

A NEW GENERALIZATION OF THE EULER TOTIENT FUNCTION

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Abstract: In this paper, for an arithmetic function f , we define the related arithmetic function φ_f which can be considered as a generalization of the Euler totient function φ , and we study some of its properties, including multiplicativity, the Dirichlet inverse, and Dirichlet series.

Keywords: arithmetic function, Euler totient function, Dirichlet product.

1 Introduction

The Euler totient function, denoted by $\varphi(n)$, is a mathematical function that counts the positive integers up to a given integer n that are coprime with n ; thus,

$$\varphi(n) = \text{card}\{x \mid 1 \leq x < n, \gcd(x, n) = 1\}.$$

It is known that (see, e.g. [1, 2]):

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product is taken over all distinct prime factors p of n .

Euler totient function is a versatile and essential tool that plays a crucial role in various branches of mathematics and its applications extend beyond number theory into diverse areas of mathematical research and computer

science. For this reasons, it has garnered significant attention from many mathematicians, and one form of this interest is to find more than ten generalizations and analogs for it (see [3, Chapter 3.7]). For example, Schemmel totient function S_k ($k \in \mathbb{N}$) is defined as:

$$S_k(n) = \text{card}\{(x, \dots, x+k-1) \mid 1 \leq x+i < n, \gcd(x+i, n) = 1 (0 \leq i \leq k-1)\},$$

and the Jordan totient function J_k ($k \in \mathbb{N}$) is defined as:

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right).$$

Clearly, we have $S_1(n) = J_1(n) = \varphi(n)$.

In the present paper, we will introduce a new generalization, associated with an arithmetic function f , of the Euler totient function. Furthermore, we will discuss some of its properties, including multiplicativity, the Dirichlet inverse, and Dirichlet series. The following arithmetic functions are also well-known and may be used throughout this paper:

- μ, λ are the Möbius and the Liouville functions, respectively.
- \mathbf{u} is the unit function: $\mathbf{u}(n) = 1$ for all n .
- \mathbf{N}_α ($\alpha \in \mathbb{C}$) is the α -th power function: $\mathbf{N}_\alpha(n) = n^\alpha$, ($\mathbf{N}_1 = \mathbf{N}$).
- ψ is the Dedekind function: $\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$.

2 Main Result

Let f be an arithmetic function. Then we define φ_f to be the arithmetic function such that:

$$\varphi_f(n) = n \prod_{p|n} \left(1 - \frac{f(p)}{p}\right), \quad \varphi_f(1) = 1.$$

Note that if $f(p) = 1$ for all prime numbers p , then $\varphi_f = \varphi$. For example:

$$\varphi_{\mathbf{u}} = \varphi, \quad \varphi_{\mu} = \varphi_{\lambda} = \psi, \quad \text{and} \quad \varphi_{\varphi} = -\varphi_{\psi}.$$

Let n and m be positive integers and let $d = \gcd(n, m)$. Then

$$\begin{aligned} \varphi_f(mn) &= mn \prod_{p|mn} \left(1 - \frac{f(p)}{p}\right) \\ &= \frac{m \prod_{p|m} \left(1 - \frac{f(p)}{p}\right) n \prod_{p|n} \left(1 - \frac{f(p)}{p}\right)}{\prod_{p|d} \left(1 - \frac{f(p)}{p}\right)} \\ &= \varphi_f(m) \varphi_f(n) \frac{d}{\varphi_f(d)}. \end{aligned}$$

In particular, if $d = 1$ then:

$$\varphi_f(mn) = \varphi_f(m)\varphi_f(n)$$

which means that φ_f is multiplicative function, so one can have

$$\varphi_f(n) = \varphi_f\left(\prod_{i=1}^r p_i^{e_i}\right) = \prod_{i=1}^r p_i^{e_i-1}(p_i - f(p_i)) \quad (n > 1).$$

It is well-known (see, e.g. [1, p. 29]) that the Euler totient function φ satisfies the following relation:

$$\varphi = \mu * \mathbf{N}, \quad (1)$$

where $*$ represents the Dirichlet product. The next theorem establishes that φ_f satisfies a relation analogous to (1).

