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PERIODIC SOLUTIONS OF THE SPATIAL EXTENSION OF A CONDITIONALLY PERIODIC SYSTEM

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Abstract. We consider a linear system of differential equations $x' = a(t)x - \mu x$ with a conditionally periodic matrix a and a parameter $\mu \in \mathbb{C}$. We prove that there exists a nonempty set $M \subset \mathbb{C}$ such that for each $\mu \in M$ the spatial periodic extension of this system, which is a system of first order partial differential equations, has a generalized (in the framework of Schwartz's theory of distributions) periodic solution.

Keywords: conditionally periodic system, quasi-periodic solution, periodic Schwartz's distribution, linear homogenous system.

0. Introduction

Consider the system of differential equations

$$x' = a(t)x - \mu x \tag{0.1_μ}$$

with a real conditionally periodic matrix $a(t) = A(et)$, where A is a continuous ω_j-periodic in φ_j (j=1,...,m) $n \times n$-matrix function of m variables $\varphi = (\varphi_1, \dots, \varphi_m) \in \mathbb{R}^m$, $\omega = (\omega_1, \dots, \omega_m)$, $e = (1, \dots, 1)$ is an m-vector, $t \in \mathbb{R}$, $\mu \in \mathbb{C}$, and the frequencies $\beta_i = 2\pi/\omega_i$ are rationally incommensurable.

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Consider also the spatial periodic extension of this system

$$\sum_{j=1}^{m} \psi'_j = A(\varphi)\psi - \mu\psi \tag{0.2_μ}$$

and remark that if system (0.2_μ) has a continuous ω_j-periodic in φ_j $(j = 1, \ldots, m)$ solution ψ, then the conditionally periodic vector $x(t) = \psi(et)$ solves system (0.1_μ).

The idea of the periodic extension of a conditionally periodic function goes back to P. Bohl [2] who was the founder of the theory of conditionally periodic functions and who proved that each continuous conditionally periodic function has the unique periodic extension.

If $m = 1$, then there exists a non-empty set $M \subset \mathbb{C}$ such that for each $\mu \in M$ system (0.1_μ) has a periodic solution.

We assume that $m > 1$. The problem is whether system (0.1_μ) has a conditionally periodic solution for some $\mu \in \mathbb{C}$ (or system (0.2_μ) has a periodic one). This is well-known to be falls, in general, even for $n = 1$. In this case $x(t) = c \exp \int_0^t (a(t) - \mu)dt$ is a solution of system (0.1_μ) with $x(0) = c$ and $\int_0^t (a(t) - \mu)dt$ may not be conditionally periodic even though μ is the mean value of a.

The aforementioned problem is related to that of existence of almost periodic solutions of a homogeneous system [1, 3, 12, 11]. In these papers some conditions have been given which ensure that bounded solutions of the homogeneous almost periodic system itself is Bohr almost periodic [1, 3, 12] or Besicovitch almost periodic [11].

Our approach is completely different.

Firstly, we assume nothing but that the matrix A is sufficiently smooth and prove that there exists a non-empty set $M \subset \mathbb{C}$ such that for any $\mu \in M$ there exists a periodic distribution ψ satisfying (0.2_μ). In a sequel to the present paper we will apply this fact to prove that for any $\mu \in M$ there exists a conditionally periodic distribution satisfying (0.1_μ)).

Secondly, we consider a conditionally periodic system, which allows us to reduce the problem to the study of difference equation (0.6). This method is unlike any attempts to solve this problem that we are aware of.

Note that we used an inhomogeneous system associated with (0.6) to prove an analogue of the Massera theorem in [10]. Namely, it was proved that if an inhomogeneous conditionally periodic system has a bounded solution, then almost every system in its H-class has a Besicovich conditionally periodic bounded solution. The proof was based on the Schauder-Tikhonov fixed point theorem. Unfortunately, we were unable to apply this theorem in the case of a homogeneous system.

Thirdly, since in the general case system (0.2_μ) does not have a continuous periodic solution, we are looking for a solution in the space of periodic distributions where apparently no one has looked for it yet.

We now present some results in [8] (see also [9]), which we are going to use. Since this paper is hardly available, we prove these results (Lemma 0.1) and Lemma 0.2 at the end of the paper.

Consider the system of integral equations

$$x(\varphi) = \int_0^{\varphi_1-\varphi_{10}} A(\varphi - e\xi)x(\varphi - e\xi)\,d\xi + x_0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})), \tag{0.3}$$

where $\hat{\varphi} = (\varphi_2, ..., \varphi_m)$, $\hat{e} = (1, \ldots, 1)$ is an $(m-1)$ -vector. For any continuous function $x^0 \colon \mathbb{R}^{m-1} \to \mathbb{R}^n$ this system has a unique continuous solution which is ω_k -periodic in φ_k if so is x^0 $(k = 2, \ldots, m)$. The vector $x = x(et; \varphi_{10}, x^0)$ is a solution of system (0.1_0) with $x(e\varphi_{10}; \varphi_{10}, x^0) = x^0(\hat{e}\varphi_{10})$.

Side by side with (0.3) it is useful to consider the associated matrix equation

$$X(\varphi) = \int_0^{\varphi_1-\varphi_{10}} A(\varphi - e\xi)X(\varphi - e\xi)\,d\xi + E, \tag{0.4}$$

where E is the identity matrix. By $X(\varphi; \varphi_{10})$ we denote the solution of this equation and $X_0(\varphi) = X(\varphi; 0)$.

Lemma 0.1 . *Let A be a continuous ω_j-periodic in φ_j (j=1,...,m) matrix function. Then matrix equation* (0.4) *has a unique continuous solution X which has the following properties.*

(a) The matrix $X(\varphi; \varphi_{10})$ is non-singular for all $\varphi \in \mathbb{R}^m$, $\varphi_{10} \in \mathbb{R}$ and ω_j-periodic in φ_j (j=2,...,m).

(b) The solution of system (0.3) *can be represented in the form*
$x(\varphi; \varphi_{10}, x^0) = X(\varphi; \varphi_{10})x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10}))$.

(c) $X(\varphi_1+\omega_1, \hat{\varphi}; \varphi_{10}) = X(\varphi; \varphi_{10})X(\varphi_{10}+\omega_1, \hat{\varphi}-\hat{e}(\varphi_1-\varphi_{10}); \varphi_{10})$*; in particular,* $X_0(\varphi_1 + \omega_1, \hat{\varphi}) = X_0(\varphi)X_0(\omega_1, \hat{\varphi} - \hat{e}\varphi_1)$ *[here the vector φ has the form $(\varphi_1, \hat{\varphi})$].*

(d) The matrix $X_0(et)$ is the normalized fundamental matrix of system (0.1_0).

