Statistical Convergence of Sequences in Gradual Normed Linear Spaces

Chiranjib Choudhury*, **Hamari Debbarma** and **Shyamal Debnath**

Abstract. In the current paper, we introduce the notion of statistical convergence of sequences in the gradual normed linear spaces as an extension of usual convergence in the same space. We study various algebraic properties and implication relations of the newly introduced convergence concept. Finally, we introduce the notion of statistical Cauchy sequence in the gradual normed linear space and prove that every statistical convergent sequence is statistical Cauchy sequence and vice versa.

Keywords. Gradual number, gradual normed linear space, natural density, statistical convergence.

AMS subject classifications. 03E72, 40A35, 40A05.

1 Introduction

The idea of fuzzy sets [21] was first introduced by Zadeh in the year 1965 which was an extension of the classical sets. Since then so many developments occurred in that direction. In 2006, Fortin et.al. [10] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are known by their respective assignment function which is defined in the interval $(0, 1]$. So in some sense every real number can be viewed as a gradual number with constant assignment function.

In 2011, Sadeqi and Azari [16] first introduced the concept of gradual normed linear space. They studied various properties of the space from both the algebraic and topological point of view. Further progress in this direction has been occurred due to Ettefagh, Azari, and Etemad (see [7, 8]) and many others. For extensive study on gradual real numbers one may refer [1, 5, 14, 20].

On the other hand, in 1951 Fast [9] and Steinhaus [19] introduced the idea of statistical convergence independently in connection with summability. Later on, it was further investigated from the sequence space point of view by Fridy [11, 12], Salat [17] and many mathematicians across the globe. Following their work several investigations and generalization have been done by Mursaleen [15], Savas [18], Esi [6], Debnath [4], and many others. In [3], Debnath et. al. investigated statistical convergent sequence spaces of intuitionistic fuzzy numbers. Statistical convergence has become one of the most active areas of research due to its wide applicability in the various branch of mathematics such as number theory, mathematical analysis, probability theory, etc.

Research on the convergence of sequences in gradual normed linear spaces has not yet gained much ground and it is still in its infant stage. The research carried out so far shows a strong analogy in the behavior of convergence of sequences in gradual normed linear spaces (for details one may refer [2, 7, 8, 16]).

Recently, the convergence of sequences in gradual normed linear spaces was introduced by Ettefagh et. al. [8]. Also, they have investigated some topological properties [7]. Therefore, study of statistical convergence of sequences in gradual normed linear spaces is significant.

2 Definitions and Preliminaries

Definition 2.1. *[10] A gradual real number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}} : (0,1] \to \mathbb{R}$ The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual real number $\tilde{r}$ is said to be non-negative if for every $\xi \in (0,1]$, $A_{\tilde{r}}(\xi) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.*

In [10], the gradual operations between the elements of $G(\mathbb{R})$ was defined as follows:

Definition 2.2. *Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$ respectively. Then, $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $A_{\tilde{r}_1 * \tilde{r}_2}$ given by $A_{\tilde{r}_1 * \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) * A_{\tilde{r}_2}(\xi)\ \forall \xi \in (0,1]$. In particular, the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r} (c \in \mathbb{R})$ are defined by*

$$A_{\tilde{r}_1 + \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) + A_{\tilde{r}_2}(\xi) \quad \textit{and} \quad A_{c\tilde{r}}(\xi) = cA_{\tilde{r}}(\xi)\ \forall \xi \in (0,1].$$

For any real number $p \in \mathbb{R}$, the constant gradual real number $\tilde{p}$ is defined by the constant assignment function $A_{\tilde{p}}(\xi) = p$ for any $\xi \in (0,1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are the constant gradual numbers defined by $A_{\tilde{0}}(\xi) = 0$ and $A_{\tilde{1}}(\xi) = 1$ respectively. One can easily verify that $G(\mathbb{R})$ with the gradual addition and scalar multiplication forms a real vector space [10].