Theorem 1. *Let f be a multiplicative function. Then*

$$\varphi_f = \mu f * \mathbf{N}. \quad (2)$$

Proof. We note that the equality holds when $n = 1$. So, we assume that $n > 1$ and let p_i ($1 \leq i \leq r$) be the distinct prime factors of n . Thus

$$\begin{aligned} \frac{\varphi_f(n)}{n} &= \prod_{i=1}^r \left(1 - \frac{f(p_i)}{p_i}\right) \\ &= 1 - \sum \frac{f(p_i)}{p_i} + \sum \frac{f(p_i)f(p_j)}{p_i p_j} + \dots + (-1)^r \frac{f(p_1)f(p_2)\dots f(p_r)}{p_1 p_2 \dots p_r} \\ &= 1 - \sum \frac{f(p_i)}{p_i} + \sum \frac{f(p_i p_j)}{p_i p_j} + \dots + (-1)^r \frac{f(p_1 p_2 \dots p_r)}{p_1 p_2 \dots p_r} \\ &= \mu(1) + \mu(p_i) \sum \frac{f(p_i)}{p_i} + \mu(p_i p_j) \sum \frac{f(p_i p_j)}{p_i p_j} + \dots + \\ &\quad + \dots + \mu(p_1 p_2 \dots p_r) \frac{f(p_1 p_2 \dots p_r)}{p_1 p_2 \dots p_r} \\ &= \sum_{d|n} \frac{\mu f(d)}{d}, \end{aligned}$$

since $\mu(d) = 0$ if d is not a square-free integer. Note that, in a term such as $\sum \frac{f(p_i p_j)}{p_i p_j}$, it is implied that we are considering all possible products $p_i p_j$ involving distinct prime factors of n . Now we can rewrite above equalities as:

$$\varphi_f(n) = \sum_{d|n} \mu f(d) \frac{n}{d} = (\mu f * \mathbf{N})(n),$$

as claimed. The proof is complete. \square

Corollary 1. *Let f be a completely multiplicative function. Then*

$$\varphi_f = f^{-1} * \mathbf{N}, \quad (3)$$

where f^{-1} is the Dirichlet inverse of f .

Proof. The result follows at once from (2) and the fact that $f^{-1} = \mu f$ (see e.g., [1, Theorem 2.17]). \square

The Dirichlet inverse of the Euler totient function is given by (see, e.g. [1, p. 37]):

$$\varphi^{-1} = \mathbf{u} * \mu \mathbf{N}.$$

The following theorem provides the Dirichlet inverse of φ_f .

Theorem 2. *If f is multiplicative function, then*

$$\varphi_f^{-1} = (\mu f)^{-1} * \mu \mathbf{N}.$$

Proof. It is well-known that [1, 2] the identity $(g*h)^{-1} = g^{-1}*h^{-1}$ holds true for all arithmetic functions g and h . Furthermore, $\mathbf{N}^{-1} = \mu \mathbf{N}$. It follows by using these facts and Theorem 1 that:

$$\begin{aligned} \varphi_f = \mu f * \mathbf{N} &\Rightarrow \varphi_f^{-1} = (\mu f)^{-1} * \mathbf{N}^{-1} \\ &\Rightarrow \varphi_f^{-1} = (\mu f)^{-1} * \mu \mathbf{N}. \end{aligned}$$

As required, the proof is finished. \square

Corollary 2. *Let f be a completely multiplicative function and let $\prod_{i=1}^r p_i^{e_i}$ be the prime factorization of the positive integer $n > 1$. Then*

$$\varphi_f^{-1}(n) = (f * \mu \mathbf{N})(n) = \prod_{i=1}^r f(p_i)^{e_i-1} (f(p_i) - p_i).$$