Lemma 0.2. *Let the matrix A be r-times continuously differentiable on $\mathbb{R}^m$; then so is the matrix X_0.*

The matrix X_0 is a solution of a differential equation which we are going to get now. Put $\varphi_1 - \xi = \zeta$ in (0.4). Then

$$X_0(\varphi) = \int_{\varphi_{10}}^{\varphi_1} A(\zeta, \hat{\varphi} + \hat{e}(\zeta - \varphi_1))X_0(\zeta, \hat{\varphi} + \hat{e}(\zeta - \varphi_1))\,d\zeta + E.$$

Let the matrix A be continuously differentiable; then so is X_0 . Differentiating both sides of the above system w.r.t. φ_j and summing w.r.t. j from 1 to m, we obtain

$$\sum_{j=1}^{m}(X_0)'_j = A(\varphi)X_0. \tag{0.5}$$

The key role in our research is played by the system

$$X_0(\omega_1, \hat{\varphi})L(\hat{\varphi} - \hat{e}\omega_1) = \lambda L(\hat{\varphi}). \tag{0.6}$$

Suppose that $L : \mathbb{R}^{m-1} \to \mathbb{C}^n$ is a continuous ω_j-periodic in φ_j $(j = 2, \ldots, m)$ vector function which satisfies (0.6) for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $\mu = \frac{1}{\omega_1}\ln\lambda$. We claim that

$$\psi = X_0(\varphi)L(\hat{\varphi} - \hat{e}\varphi_1)\exp(-\mu\varphi_1)$$

is periodic, $\psi(et)$ is a conditionally periodic solution of (0.1_μ), and if L is differentiable, then ψ satisfies (0.2_μ). Indeed, ψ is ω_j-periodic in φ_j (j=2,...,m) since X

and L are so. Let us prove that it is periodic in φ_1. By item (c) of Lemma 0.1, we have

$$\begin{aligned} \psi(\varphi_1+\omega_1,\hat\varphi) &= X_0(\varphi_1+\omega_1,\hat\varphi)L(\hat\varphi-\hat e(\varphi_1+\omega_1))e^{-\mu\omega_1}e^{-\mu\varphi_1} \\ &= X_0(\varphi)X_0(\omega_1,\hat\varphi-\hat e\varphi_1)L(\hat\varphi-\hat e(\varphi_1+\omega_1))\lambda^{-1}e^{-\mu\varphi_1} \\ &\overset{(0.6)}{=} X_0(\varphi)L(\hat\varphi-\hat e\varphi_1)\lambda\lambda^{-1}e^{-\mu\varphi_1} = \psi(\varphi). \end{aligned} \tag{0.7}$$

In view of item (d) of Lemma 0.1, it is clear that $\psi(et) = X_0(et)L(\hat 0)e^{-\mu t}$ is a solution of (0.1_μ). That ψ solves (0.2_μ) one can check by direct calculation using (0.5) and the relation $\sum\limits_{j=1}^{m}(L(\hat\varphi-\hat e\varphi_1))'_j = 0$.

In the first section of this paper, we present the basic facts about periodic distributions and show that if there exists a periodic distribution L that solves (0.6) for some $\lambda \neq 0$, then $\psi = X_0(\varphi)L(\hat\varphi-\hat e\varphi_1)\exp(-\mu\varphi_1)$ is a generalized periodic solution of system (0.2_μ).

In the second section, we prove that there exists a non-empty set $\Lambda \subset \mathbb{C}$ such that for each $\lambda \in \Lambda$ the distribution L exists. And in the third section we prove Lemmas 0.1, 0.2.

1. Periodic distributions

The definition and properties of periodic distributions, which we use, can be found in the monographs [14, 15, 6]. We introduce vector periodic distributions following [6] (see also [4, Ch.1]).

Let $P_n^q(\omega)$ be the Banach space of ω_j-periodic in φ_j $(j = 1,\dots,m)$ q-times continuously differentiable functions $y\colon \mathbb{R}^m \to \mathbb{C}^n$ with the norm $\|y\|_{P_n^q(\omega)} = \max\limits_{|r|\le q}\max\limits_{\varphi\in\mathbb{R}^m}\|D^r y(\varphi)\|$, where $\|\cdot\|$ is a norm on a finite-dimensional space, $r = (r_1,\dots,r_m)$, r_j are non-negative integers, $D^r y(\varphi) = \frac{\partial^{|r|}y(\varphi)}{\partial\varphi_1^{r_1}\dots\partial\varphi_m^{r_m}}$, and $|r| = r_1 + \cdots + r_m$.

Let $P_{n^2}^q(\omega)$ be the Banach space of periodic q- times continuously differentiable $n\times n$ matrices with the norm $\|A\|_{P_{n^2}^q(\omega)} = \max\limits_{|s|\le n}\|a_s\|_{P_n^r(\omega)}$, where $a_s \in P_n^q(\omega)$ $(s = 1,\dots,n)$ are columns of A.

Denote by $P_n'^q(\omega)$ the space of linear continuous maps $x\colon P_n^q(\omega)\to\mathbb{C}^n$, where $x = (x_1,\dots,x_n)^T$, x^T is the transpose of x and x_k are linear continuous functionals on $P_1^q(\omega)$.

Every $x \in P_n'^q(\omega)$ determines a linear continuous functional on $P_n^q(\omega)$ by the rule $\langle x,y\rangle_\omega = \sum\limits_{k=1}^{n}\langle x_k,y_k\rangle_\omega$, where $y = (y_1,\dots,y_n)^T \in P_n^q(\omega)$ and $\langle x_k,y_k\rangle_\omega$ is the value of x_k at the point $y_k \in P_1^q(\omega)$.

The space $P_n'^q(\omega)$, endowed with the norm $\|x\|_{P_n'^q(\omega)} = \sup\limits_{\|y\|_{P_n^q(\omega)}=1}|\langle x,y\rangle_\omega|$, is the Banach space, and it is isomorphic to the dual space of $P_n^q(\omega)$ [6, p.267].

Consider the countably normed space $P_n(\omega) = \bigcap_{q=0}^\infty P_n^q(\omega)$ and its dual $P_n'(\omega) = \bigcup_{q=0}^\infty P_n'^q(\omega)$. By definition, a sequence $\{y^k\}$ in $P_n(\omega)$ converges to $y \in P_n(\omega)$ if $\|y^k - y\|_{P_n^q(\omega)} \to 0$ for every non-negative integer q, and $\{x^k\}$ in $P_n'(\omega)$ converges weakly to $x \in P_n'(\omega)$ if $\langle x^k - x, y\rangle_\omega \to 0$ for every $y \in P_n(\omega)$ as $k\to\infty$. The latter is equivalent to the fact that $x^k \to x$ weakly in some $P_n'^q(\omega)$ [4, Ch.1].

The space $P_1'(\omega)$ is isomorphic to the space of the Schwartz periodic distributions [6, 14] that is why $x \in P_1'(\omega)$ is called an *ω-periodic distribution*, and therefore $x \in P_n'(\omega)$ we call an *ω-periodic vector distribution* [15, 6]. Note that the term ω-periodic is used for brevity. In fact $x \in P_n'(\omega)$ is ω_j-periodic in φ_j $(j = 1, \dots, m)$, i.e.

$$x(\varphi_1, \dots, \varphi_{j-1}, \varphi_j + \omega_j, \varphi_{j+1}, \dots, \varphi_m) = x(\varphi).$$

Indeed, by the definitions of equality and translation (see below on this page) we have $\langle x(\varphi_1, \dots, \varphi_{j-1}, \varphi_j + \omega_j, \varphi_{j+1}, \dots, \varphi_m), y(\varphi)\rangle_\omega = \langle x(\varphi), y(\varphi_1, \dots, \varphi_{j-1}, \varphi_j - \omega_j, \varphi_{j+1}, \dots, \varphi_m)\rangle_\omega = \langle x(\varphi), y(\varphi)\rangle_\omega$ for any $y \in P_n(\omega)$.