Definition 2.3. *[16] Let X be a real vector space. The function $||\cdot||_G : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X if for every $\xi \in (0,1]$, following three conditions are true for any $x, y \in X$:*
(G_1) $A_{||x||_G}(\xi) = A_{\tilde{0}}(\xi)$ iff $x = 0$;
(G_2) $A_{||\lambda x||_G}(\xi) = |\lambda| A_{||x||_G}(\xi)$ for any $\lambda \in \mathbb{R}$;
(G_3) $A_{||x+y||_G}(\xi) \leq A_{||x||_G}(\xi) + A_{||y||_G}(\xi)$.

The pair $(X, ||\cdot||_G)$ is called a gradual normed linear space (GNLS).

Example 2.4. *[16] Let $X = \mathbb{R}^n$ and for $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, \xi \in (0,1]$, define $||\cdot||_G$ by*

$$A_{||x||_G}(\xi) = e^{\xi} \sum_{i=1}^{n} |x_i|.$$

Then, $||\cdot||_G$ is a gradual norm on $\mathbb{R}^n$ and $(\mathbb{R}^n, ||\cdot||_G)$ is a gradual normed linear space.

Definition 2.5. *[16] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, (x_k) is said to be gradual convergent to $x \in X$ if for every $\xi \in (0,1]$ and $\varepsilon > 0$, there exists $N(\xi) \in \mathbb{N}$ such that*

$$A_{||x_k - x||_G}(\xi) < \varepsilon, \ \forall k \geq N(\xi).$$

Symbolically, $x_k \xrightarrow{||\cdot||_G} x$.

Definition 2.6. *[8] Let $(X, ||\cdot||_G)$ be a GNLS. Then, a sequence (x_k) in X is said to be gradual bounded if for every $\xi \in (0,1]$, there exists $M = M(\xi) > 0$ such that*

$$A_{||x_k||_G}(\xi) < M, \ \forall k \in \mathbb{N}.$$

Definition 2.7. *[16] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, (x_k) is said to be gradual Cauchy if for every $\xi \in (0,1]$ and $\varepsilon > 0$, there exists $N(\xi, \varepsilon) \in \mathbb{N}$ such that*

$$A_{||x_k - x_j||_G}(\xi) < \varepsilon \ \forall k, j \geq N(\xi).$$

Theorem 2.8. *([16], Theorem 3.6) Let $(X, ||\cdot||_G)$ be a GNLS, then every gradual convergent sequence in X is also a gradual Cauchy sequence.*

Definition 2.9. *[12] If K is a subset of the positive integers $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$. The natural density of K is defined by*

$$d(K) = \lim_{n \to \infty} \tfrac{|K_n|}{n}.$$

Remark 2.10. *[12] If (x_k) is a sequence such that x_k satisfies property P for every $k \in \mathbb{N}$ except at a set of natural density zero, then x_k is said to satisfy P for "almost all k"which in short termed as "x_k satisfies P a.a.k".*

Definition 2.11. *[12] A sequence (x_k) is said to be statistical convergent to l if for each $\varepsilon > 0$,*

$$d(\{k \in \mathbb{N} :| x_k - l |\geq \varepsilon\}) = 0.$$

In this case l is called the statistical limit of the sequence (x_k) and we write $st - \lim\limits_{k\to\infty} x_k = l$.

Definition 2.12. *[13] A sequence (x_k) is said to be statistical bounded if there exists a number M such that*

$$d(\{k \in \mathbb{N} : |x_k| > M\}) = 0.$$

Definition 2.13. *[12] A sequence (x_k) is said to be statistical Cauchy if for every $\varepsilon > 0$, there exists a natural number $N(= N_\varepsilon)$ such that*

$$|x_k - x_N| < \varepsilon \ a.a.k.$$

In other words, $\lim\limits_{n\to\infty} \frac{1}{n} \mid \{k \leq n : |x_k - x_N| \geq \varepsilon\} \mid = 0$.

