostrowski like inequalities for $(\alpha, \beta, \gamma, \delta)-$convex functions via fuzzy Riemann integrals, *J. Math. Computer Sci.*, **31**(2) (2023), 137—149.

[16] FARAZ MEHMOOD AND AKHMADJON SOLEEV, Generalisation of Hermite-Hadamard Type Inequalities for $(\alpha, \beta, \gamma, \mu)-$Convex Functions in Mixed Kind with Applications, Submitted.

[17] FARAZ MEHMOOD, ASIF R. KHAN AND AKHMADJON SOLEEV, Generalisation of Hermite-Hadamard Type Inequalities for $(m, \alpha, \beta, \gamma, \mu)-$Convex Functions in Mixed Kind with Applications, Submitted.

[18] V. G. MIHESAN, *A generalization of the convexity*, Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, (Romania), 1993.

[19] D. S. MITRINOVIĆ, I. B. LACKOVIĆ, Hermite and convexity, *Aequationes Math.*, **28**(1) (1985), 229–232.

[20] M. A. NOOR AND M. U. AWAN, Some integral inequalities for two kinds of convexities via fractional integrals, *TJMM*, **55** (2013), 129–136.

[21] ANDREW OWUSU-HEMENG, PETER KWASI SARPONG AND JOSEPH ACKORA-PRAH, The Role of Concave and Convex Functions in the Study of Linear & Non-Linear Programming, *Dama International Journal of Researchers*, **3**(5) (2018), 15–29.

[22] C. E. M. PEARCE AND J. PEČARIĆ, Inequalities for differentiable mappings with application to special means and quadrature formula, *Appl. Math. Lett.*, **13**(2) (2000), 51—55.

[23] J. E. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex Functions, Partial Ordering and Statistical Applications*, Academic Press, New York, 1991.

[24] M. Z. SARIKAYA, E. SET AND M. E. ÖZDEMIR, On some new inequalities of Hadamard type involving $h-$convex functions, *Acta Math. Univ. Comenian. (N.S.)*, **79** (2010), 265–272.

1-DEPARTMENT OF MATHEMATICS, SAMARKAND STATE UNIVERSITY, UNIVERSITY BOULEVARD 15, SAMARKAND 140104, UZBEKISTAN
Email address: faraz.mehmood@duet.edu.pk

2-DEPARTMENT OF MATHEMATICS, DAWOOD UNIVERSITY OF ENGINEERING AND TECHNOLOGY, NEW M. A. JINNAH ROAD, KARACHI-74800, PAKISTAN