We are going to mainly deal with the spaces of vector distributions ${P'}_n^q(\omega)$ and spaces of test functions $P_n^q(\omega)$. Suppose $x, z \in {P'}_n^q(\omega)$. Let us remember the definitions of

equality : $x = z \;\Leftrightarrow\; \langle x, y\rangle_\omega = \langle z, y\rangle_\omega, \quad \forall y \in P_n^q(\omega)$;
derivative : $\langle x_j', y\rangle_\omega = -\langle x, y_j'\rangle_\omega, \quad \forall y \in P_n^{q+1}(\omega)$;
multiplication by matrix $A \in P_{n^2}^q(\omega)$: $\langle Ax, y\rangle_\omega = \langle x, A^T y\rangle_\omega, \quad \forall y \in P_n^q(\omega)$;
translation by τ : $\langle x(\varphi - \tau), y(\varphi)\rangle_\omega = \langle x(\varphi), y(\varphi + \tau)\rangle_\omega, \quad \forall y \in P_n^q(\omega)$, $\tau \in \mathbb{R}^m$.

It was proved in the monographs [14, p.225], [15, p.130] that every $x \in P_n'(\omega)$ has the unique Fourier-series expansion

$$x(\varphi) = \sum_{k\in\mathbb{Z}^m} c_k \exp(i(k\beta\varphi)), \tag{1.1}$$

where
$c_k = (\langle x_1(\varphi), \exp(-i(k\beta\varphi))\rangle_\omega, \dots, \langle x_n(\varphi), \exp(-i(k\beta\varphi))\rangle_\omega)^T/(\omega_1 \dots \omega_m)$,
$k = (k_1, \dots, k_m) \in \mathbb{Z}^m$, $\beta = (\beta_1, \dots, \beta_m)$, $\beta_r = 2\pi/\omega_r$, $(k\beta\varphi) = \sum\limits_{j=1}^m k_j\beta_j\varphi_j$.

Besides, there exist $K > 0$ and $p \in \mathbb{N}$ such that

$$\|c_k\| \le K\|k\|^p. \tag{1.2}$$

Inversely, if there exist $K > 0$, $p \in \mathbb{N}$ such that for any $k \in \mathbb{Z}^m$ inequality (1.2) holds, then series (1.1) converges weakly and $x \in P_n'(\omega)$ [6, pp.265,266], [14, p.225], [15, p.130].

It was shown in the previous section that if (0.6) has a continuous periodic solution L, then $\psi(\varphi) = X_0(\varphi)L(\hat\varphi - \hat e\varphi_1)\exp(-\mu\varphi_1)$ is continuous and periodic. Now suppose that $L \in {P'}_n^r(\hat\omega)$ and $X_0 \in P_{n^2}^r(\omega)$ (for some non-negative integer r); then $\psi \in {P'}_n^r(\hat\omega)$ for any $\varphi_1 \in \mathbb{R}$. Likewise (see (0.7)), it can be proved that ψ is ω_1-periodic in φ_1. That is why we are going to regard ψ as a distribution given by $\langle \psi, y\rangle_\omega = \int\limits_0^{\omega_1} \langle \psi, y\rangle_{\hat\omega}\, d\varphi_1$, $y \in P_n^r(\omega)$. Let us prove that $\psi \in {P'}_n^r(\omega)$.

In this proof we use a standard fact in calculus : *Let* $y : [\alpha, \beta] \times K \to \mathbb{C}^n$ *be uniformly continuous for given* $\alpha, \beta \in \mathbb{R}$ *and* $K \subset \mathbb{R}^{m-1}$, *let* $\{\varphi_{1k}\} \subset [\alpha, \beta]$; *then* $\varphi_{1k} \to \varphi_{10}$ *implies* $y(\varphi_{1k}, \hat\varphi) \to y(\varphi_{10}, \hat\varphi)$ *uniformly for* $\hat\varphi \in K$ *as* $k \to \infty$.

Lemma 1.1. *Let* $A \in P_{n^2}^r(\omega)$, $L \in {P'}_n^r(\hat\omega)$, *and* $r \ge 0$; *then* $\psi(\varphi) = X_0(\varphi)L(\hat\varphi - \hat e\varphi_1)\exp(-\mu\varphi_1) \in {P'}_n^r(\hat\omega)$ *for any* $\varphi_1 \in \mathbb{R}$ *and the function* $g(\varphi_1) = \langle \psi, y\rangle_{\hat\omega}$ *is continuous w.r.t.* $\varphi_1 \in [0, \omega_1]$ *for every* $y \in P_n^r(\hat\omega)$.

Proof. By Lemma 0.2, it follows that $X_0 \in P_{n^2}^r(\omega)$. Hence, $\psi \in P_n'^r(\hat\omega)$ for each $\varphi_1 \in \mathbb{R}$. Put $X_{0\mu}(\varphi) = X_0(\varphi)\exp(-\mu\varphi_1)$ and take into account that then $\psi(\varphi) = X_{0\mu}(\varphi)L(\hat\varphi - \hat e\varphi_1)$ and g can be written as $g(\varphi_1) = \langle L(\hat\varphi), X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)y(\hat\varphi +$

$\hat{e}\varphi_1))_{\hat{\omega}}$. The vector $v(\varphi) = X^T_{0\mu}(\varphi_1, \hat{\varphi}+\hat{e}\varphi_1)y(\hat{\varphi}+\hat{e}\varphi_1)$ and its partial derivatives of $|j|th$ order ($|j| = 1, \dots, r$) are uniformly continuous on $[0, \omega_1] \times \mathbb{R}^{m-1}$. Therefore, if $[0, \omega_1] \ni \varphi_{1k} \to \varphi_{10} \in [0, \omega_1]$ as $k \to \infty$, then $v(\varphi_{1k}, \hat{\varphi})$ and $D^j v(\varphi_{1k}, \hat{\varphi})$ tends uniformly w.r.t. $\hat{\varphi} \in \mathbb{R}^{m-1}$ to $v(\varphi_{10}, \hat{\varphi})$ and $D^j v(\varphi_{10}, \hat{\varphi})$, respectively. From this, remembering that L is a continuous mapping, we conclude that $g(\varphi_{1k}) \to g(\varphi_{10})$, hence g is continuous. □

Lemma 1.2. *Let $A \in P^r_{n^2}(\omega)$, $L \in P'^r_n(\hat{\omega})$, and $r \geq 0$; then $\psi \in P'^r_n(\omega)$.*

Proof. First, by using the fact that the mapping $\langle \cdot, \cdot \rangle_{\hat{\omega}} \colon P'^r_n(\hat{\omega}) \times P^r_n(\hat{\omega}) \to \mathbb{C}$ is bilinear and separately continuous, we prove that $\langle \psi, y \rangle_{\hat{\omega}}$ is continuous in $\varphi_1 \in [0, \omega_1]$ if $y \in P^r_n(\omega)$.