3 Main Results

Definition 3.1. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, (x_k) is said to be gradual statistical convergent to $x \in X$ if for every $\xi \in (0, 1]$ and $\varepsilon > 0$,*

$$d(\{k \in \mathbb{N} : A_{||x_k-x||_G}(\xi) \geq \varepsilon\}) = 0.$$

Symbolically we write, $x_k \xrightarrow{st-||\cdot||_G} x$.

Theorem 3.2. *Let $(X, ||\cdot||_G)$ be a GNLS. If a sequence (x_k) is gradual convergent to $x \in X$, then (x_k) is gradual statistical convergent to $x \in X$.*

Proof. Since (x_k) is gradual convergent to x, so the set $\{k \in \mathbb{N} : A_{||x_k-x||_G}(\xi) \geq \varepsilon\}$ is a finite set and so has density zero. □

But the converse of Theorem 3.2 is not true. Example 3.3 illustrates the fact.

Example 3.3. *Let $X = \mathbb{R}^n$ and $||\cdot||_G$ be the norm defined in Example 2.4. Consider the sequence (x_k) in $\mathbb{R}^n$ defined as*

$$x_k = \begin{cases} (0,0,...,0,n) & if\ k = p^2, p \in \mathbb{N} \\ (0,0,....0,0) & otherwise. \end{cases}$$

Let $\mathbf{0}$ denotes the element $(0,0,....0,0) \in \mathbb{R}^n$. Then, for any $\varepsilon > 0$ and $\xi \in (0,1]$, $\{k \in \mathbb{N} : A_{||x_k - \mathbf{0}||_G}(\xi) \geq \varepsilon\} \subseteq \{1,4,9,..\}$ and eventually $d(\{k \in \mathbb{N} : A_{||x_k - \mathbf{0}||_G}(\xi) \geq \varepsilon\}) = 0$. In other words, $x_k \xrightarrow{st-||\cdot||_G} \mathbf{0}$ in $\mathbb{R}^n$.

Theorem 3.4. *Let (x_k) be any sequence in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{st-||\cdot||_G} x$ in X. Then, x is uniquely determined.*

Proof. If possible suppose $x_k \xrightarrow{st-||\cdot||_G} x$ and $x_k \xrightarrow{st-||\cdot||_G} y$ for some $x \neq y$ in X. Let $\varepsilon > 0$ be arbitrary. Then, by Definition 3.1 we have, for any $\varepsilon > 0$ and $\xi \in (0,1]$,

$$d(B_1(\xi,\varepsilon)) = d(B_2(\xi,\varepsilon)) = 1,$$

where $B_1(\xi,\varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) < \varepsilon\}$ and $B_2(\xi,\varepsilon) = \{k \in \mathbb{N} : A_{||x_k - y||_G}(\xi) < \varepsilon\}$. Choose $m \in B_1(\xi,\varepsilon) \cap B_2(\xi,\varepsilon)$, then $A_{||x_m - x||_G}(\xi) < \varepsilon$ and $A_{||x_m - y||_G}(\xi) < \varepsilon$. Hence, $A_{||x-y||_G}(\xi) \leq A_{||x_m - x||_G}(\xi) + A_{||x_m - y||_G}(\xi) < \varepsilon + \varepsilon = 2\varepsilon$. Since ε is arbitrary, so $A_{||x-y||_G}(\xi) = A_{\tilde{0}}(\xi)$ and so we must have $x = y$. □

Theorem 3.5. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, $x_k \xrightarrow{st-||\cdot||_G} x$ if and only if there exists $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that $d(M) = 1$ and $x_{m_k} \to x(G)$.*