Proof. Because f is completely multiplicative we have $f^{-1} = \mu f$, so

$$\begin{aligned} \varphi_f^{-1}(n) &= ((f^{-1})^{-1} * \mu \mathbf{N})(n) \\ &= (f * \mu \mathbf{N})(n) \\ &= \sum_{d|n} d\mu(d) f\left(\frac{n}{d}\right). \end{aligned}$$

We know that φ_f^{-1} is multiplicative, which means that it suffices to compute $\varphi_f^{-1}(p_i^{e_i})$ to determine $\varphi_f^{-1}(n)$. Thus

$$\begin{aligned} \varphi_f^{-1}(p_i^{e_i}) &= \sum_{d|p_i^{e_i}} d\mu(d) f\left(\frac{p_i^{e_i}}{d}\right) \\ &= \sum_{k=0}^{e_i} p_i^k \mu(p_i^k) f\left(\frac{p_i^{e_i}}{p_i^k}\right) \\ &= f(p_i^{e_i}) - p_i f(p_i^{e_i-1}) \\ &= f(p_i)^{e_i-1} (f(p_i) - p_i), \end{aligned}$$

since f is completely multiplicative. Hence

$$\varphi_f^{-1}(n) = \prod_{i=1}^r f(p_i)^{e_i-1} (f(p_i) - p_i).$$

This completes the proof. \square

Example 1. (1) If $f = \mathbf{u}$, then

$$\varphi_f^{-1}(n) = \varphi^{-1}(n) = \prod_{i=1}^r (1 - p_i).$$

(2) If $f = \lambda$, then

$$\varphi_f^{-1}(n) = \psi^{-1}(n) = \prod_{i=1}^r (-1)^{e_i} (1 + p_i).$$

(3) If $f = \mathbf{N}_\alpha$, then

$$\varphi_f^{-1}(n) = n^\alpha \prod_{i=1}^r \left(1 - \frac{1}{p_i^{\alpha-1}}\right).$$

The Dirichlet series associated with an arithmetic function f is an infinite series of the form [1, 2]:

$$D(f, s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad (s \in \mathbb{C}).$$

The most famous example is the Riemann ζ function $\zeta(s) = D(\mathbf{u}, s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ ($\Re(s) > 1$). Since $\varphi_f = \mu f * \mathbf{N}$, we have according to [1, Theorem 11.5]:

$$D(\varphi_f, s) = D(\mu f, s)D(\mathbf{N}, s), \quad (4)$$

which holds in the half-plane where both series on the right-hand side of (4) converge absolutely. In particular, if f is completely multiplicative, then we have the following theorem:

Theorem 3. Let $D(f, s)$ be the Dirichlet series associated with a completely multiplicative function f . Suppose that $D(f, s)$ is converge absolutely in the half-plane $\Re(s) > a$, then:

$$D(\varphi_f, s) = \frac{\zeta(s-1)}{D(f, s)} \quad (\Re(s) > \max\{2, a\}).$$

Proof. This follows at once from (3), (4), and [1, Examples 2,3 p. 229]. \square

Example 2. (1) If $f = \mathbf{u}$, then

$$D(\varphi_f, s) = D(\varphi, s) = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \quad (\Re(s) > 2).$$

(2) If $f = \lambda$, then

$$D(\varphi_f, s) = D(\psi, s) = \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)} \quad (\Re(s) > 2).$$

(3) If $f = \mathbf{N}_\alpha$, then

$$D(\varphi_f, s) = \sum_{n=1}^{\infty} \frac{\varphi_f(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s-\alpha)} \left(\Re(s) > \max\{2, 1 + \Re(\alpha)\} \right).$$

(4) If $f = \chi$ is a Dirichlet character, then

$$D(\varphi_f, s) = \sum_{n=1}^{\infty} \frac{\varphi_f(n)}{n^s} = \frac{\zeta(s-1)}{L(s, \chi)} \left(\Re(s) > 2 \right),$$

where $L(s, \chi)$ is the well know Dirichlet L -function associated with χ .

References

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