It follows from Lemma 1.1 that the function $\psi : [0, \omega_1] \to P'^r_n(\hat{\omega})$ is continuous, i.e. $[0, \omega_1] \ni \varphi_{1k} \to \varphi_{10}$ implies

$$\langle \psi(\varphi_{1k}, \cdot), z \rangle_{\hat{\omega}} \to \langle \psi(\varphi_{10}, \cdot), z \rangle_{\hat{\omega}}, \tag{1.3}$$

for every $z \in P^r_n(\hat{\omega})$.

Besides, if $y \in P^r_n(\omega)$, then $\|y(\varphi_{1k}, \hat{\varphi}) - y(\varphi_{10}, \hat{\varphi})\|_{P^r_n(\hat{\omega})} \to 0$ since the derivatives $D^j y(\varphi)$, $0 \leq |j| \leq r$ are uniformly continuous on $\mathbb{R}^m$, and hence

$$\langle x, y(\varphi_{1k}, \cdot) \rangle_{\hat{\omega}} \to \langle x, y(\varphi_{10}, \cdot) \rangle_{\hat{\omega}}, \tag{1.4}$$

for each $x \in P'^r_n(\hat{\omega})$.

Therefore, by the bilinear mapping continuity theorem [13, Th. 2.17], in view of relations (1.3), (1.4), we get

$$\langle \psi(\varphi_{1k}, \cdot), y(\varphi_{1k}, \cdot) \rangle_{\hat{\omega}} \to \langle \psi(\varphi_{10}, \cdot), y(\varphi_{10}, \cdot) \rangle_{\hat{\omega}}, \quad y \in P^r_n(\omega).$$

This proves that $\langle \psi, y \rangle_{\hat{\omega}}$ is continuous, and hence integrable w.r.t. $\varphi_1 \in [0, \omega_1]$.

To conclude the proof, we note that

$$\Big| \int_0^{\omega_1} \langle \psi, y \rangle_{\hat{\omega}} \, d\varphi_1 \Big| \leq \omega_1 \sup_{\varphi_1 \in [0, \omega_1]} |\langle \psi, y \rangle_{\hat{\omega}}| \leq \omega_1 K \sup_{\varphi_1 \in [0, \omega_1]} \|y\|_{P^r_n(\hat{\omega})} \leq \omega_1 K \|y\|_{P^r_n(\omega)}.$$

The existence of $K = \sup_{\varphi_1 \in [0, \omega_1]} \|\psi\|_{P'^r_n(\hat{\omega})}$ follows from the Principle of Uniform Boundedness. Indeed, due to Lemma 1.1, $\langle \psi, y \rangle_{\hat{\omega}}$ is bounded in $\varphi_1 \in [0, \omega_1]$ at each point $y \in P^r_n(\hat{\omega})$. Hence, the collection $\{\langle \psi, \cdot \rangle_{\hat{\omega}} : \varphi_1 \in [0, \omega_1]\}$ of the continuous linear mappings from $P^r_n(\hat{\omega})$ into $\mathbb{C}$ is bounded. The inequality just obtained proofs the lemma. □

We say that $\psi \in P'^r_n(\omega)$ is a solution of system (0.2_μ) if

$$\langle \sum_{i=1}^m \psi'_i, y \rangle_\omega = \langle A\psi - \mu\psi, y \rangle_\omega, \tag{1.5}$$

for any $y \in P^{r+1}_n(\omega)$.

Lemma 1.3. *Let $A \in P^r_{n^2}(\omega)$, $L \in P'^r_n(\hat{\omega})$ be a solution of (0.6) for some $\lambda \neq 0$, $r \geq 1$, and $\mu = \omega_1^{-1} \ln \lambda$; then the distribution $\psi(\varphi) = X_{0\mu}(\varphi) L(\hat{\varphi} - \hat{e}\varphi_1) \in P'^r_n(\omega)$ is a solution of system (0.2_μ).*

Proof. It follows from Lemma 1.2 that $\psi \in P'^r_n(\omega)$, and hence $\psi'_j \in P'^{r+1}_n(\omega)$, $j = 1, \dots, m$.

Let $y \in P_n^{r+1}(\omega)$. We are going to use the relations

$$\begin{aligned}
&\int_0^{\omega_1} \langle L(\hat\varphi - \hat e\varphi_1), (X_{0\mu}^T(\varphi)A^T(\varphi) - \mu X_{0\mu}^T(\varphi))y(\varphi)\rangle_{\hat\omega} d\varphi_1 \\
&= \int_0^{\omega_1} \langle (A(\varphi)X_{0\mu}(\varphi) - \mu X_{0\mu}(\varphi))L(\hat\varphi - \hat e\varphi_1), y(\varphi)\rangle_{\hat\omega} d\varphi_1 \\
&= \langle A\psi - \mu\psi, y\rangle_\omega
\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}
&\partial X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)/\partial\varphi_1 = \sum_{i=1}^m (X_{0\mu}^T)_i'(\varphi_1, \hat\varphi + \hat e\varphi_1) \\
&= X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)A^T(\varphi_1, \hat\varphi + \hat e\varphi_1) - \mu X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1).
\end{aligned} \tag{1.7}$$

The latter follows from (0.5).

So, remembering that $y \in P_n^{r+1}(\omega)$, we have

$$\begin{aligned}
&\langle \sum_{i=1}^m \psi_i', y\rangle_\omega = -\langle X_{0\mu}(\varphi)L(\hat\varphi - \hat e\varphi_1), \sum_{i=1}^m y_i'(\varphi)\rangle_\omega \\
&= -\int_0^{\omega_1} \langle L(\hat\varphi - \hat e\varphi_1), X_{0\mu}^T(\varphi)\sum_{i=1}^m y_i'(\varphi)\rangle_{\hat\omega}\, d\varphi_1 \\
&= -\int_0^{\omega_1} \langle L(\hat\varphi), X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)y_1'(\varphi_1, \hat\varphi + \hat e\varphi_1)\rangle_{\hat\omega}\, d\varphi_1 \\
&-\int_0^{\omega_1} \langle L(\hat\varphi), X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)\sum_{i=2}^m y_i'(\varphi_1, \hat\varphi + \hat e\varphi_1)\rangle_{\hat\omega}\, d\varphi_1.
\end{aligned}$$

Denote the first summand at the right in the above relation by I_1, the second by I_2, and transform I_1.

$$\begin{aligned}
&I_1 = -\int_0^{\omega_1} \langle L(\hat\varphi), X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)\partial y(\varphi_1, \hat\varphi + \hat e\varphi_1)/\partial\varphi_1\rangle_{\hat\omega}\, d\varphi_1 \\
&+\int_0^{\omega_1} \langle L(\hat\varphi), X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)\sum_{i=2}^m y_i'(\varphi_1, \hat\varphi + \hat e\varphi_1)\rangle_{\hat\omega}\, d\varphi_1 \\
&= \int_0^{\omega_1} \langle L(\hat\varphi), \partial X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)/\partial\varphi_1 y(\varphi_1, \hat\varphi + \hat e\varphi_1)\rangle_{\hat\omega}\, d\varphi_1 \\
&-\int_0^{\omega_1} \langle L(\hat\varphi), \partial(X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)y(\varphi_1, \hat\varphi + \hat e\varphi_1))/\partial\varphi_1\rangle_{\hat\omega}\, d\varphi_1 \\
&+\int_0^{\omega_1} \langle L(\hat\varphi), X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)\sum_{i=2}^m y_i'(\varphi_1, \hat\varphi + \hat e\varphi_1)\rangle_{\hat\omega}\, d\varphi_1, \\
&I_2 = -\int_0^{\omega_1} \langle L(\hat\varphi), X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)\sum_{i=2}^m y_i'(\varphi_1, \hat\varphi + \hat e\varphi_1)\rangle_{\hat\omega} d\varphi_1.
\end{aligned}$$