Proof. Firstly, we assume that there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ satisfying

$$d(M) = 1 \text{ and } x_{m_k} \to x(G).$$

Then, for every $\xi \in (0,1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that

$$A_{||x_{m_k} - x||_G}(\xi) < \varepsilon,\ \forall k \geq N.$$

Let $B(\xi,\varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}$. Then, the inclusion

$$B(\xi,\varepsilon) \subset \mathbb{N} \setminus \{m_{N+1}, m_{N+2}, ...\}$$

holds and as a consequence we have $d(B(\xi,\varepsilon)) = 0$. Hence, $x_k \xrightarrow{st-||\cdot||_G} x$

For the converse part, assume that $x_k \xrightarrow{st-||\cdot||_G} x$ holds. Then, for every $\xi \in (0,1]$ and $j \in \mathbb{N}$, $d(M_j) = 1$, where

$$M_j = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) < \tfrac{1}{j}\}.$$

From the construction of M_j's, it is clear that

$$M_1 \supset M_2 \supset ... \supset M_j \supset M_{j+1} \supset ... \tag{1}$$

Let us choose $v_1 \in M_1$ be an arbitrary element. Then, there exists $v_2 \in M_2$ such that for all $n \geq v_2$,

$$\tfrac{1}{n} \mid \{k \leq n : k \in M_2\} \mid > \tfrac{1}{2},$$

holds. In a similar way, there exists $v_3 \in M_3$ such that for all $n \geq v_3$,

$$\tfrac{1}{n} \mid \{k \leq n : k \in M_3\} \mid > \tfrac{2}{3},$$

satisfies. Proceeding like this, we can construct a increasing sequence (v_j) of positive integers such that $v_j \in M_j$ and for all $n \geq v_j$,

$$\frac{1}{n} \mid \{k \leq n : k \in M_j\} \mid > 1 - \frac{1}{j}, \tag{2}$$

is true. Let us construct M as follows: each natural number of the interval $[1, v_1]$ belong to M and any natural number of the interval $[v_j, v_{j+1}]$ belongs to M if and only if it belongs to M_j $(j \in \mathbb{N})$.

From (1) and (2), we have for each $v_j \leq n < v_{j+1}$,

$$\frac{\mid \{k \leq n : k \in M\} \mid}{n} \geq \frac{\mid \{k \leq n : k \in M_j\} \mid}{n} > 1 - \frac{1}{j}.$$

Consequently, $d(M) = 1$. Let $\varepsilon > 0$ be given. By Archimedean property, choose $j \in \mathbb{N}$ such that $\frac{1}{j} < \varepsilon$. Further, let $k \in M$ be such that $k \geq v_j$. Then, there exists $t \geq j$ such that $v_t \leq k \leq v_{t+1}$. But by the definition of M, $k \in M_t$.

Therefore,

$$A_{||x_k - x||_G}(\xi) < \tfrac{1}{t} \leq \tfrac{1}{j} < \varepsilon.$$

Hence, $x_{m_k} \to x(G)$ holds and the proof is complete. □

Theorem 3.6. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{st-||\cdot||_G} x$ and $y_k \xrightarrow{st-||\cdot||_G} y$. Then,*
(i) $x_k + y_k \xrightarrow{st-||\cdot||_G} x + y$ and (ii) $cx_k \xrightarrow{st-||\cdot||_G} cx$, $c \in \mathbb{R}$.

Proof. (i) Suppose $x_k \xrightarrow{st-||\cdot||_G} x$ and $y_k \xrightarrow{st-||\cdot||_G} y$. Then, by Definition 3.1, for given $\varepsilon > 0$, $d(C_1) = d(C_2) = 0$, where $C_1 = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \frac{\varepsilon}{2}\}$ and $C_2 = \{k \in \mathbb{N} : A_{||y_k - y||_G}(\xi) \geq \frac{\varepsilon}{2}\}$. Now as the inclusion

$$(\mathbb{N}\setminus C_1)\cap(\mathbb{N}\setminus C_2)\subseteq\{k\in\mathbb{N}: A_{||x_k+y_k-x-y||_G}(\xi)<\varepsilon\}$$