Hence,

$$\begin{aligned}
&I_1 + I_2 = \int_0^{\omega_1} \langle L(\hat\varphi), \partial X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)/\partial\varphi_1 y(\varphi_1, \hat\varphi + \hat e\varphi_1)\rangle_{\hat\omega}\, d\varphi_1 \\
&-\int_0^{\omega_1} \langle L(\hat\varphi), \partial(X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)y(\varphi_1, \hat\varphi + \hat e\varphi_1))/\partial\varphi_1\rangle_{\hat\omega}\, d\varphi_1.
\end{aligned}$$

We denote the first integral in the previous relation by J_1 [and the second by J_2] and transform it using (1.7):

$$\begin{aligned}
&J_1 = \int_0^{\omega_1} \langle L(\hat\varphi), (X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)A^T(\varphi_1, \hat\varphi + \hat e\varphi_1) \\
&-\mu X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1))y(\varphi_1, \hat\varphi + \hat e\varphi_1)\rangle_{\hat\omega}\, d\varphi_1 = \int_0^{\omega_1} \langle L(\hat\varphi - \hat e\varphi_1), (X_{0\mu}^T(\varphi)A^T(\varphi) \\
&-\mu X_{0\mu}^T(\varphi))y(\varphi)\rangle_{\hat\omega}\, d\varphi_1 \overset{(1.6)}{=} \langle A\psi - \mu\psi, y\rangle_\omega.
\end{aligned}$$

To transform J_2, we use differentiation of $\langle\cdot,\cdot\rangle_{\hat\omega}$ w.r.t. parameter φ_1 [14, p.105]:
$\langle L(\hat\varphi), \partial(X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)y(\varphi_1, \hat\varphi + \hat e\varphi_1))/\partial\varphi_1\rangle_{\hat\omega}$
$= \partial\langle L(\hat\varphi), X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1)y(\varphi_1, \hat\varphi + \hat e\varphi_1))\rangle_{\hat\omega}/\partial\varphi_1.$

As a result we get

$$\begin{aligned} J_2 &= \int_0^{\omega_1} \partial \langle L(\hat\varphi), X_{0\mu}^T(\varphi_1, \hat\varphi + \hat e\varphi_1) y(\varphi_1, \hat\varphi + \hat e\varphi_1)\rangle_{\hat\omega} / \partial\varphi_1 \, d\varphi_1 \\ &= \langle L(\hat\varphi), (X_{0\mu}^T(\omega_1, \hat\varphi + \hat e\omega_1) y(\omega_1, \hat\varphi + \hat e\omega_1) - X_{0\mu}^T(0,\hat\varphi) y(0,\hat\varphi))\rangle_{\hat\omega} \\ &= \langle X_{0\mu}(\omega_1,\hat\varphi) L(\hat\varphi - \hat e\omega_1) - X_{0\mu}(0,\hat\varphi) L(\hat\varphi), y(0,\hat\varphi)\rangle_{\hat\omega}. \end{aligned}$$

Remembering that L satisfies (0.6), $X_{0\mu}(\omega_1,\hat\varphi) = X_0(\omega_1,\hat\varphi)\lambda^{-1}$, and $X_{0\mu}(0,\hat\varphi) = X_0(0,\hat\varphi) = E$, we get $J_2 = 0$.

Summarizing all the above, we get (1.5). □

2. Generalized solutions of systems (0.6) and ($\mathbf{0.2}_\mu$)

Consider the system

$$X_0(\omega_1,\hat\varphi) L(\hat\varphi - \alpha_s) = \lambda L(\hat\varphi), \tag{2.1}$$

where $\alpha_s = (p_{2s}\omega_2/q_{2s}, \dots, p_{ms}\omega_m/q_{ms})$, $p_{js} \in \mathbb{Z}$, $q_{js} \in \mathbb{N}$. Due to the incommensurability of the frequencies $\beta_i = 2\pi/\omega_i$ we can choose p_{js}, q_{js} such that $\alpha_s \to \hat e\omega_1$ as $s \to \infty$.

We have now to prove that, for each $s \in \mathbb{N}$, there exists a distribution L_s which solves this system for some $\lambda_s \in \mathbb{C}$ and that the sequence $\{L_s\}$ has a week partial limit L which satisfies (0.6) for some $\lambda \in \mathbb{C}$.

Lemma 2.1. *Let $A \in P^0_{n^2}(\omega)$; then for any $s \in \mathbb{N}$ there exists a distribution $L_s \in P'^0_n(\hat\omega)$ and a complex number $\lambda_s \neq 0$ which satisfy* (2.1).

Proof. Let us endeavour to satisfy (2.1) by the distribution

$$L_s(\hat\varphi) = \sum_{j=0}^{\nu_s - 1} f_{sj}\delta(\hat\varphi - j\alpha_s), \tag{2.2}$$

where ν_s is the least common multiple of q_{js} $(j = 2,\dots,m)$, f_{sj} are constant column vectors, $\delta \in P'^0_1(\hat\omega)$ is the Dirac distribution: $\langle f_{sj}\delta(\hat\varphi - \hat\varphi^0), y\rangle_{\hat\omega} = f_{sj}^T y(\hat\varphi^0)$, $y \in P^0_n(\hat\omega)$.

Note that in our case δ is a periodic functional since it is defined on the space of periodic functions.

Recall that L_s satisfies (2.1) if, for any $y \in P^0_n(\hat\omega)$, the following relation holds

$$\langle X_0(\omega_1,\hat\varphi) L_s(\hat\varphi - \alpha_s), y\rangle_{\hat\omega} = \langle \lambda_s L_s(\hat\varphi), y\rangle_{\hat\omega}.$$

Upon substituting L_s into the left and right sides of this equation, we find

$\langle X_0(\omega_1,\hat\varphi) L_s(\hat\varphi - \alpha_s), y(\hat\varphi)\rangle_{\hat\omega} = \langle L_s(\hat\varphi - \alpha_s), (X_0(\omega_1,\hat\varphi))^T y(\hat\varphi)\rangle_{\hat\omega} =$

$\langle L_s(\hat\varphi), (X_0(\omega_1,\hat\varphi + \alpha_s))^T y(\hat\varphi + \alpha_s)\rangle_{\hat\omega} = \sum_{j=0}^{\nu_s-1} \langle f_{sj}\delta(\hat\varphi - j\alpha_s), (X_0(\omega_1,\hat\varphi + \alpha_s))^T y(\hat\varphi +$

$\alpha_s)\rangle_{\hat\omega} =$

$\sum_{j=0}^{\nu_s-1} f_{sj}^T (X_0(\omega_1,(j+1)\alpha_s))^T y((j+1)\alpha_s) = \sum_{j=0}^{\nu_s-1} (X_0(\omega_1,(j+1)\alpha_s) f_{sj})^T y((j+1)\alpha_s)$,

and $\langle \lambda_s L_s(\hat\varphi), y(\hat\varphi)\rangle_{\hat\omega} = \lambda_s \sum_{j=0}^{\nu_s-1} \langle f_{sj}\delta(\hat\varphi - j\alpha_s), y(\hat\varphi)\rangle_{\hat\omega} =$

$\lambda_s \sum_{j=0}^{\nu_s-1} f_{sj}^T y(j\alpha_s)$.

Thus

$$\sum_{j=0}^{\nu_s-1}(X_0(\omega_1,(j+1)\alpha_s)f_{sj})^T y((j+1)\alpha_s) = \lambda_s \sum_{j=0}^{\nu_s-1} f_{sj}^T y(j\alpha_s).$$

This relation holds if the vectors f_{sj} satisfy the system

$$\begin{cases} X_0(\omega_1,\alpha_s)f_{s0} & = & \lambda_s f_{s1} \\ X_0(\omega_1,2\alpha_s)f_{s1} & = & \lambda_s f_{s2} \\ \dots & \dots & \dots \\ X_0(\omega_1,(\nu_s-1)\alpha_s)f_{s\nu_s-2} & = & \lambda_s f_{s\nu_s-1} \\ X_0(\omega_1,\nu_s\alpha_s)f_{s\nu_s-1} & = & \lambda_s f_{s0}. \end{cases} \tag{2.3}$$

Indeed, let $y \in P_n^0(\hat\omega)$. Then it follows from (2.3) that

$$\begin{cases} (X_0(\omega_1,\alpha_s)f_{s0})^T y(\alpha_s) & = & \lambda_s (f_{s1})^T y(\alpha_s) \\ (X_0(\omega_1,2\alpha_s)f_{s1})^T y(2\alpha_s) & = & \lambda_s (f_{s2})^T y(2\alpha_s) \\ \dots & \dots & \dots \\ (X_0(\omega_1,(\nu_s-1)\alpha_s)f_{s\nu_s-2})^T y((\nu_s-1)\alpha_s) & = & \lambda_s (f_{s\nu_s-1})^T y((\nu_s-1)\alpha_s) \\ (X_0(\omega_1,\nu_s\alpha_s)f_{s\nu_s-1})^T y(\nu_s\alpha_s) & = & \lambda_s (f_{s0})^T y(\nu_s\alpha_s). \end{cases}$$

Putting $y(\nu_s\alpha_s) = y(\hat 0)$ in the right-hand side of the last equation and summing these equations we obtain the desired relation.

It follows from (2.3) that

$$X_0(\omega_1,\nu_s\alpha_s)X_0(\omega_1,(\nu_s-1)\alpha_s)\dots X_0(\omega_1,\alpha_s)f_{s0} = \lambda_s^{\nu_s} f_{s0}. \tag{2.4}$$

Consider some root $G_s = (X_0(\omega_1,\nu_s\alpha_s)X_0(\omega_1,(\nu_s-1)\alpha_s)\dots X_0(\omega_1,\alpha_s))^{\frac{1}{\nu_s}}$, some eigenvalue λ_s of G_s, and the corresponding eigenvector f_{s0}; then λ_s and f_{s0} satisfy (2.4). Substituting λ_s and f_{s0} into (2.3), we successively find vectors $f_{sk}, k = 1,\dots,\nu_s-1$. Hence, we get the distribution L_s which is the solution of (2.1). It can be seen from (2.2) that $L_s \in {P'_n}^0(\hat\omega)$ since $f_{sj}\delta(\hat\varphi - j\alpha_s) \in {P'_n}^0(\hat\omega)$. □

Lemma 2.2. *Let $A \in P_{n^2}^0(\omega)$ and let λ_s be an eigenvalue of G_s, $s \in \mathbb{N}$; then there exists a convergent subsequence $\{\lambda_{s_k}\}$ of the sequence $\{\lambda_s\}$ with nonzero limit.*

Proof. The matrix function $X_0(\omega_1,\cdot)$ is non-singular continuous and $\hat\omega$-periodic in $\hat\varphi \in \mathbb{R}^{m-1}$. Hence, there exist $\alpha,\beta > 0$ such that for any $x \in \mathbb{R}^n$, $\hat\varphi \in \mathbb{R}^{m-1}$ we have $\alpha\|x\| \le \|X_0(\omega_1,\hat\varphi)x\| \le \beta\|x\|$. From these inequalities we get $\|G_s^{\nu_s}\| \le \beta^{\nu_s}$, $\|G_s^{-\nu_s}\| \le \alpha^{-\nu_s}$ (the matrix norm induced by the vector norm $\|\cdot\|$ is also denoted by $\|\cdot\|$). Since $\lambda_s^{\nu_s}$ and $\lambda_s^{-\nu_s}$ are eigenvalues of the matrices $G_s^{\nu_s}$ and $G_s^{-\nu_s}$ respectively, it follows that $|\lambda_s^{\nu_s}| \le \beta^{\nu_s}$, $|\lambda_s^{-\nu_s}| \le \alpha^{-\nu_s}$ and, finally, $\alpha \le |\lambda_s| \le \beta$, $s \in \mathbb{N}$. On the strength of that there exists a partial limit λ of $\{\lambda_s\}$ such that $\alpha \le |\lambda| \le \beta$. Hence, the desired subsequence exists. □

Now we want to show that there exist sequences $\{\lambda_s\}$ and $\{L_s\}$ such that $\{L_s\}$ has a weak limit L which is a solution of (0.6) with $\lambda = \lim\lambda_s$.

Consider the Furier-series of L_s:

$$L_s(\hat\varphi) = \sum_{\hat k\in\mathbb{Z}^{m-1}} t_{s\hat k}\exp(i(\hat k\hat\beta\hat\varphi)),$$

$$t_{s\hat k} = (\omega_2\dots\omega_m)^{-1}\sum_{j=0}^{\nu_s-1} f_{sj}\exp(-ij(\hat k\hat\beta\alpha_s)),\ (\hat k\hat\beta\alpha_s) = \sum_{r=2}^{m} k_r\beta_r p_{sr}\omega_r/q_{sr}.$$

Lemma 2.3. *For each $s \in \mathbb{N}$ the mapping $\hat{k} \to t_{s\hat{k}}$ is ν_s-periodic w.r.t. k_r $(r = 2, \ldots, m)$.*

Proof. Recall that $\hat{k} = (k_2, \ldots, k_m)$, $\beta_r = 2\pi/\omega_r$, ν_s is the least common multiple of q_{rs}, and $\alpha_s = (p_{2s}\omega_2/q_{2s}, \ldots, p_{ms}\omega_m/q_{ms})$. Since $\nu_s/q_{rs} \in \mathbb{Z}$ $(r = 2, \ldots, m)$, we have

$$t_{s\,(k_2\ldots,k_r+\nu_s,\ldots,k_m)} = \sum_{j=0}^{\nu_k-1} f_{sj} \exp(-ij(\hat{k}\hat{\beta}\alpha_s + \nu_s(2\pi/\omega_r)p_{rs}\omega_r/q_{rs}))$$
$$= \sum_{j=0}^{\nu_k-1} f_{sj} \exp(-ij(\hat{k}\hat{\beta}\alpha_s)) = t_{s\hat{k}}.$$