holds, so we must have

$$d(\{k\in\mathbb{N}: A_{||x_k+y_k-x-y||_G}(\xi)\geq\varepsilon\})\leq d(C_1\cup C_2)=0;$$

and consequently, $x_k+y_k\xrightarrow{st-||\cdot||_G}x+y$.
(ii) If $c=0$, then there is nothing to prove. So let us assume $c\neq 0$. Then, since $x_k\xrightarrow{st-||\cdot||_G}x$, we have for given $\varepsilon>0$, $d(C_1)=0$, where $C_1=\{k\in\mathbb{N}: A_{||x_k-x||_G}(\xi)\geq\frac{\varepsilon}{|c|}\}$. Now since

$$A_{||cx_k-cx||_G}(\xi)=|c|A_{||x_k-x||_G}(\xi)$$

holds for any $c\in\mathbb{R}$, we must have $C_2\subseteq C_1$, where $C_2=\{k\in\mathbb{N}: A_{||cx_k-cx||_G}(\xi)\geq\varepsilon\}$, which as a consequence implies $d(C_2)=0$. This completes the proof. □

Theorem 3.7. *Let (x_k) be any sequence in the GNLS $(X,||\cdot||_G)$. If every subsequence of (x_k) is gradual statistical convergent to x, then (x_k) is also gradual statistical convergent to x.*

Proof. If possible suppose (x_k) is not gradual statistical convergent to x. Then, there exists some $\varepsilon>0$ and $\xi\in(0,1]$ such that $d(C)\neq 0$, where $C=\{k\in\mathbb{N}: A_{||x_k-x||_G}(\xi)\geq\varepsilon\}$. Since density of a finite set is zero, so C must be an infinite set. Let $C=\{k_1<k_2<...<k_j<...\}$. Now define a sequence (y_j) as $y_j=x_{k_j}$ for $j\in\mathbb{N}$. Then, (y_j) is a subsequence of (x_k) which is not gradual statistical convergent to x, a contradiction. □

Remark 3.8. *Converse of the above theorem is not true. One can easily verify this from Example 3.3.*

Theorem 3.9. *Let $(X,||\cdot||_G)$ be a GNLS. A subsequence (x_{k_j}) of a gradual statistical convergent sequence (x_k) in X is gradual statistical convergent if and only if $d(\{k_j: j\in\mathbb{N}\})=1$.*

Proof. Proof is easy and so is omitted. □

In ([8], Theorem 3.5), Ettefagh et. al. proved that in a GNLS every gradual convergent sequence is gradual bounded. But this result is not true in case of gradual statistical convergence. Consider the gradual normed space $(\mathbb{R}^2,||\cdot||_G)$ with the gradual norm defined in Example 2.4. Consider the sequence (x_k) in $\mathbb{R}^2$ defined as

$$x_k=\begin{cases}(0,k) & if\ k=p^2, p\in\mathbb{N};\\ (0,0) & otherwise.\end{cases}$$

Then, it is clear that (x_k) is not gradual bounded but gradual statistical convergent to $(0,0)\in\mathbb{R}^2$.

Definition 3.10. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, (x_k) is said to be gradual statistical bounded if for every $\xi \in (0,1]$, there exists $M(= M(\xi)) > 0$ such that*

$$d(\{k \in \mathbb{N} : A_{||x_k||_G}(\xi) > M\}) = 0.$$

Theorem 3.11. *Let $(X, ||\cdot||_G)$ be a GNLS and suppose (x_k) be a gradual statistical convergent sequence. Then, (x_k) is gradual statistical bounded.*

Proof. Proof is straightforward so omitted. □

Definition 3.12. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, (x_k) is said to be gradual statistical Cauchy if for every $\varepsilon > 0$ and $\xi \in (0,1]$, there exists a natural number $N(= N(\xi, \varepsilon))$ such that $A_{||x_k - x_N||_G}(\xi) < \varepsilon$ a.a.k.*