□

Lemma 2.4. *For any $\hat{j} \in \mathbb{Z}^{m-1}$, side by side with L_s, λ_s, system (2.1) has the solution $L_s^{\hat{j}}(\hat{\varphi}) = \sum_{\hat{k}\in\mathbb{Z}^{m-1}} t_{s\hat{k}-\hat{j}} \exp(i(\hat{k}\hat{\beta}\hat{\varphi}))$ for $\lambda_s^{\hat{j}} = \lambda_s \exp(-i(\hat{j}\hat{\beta}\alpha_s))$.*

Proof. Making a change $\hat{k} - \hat{j} = \hat{r}$ in the Fourier-series of $L_s^{\hat{j}}$, we obtain $L_s^{\hat{j}} = L_s \exp i(\hat{\beta}\hat{j}\hat{\varphi})$. Then, putting $L_s = L_s^{\hat{j}} \exp(-i(\hat{\beta}\hat{j}\hat{\varphi}))$ into (2.1), we conclude that the lemma is true. □

Lemma 2.5. *Let, for any $\hat{k} \in \mathbb{Z}^{m-1}$, there exists $t_{\hat{k}} = \lim_{s\to\infty} t_{s\hat{k}}$, and let there exists $C > 0$ such that for all $s \in \mathbb{N}$, $\hat{k} \in \mathbb{Z}^{m-1}$ inequality $\|t_{s\hat{k}}\| \leq C$ holds. Then $L_s \to L = \sum_{\hat{k}\in\mathbb{Z}^{m-1}} t_{\hat{k}} \exp(i(\hat{k}\hat{\beta}\hat{\varphi}))$ and $L_s(\hat{\varphi} - \alpha_s) \to L(\hat{\varphi} - e\omega_1)$ weakly in $P_n'^r(\hat{\omega})$ for $r \geq [\frac{m-1}{2}] + 1$ ($[t]$ stands for the integral part of t).*

Proof. Let $y \in P_n^r(\hat{\omega})$, y_k be the Fourier constants of y, and $r \geq [\frac{m-1}{2}] + 1$; then $\sum_{\hat{k}\in\mathbf{Z^{m-1}}} \|y_{\hat{k}}\|$ converges [5]. We say that $\langle L_s, y\rangle_{\hat{\omega}} = \sum_{\hat{k}\in\mathbb{Z}^{m-1}} t_{s\hat{k}}^T y_{-\hat{k}}$ and $\langle L, y\rangle_{\hat{\omega}} = \sum_{\hat{k}\in\mathbb{Z}^{m-1}} t_{\hat{k}}^T y_{-\hat{k}}$. The proofs of these relations are similar since the boundedness of $\{t_{s\hat{k}}\}$ implies $\|t_{\hat{k}}\| \leq C$ for $\hat{k} \in \mathbb{Z}^{m-1}$, so we prove the first one. It holds if $y = y^h = \sum_{\|\hat{k}\|\leq h} y_{\hat{k}} \exp(i(\hat{k}\hat{\beta}\hat{\varphi}))$ is a trigonometrical polynomial [15, p.132], besides, the series $\sum_{\hat{k}\in\mathbb{Z}^{m-1}} t_{s\hat{k}}^T y_{-\hat{k}}$ converges because $|t_{s\hat{k}}^T y_{-\hat{k}}| \leq C\|y_{-k}\|$. Therefore, we can take the limit in $\langle L_s, y^h\rangle_{\hat{\omega}} = \sum_{\|\hat{k}\|\leq h} t_{s\hat{k}}^T y_{-\hat{k}}$ as $h \to \infty$ and get the desired relation.

To prove the weak convergence of $\{L_s\}$ to $L = \sum_{\hat{k}\in\mathbb{Z}^{m-1}} t_{\hat{k}} \exp(i(\hat{k}\hat{\beta}\hat{\varphi}))$, we consider $\langle L_s - L, y\rangle_{\hat{\omega}} = \sum_{\hat{k}\in\mathbb{Z}^{m-1}} (t_{s\hat{k}} - t_{\hat{k}})^T y_{-\hat{k}}$. For given $\varepsilon > 0$ we find M_1, M_2 such that $\sum_{\|\hat{k}\|>M_1} \|y_{\hat{k}}\| < \varepsilon/(4C)$ and $\sum_{\|\hat{k}\|\leq M_1} \|t_k - t_{ks}\| < \varepsilon/(2C_1)$ for $s > M_2$, where $C_1 = \max\{\|y_{\hat{k}}\| : \|\hat{k}\| < M_1\}$, then

$$|\sum_{\hat{k}\in\mathbb{Z}^{m-1}} (t_{s\hat{k}} - t_{\hat{k}})^T y_{-\hat{k}}| < C_1 \sum_{\|\hat{k}\|\leq M_1} \|t_k - t_{ks}\| + 2C \sum_{\|\hat{k}\|>M_1} \|y_{\hat{k}}\| < \varepsilon$$

for $s > M_2$. Hence $L_s \to L$ weekly as $s \to \infty$. By the same argument we have

$$\langle L_s(\hat\varphi - \alpha_s) - L(\hat\varphi - e\omega_1), y\rangle_{\hat\omega} = \sum_{\hat k \in \mathbb{Z}^{m-1}} (t_{s\hat k} \exp(-i(\hat k \hat\beta \alpha_s)) - t_{\hat k} \exp(-i(\hat k \hat\beta \omega_1)))^T y_{-\hat k} \to 0$$

as $s \to \infty$.

Owing to the weak* sequential completeness of $P_n'^r(\hat\omega)$, we obtain $L \in P_n'^r(\hat\omega)$ since $L_s \in P_n'^0(\hat\omega) \subset P_n'^r(\hat\omega)$. □

Theorem 2.1. *Let $A \in P_{n^2}^{r_0}(\omega)$, $r_0 = [\frac{m-1}{2}] + 1$; then there exist a distribution $L \in P_n'^{r_0}(\hat\omega)$ and $\lambda \neq 0$ satisfying* (0.6).

Proof. We claim that there exists a sequence $\{L_s\}$ such that for Fourier constants of L_s the inequalities

$$\|t_{s\hat 0}\| = 1 \geq \|t_{s\hat k}\|, \hat k \neq \hat 0, \tag{2.5}$$

are true for any $s \in \mathbb{N}$. Indeed, due to Lemma 2.3, for each $s \in \mathbb{N}$, there exists $t_{s\hat k_s}$ such that $\|t_{s\hat k_s}\| \geq \|t_{s\hat k}\|$, $\hat k_s \neq \hat k$. Owing to Lemma 2.4, the distribution $L_s^{-\hat k_s}/\|t_{s\hat k_s}\| = \sum\limits_{\hat k \in \mathbb{Z}^{m-1}} \tilde t_{s\hat k} \exp(i(\hat k \hat\beta \hat\varphi))$ satisfies (2.1) for $\lambda = \lambda_s \exp(i(\hat k_s \hat\beta \alpha_s))$; then $\tilde t_{s\hat k} = t_{s\hat k + \hat k_s}/\|t_{s\hat k_s}\|$ satisfy (2.5).

Let the Fourier constants of L_s satisfy (2.5). Taking a subsequence if necessary, we may assume in view of Lemma 2.2 that $\lambda_s \to \lambda \neq 0$. It follows from (2.5) that for each $\hat k \in \mathbb{Z}^{m-1}$ the sequence $\{t_{s\hat k}\}$ is bounded. Hence, there exists a sequence $\{s_p\}$ of positive integers such that $\lim\limits_{p\to\infty} t_{s_p \hat k} = t_k$ exists and $\|t_k\| \leq 1$ for each $\hat k \in \mathbb{Z}^{m-1}$. By Lemma 2.5, we get $L_{s_p} \to L$ weakly in $P_n'^{r_0}(\hat\omega)$. Besides, by Lemma 0.2, $X_0(\omega_1, \cdot) \in P_{n^2}^{r_0}(\hat\omega)$ therefore $X_0(\omega_1, \cdot) L_{s_p} \to X_0(\omega_1, \cdot) L$ weakly in $P_n'^{r_0}(\hat\omega)$. Then taking limit in (2.1), we get $X_0(\omega_1, \hat\varphi) L(\hat\varphi - \hat e \varphi_1) = \lambda L(\hat\varphi)$, moreover, $L \neq 0$ since $t_{\hat 0} \neq 0$. □

Consider the set of sequences $\{\lambda_s\}$, where λ_s is an eigenvalue of G_s and denote by Λ the set of partial limits of these sequences.

Theorem 2.2. *Let $\lambda \in \Lambda$, $\mu = \omega_1^{-1} \ln \lambda$, and $A \in P_{n^2}^{r_0}(\omega)$; then there exists a periodic distribution $\psi \in P_n^{r_0}(\omega)$ that satisfies* (0.2_μ).

Proof. If $A \in P_{n^2}^{r_0}(\omega)$, then, by Theorem 2.1, there exists $L \in P_n'^{r_0}(\hat\omega)$ satisfying (0.6). Therefore, Theorem 2.2 follows from Lemma 1.3. □

3. Proof of Lemmas 0.1 and 0.2

Lemma 0.1.

Proof. System (0.5) has the unique continuous solution

$$X(\varphi; \varphi_{10}) = E + \int_0^{\varphi_1 - \varphi_{10}} A(\varphi - e\xi)\, d\xi + \sum_{k=2}^{\infty} \int_0^{\varphi_1 - \varphi_{10}} \int_0^{\varphi_1 - \varphi_{10} - \xi_k} \cdots \int_0^{\varphi_1 - \varphi_{10} - \xi_2 - \ldots - \xi_k} A(\varphi - e\xi_k) A(\varphi - e(\xi_k + \xi_{k-1})) \ldots A\left(\varphi - e \sum_{i=1}^{k} \xi_i\right) d\xi_1 \ldots d\xi_k \tag{3.1}$$

which one can get by the method of successive approximations. This series converges uniformly w.r.t. $\varphi_1 \in [\alpha, \beta]$, $\hat{\varphi} \in \mathbb{R}^{m-1}$, where $\alpha, \beta \in \mathbb{R}$ are arbitrary, because it has the majorant

$$\sum_{s=0}^{\infty} \|A\|^s_{P^0_{n^2}(\omega)} |\varphi_1 - \varphi_{10}|^s / s!.$$

This solution is ω_j-periodic in φ_j $(j = 2, \dots, m)$ since so is the right part of (3.1). Besides,

$$X(\varphi + et; \varphi_{10}) = \int_0^{\varphi_1 + t - \varphi_{10}} A(\varphi + e(t - \xi)) X(\varphi + e(t - \xi)); \varphi_{10})\, d\xi + E$$

$$= [t - \xi = \zeta] = \int_{\varphi_{10} - \varphi_1}^{t} A(\varphi + e\zeta) X(\varphi + e\zeta; \varphi_{10})\, d\zeta + E.$$

Therefore,

$$dX(\varphi + et; \varphi_{10})/dt = A(\varphi + et) X(\varphi + et; \varphi_{10}) \tag{3.2}$$

and $X(\varphi_{10}, \hat{\varphi}; \varphi_{10}) = E$. Consequently, the matrix $X(\varphi + et, \varphi_{10})$, being a fundamental matrix of the system

$$x' = A(\varphi + et)x,$$

is non-singular for $\varphi \in \mathbb{R}^m$ and $\varphi_{10} \in \mathbb{R}$. In particular, $X_0(et)$ is the normalized fundamental matrix of system (0.1_0).

The fact that the function $x(\varphi; \varphi_{10}, x^0) = X(\varphi; \varphi_{10}) x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10}))$ is a solution of system (0.3) we prove by substituting it into this system. Taking into account that $x(\varphi - e\xi; \varphi_{10}, x^0) = X(\varphi - e\xi; \varphi_{10}) x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10}))$ we get

$$X(\varphi; \varphi_{10}) x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})) = \int_0^{\varphi_1 - \varphi_{10}} A(\varphi - e\xi) X(\varphi - e\xi; \varphi_{10}) x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10}))\, d\xi$$

$$+ x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})) = \Big(\int_0^{\varphi_1 - \varphi_{10}} A(\varphi - e\xi) X(\varphi - e\xi; \varphi_{10})\, d\xi + E\Big) x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})),$$

which is true since $X(\varphi; \varphi_{10})$ satisfies (0.4).

To prove (c) note that $X(t + \omega_1, \hat{\varphi} + \hat{e}t; \varphi_{10})$ solves matrix equation (3.2). Hence,

$$X(t + \omega_1, \hat{\varphi} + \hat{e}t; \varphi_{10}) = X(t, \hat{\varphi} + \hat{e}t; \varphi_{10}) Q. \tag{3.3}$$

Putting $t = \varphi_{10}$ in this relation we get $Q = X(\varphi_{10} + \omega_1, \hat{\varphi} + \hat{e}\varphi_{10}; \varphi_{10})$. Then substituting φ_1 for t and $\hat{\psi}$ for $\hat{\varphi} + \hat{e}\varphi_1$ in (3.3) we get $X(\varphi_1 + \omega_1, \hat{\psi}; \varphi_{10}) = X(\varphi_1, \hat{\psi}; \varphi_{10}) X(\varphi_{10} + \omega_1, \hat{\psi} + \hat{e}(\varphi_{10} - \varphi_1); \varphi_{10})$. □

Lemma 0.2.

Proof. Let the matrix A be r-times continuously differentiable on $\mathbb{R}^m$. Then the matrix X, being a solution of system (3.2), is r-times continuously differentiable by the theorem on differentiability of the solution w.r.t. parameter φ [7, p.126].

□

It is noteworthy that if $\lambda_0 \in \Lambda$ and $\mu_0 = \omega_1^{-1} \ln \lambda_0$, then some system in the hull of (0.1_{μ_0}) has a bounded solution. It follows from this that μ_0 belongs to the Sacker-Sell spectrum of this system and λ_0 belongs to the spectrum of the monodromy operator of this system. Besides, system (0.1_{μ_0}) has a generalized conditionally periodic solution. But these will be the objects of another paper.

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