Theorem 3.13. *Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then, the following statements are equivalent:*
(i) $x_k \xrightarrow{st-||\cdot||_G} x.$;
(ii) (x_k) is a gradual statistical Cauchy sequence;
(iii) (x_k) is a sequence for which there is a gradual convergent sequence (y_k) such that

$$d(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0.$$

Proof. $(i) \Rightarrow (ii)$ Let (x_k) be a sequence in X such that $x_k \xrightarrow{st-||\cdot||_G} x$. Then, for every $\varepsilon > 0$ and $\xi \in (0,1]$,

$$d(B_1(\xi, \varepsilon)) = 0, \tag{3}$$

where

$$B_1(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}.$$

Choose $N \in \mathbb{N} \setminus B_1(\xi, \varepsilon)$. Then, we have $A_{||x_N - x||_G}(\xi) < \varepsilon$.

Let $B_2(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x_N||_G}(\xi) \geq 2\varepsilon\}$. Now we prove that the following inclusion is true

$$B_2(\xi, \varepsilon) \subseteq B_1(\xi, \varepsilon). \tag{4}$$

For if $p \in B_2(\xi, \varepsilon)$ we have

$$2\varepsilon \leq A_{||x_p - x_N||_G}(\xi) \leq A_{||x_p - x||_G}(\xi) + A_{||x - x_N||_G}(\xi) < A_{||x_p - x||_G}(\xi) + \varepsilon,$$

which implies $p \in B_1(\xi, \varepsilon)$ and so (4) is true. From (3) and (4) we conclude that $d(B_2(\xi, \varepsilon)) = 0$, which means that (x_k) is gradual statistical Cauchy sequence.
$(ii) \Rightarrow (iii)$ Let (x_k) be a gradual statistical Cauchy sequence. Choose N so that

$$d(\{k \in \mathbb{N} : A_{||x_k||_G}(\xi) \notin I\}) = 0,$$

where

$$I = [A_{||x_N||_G}(\xi) - 1, A_{||x_N||_G}(\xi) + 1].$$

Again choose M so that $d(\{k \in \mathbb{N} : A_{||x_k||_G}(\xi) \notin I'\}) = 0$, where

$$I' = [A_{||x_M||_G}(\xi) - \frac{1}{2}, A_{||x_M||_G}(\xi) + \frac{1}{2}].$$

Now as the equality

$$\{k \leq n : A_{||x_k||_G}(\xi) \notin I \cap I'\} = \{k \leq n : A_{||x_k||_G}(\xi) \notin I\} \cup \{k \leq n : A_{||x_k||_G}(\xi) \notin I'\}$$

holds, so we must have

$$d(\{k \in \mathbb{N} : A_{||x_k||_G}(\xi) \notin I \cap I'\}) = 0.$$

Denote $I \cap I'$ by I_1. Then, it is clear that I_1 is a closed interval with $diam(I_1) \leq 1$, where $diam(I_1)$ represents the length of the interval I_1. Proceeding like this, we choose $N(2)$ so that $d(\{k \in \mathbb{N} : A_{||x_k||_G}(\xi) \notin I''\}) = 0$, where

$$I'' = [A_{||x_{N(2)}||_G}(\xi) - \frac{1}{4}, A_{||x_{N(2)}||_G}(\xi) + \frac{1}{4}].$$

Let us denote $I_1 \cap I''$ by I_2. Then, by the previous argument we can say that I_2 is a closed interval with $diam(I_2) \leq \frac{1}{2}$ satisfying

$$d(\{k \in \mathbb{N} : A_{||x_k||_G}(\xi) \notin I_2\}) = 0.$$

Continuing in this way, we obtain a sequence (I_m) of closed intervals such that

$$I_1 \supseteq I_2 \supseteq ... \supseteq I_m \supseteq I_{m+1} \supseteq ...$$

and

$$diam(I_m) \leq 2^{1-m}.$$

By Nested Interval Theorem, there exists a $\lambda \in \mathbb{R}$ such that

$$\bigcap_{m=1}^{\infty} I_m = \{\lambda\}.$$

Now we choose an increasing sequence of natural numbers (T_m) such that

$$\frac{1}{n} \mid \{k \leq n : A_{||x_k||_G}(\xi) \notin I_m\} \mid < \frac{1}{m} \tag{5}$$

if $n > T_m$. Define a subsequence (z_k) of (x_k) consisting of all terms x_k such that $k > T_1$ and if $T_m < k \leq T_{m+1}$ then $A_{||x_k||_G}(\xi) \notin I_m$. Define the sequence (y_k) as follows:

$$y_k = \begin{cases} \tilde{\lambda}, \; if \; x_k \; is \; a \; term \; of \; (z_k) \\ x_k, \; otherwise, \end{cases}$$

where $A_{\tilde{\lambda}}(\xi) = \lambda$, $\forall\, \xi \in (0, 1]$. Then, $y_k \to \tilde{\lambda}$; for, if $\varepsilon > \frac{1}{m} > 0$ and $k > T_m$ then either x_k is a term of (z_k), which means $y_k = \tilde{\lambda}$ or $y_k = x_k$, $A_{||x_k||_G}(\xi) \in I_m$ and $A_{||y_k - \tilde{\lambda}||_G}(\xi) \leq diam(I_m) \leq 2^{1-m}$.

We also claim that $d(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$. Because if $T_m < n < T_{m+1}$, then the inclusion

$$\{k \leq n : x_k \neq y_k\} \subseteq \{k \leq n : A_{||x_k||_G}(\xi) \notin I_m\}$$

holds and consequently by (5),

$$\tfrac{1}{n} \mid \{k \leq n : x_k \neq y_k\} \mid < \tfrac{1}{n}.$$

Letting $n \to \infty$ on both sides of the above inequation, we obtain

$$\lim_{n \to \infty} \frac{1}{n} \mid \{k \leq n : x_k \neq y_k\} \mid = 0,$$

i.e.,

$$d(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0.$$

(iii)$\Rightarrow$ (i) Finally we assume that $d(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$ and $y_k \xrightarrow{||\cdot||_G} x$. Then, by definition for any $\varepsilon > 0$ and $\xi \in (0, 1]$, the set

$$\{k \leq n : A_{||y_k - x||_G}(\xi) \geq \varepsilon\}$$

contains a finite number of elements say N_0. Now as the inclusion

$$\{k \leq n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \subseteq \{k \leq n : x_k \neq y_k\} \cup \{k \leq n : A_{||y_k - x||_G}(\xi) \geq \varepsilon\}$$

holds, so we must have,

$$\frac{1}{n} \mid \{k \leq n : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \mid \leq \frac{1}{n} \mid \{k \leq n : x_k \neq y_k\} \mid + \frac{N_0}{n}.$$

Letting $n \to \infty$ on both sides of the above inequality and using the fact that

$$d(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$$

we obtain $x_k \xrightarrow{st-||\cdot||_G} x$. This completes the proof. □

Acknowledgment: The first author is grateful to the **University Grants Commission, India** for their fellowships funding under the **UGC-SRF** scheme **(F. No. 16-6(DEC. 2018)/2019(NET/CSIR))** during the preparation of this paper.

References

[1] F. Aiche and D. Dubois, *Possibility and gradual number approaches to ranking methods for random fuzzy intervals*, Commun. Comput. Inf. Sci., **299**, 9-18, 2012.

[2] C. Choudhury and S. Debnath, On $\mathcal{I}-$convergence of sequences in gradual normed linear spaces, Facta Univ. Ser. Math. Inform., **36**(3), 595-604, 2021.

[3] S. Debnath, V. N. Mishra and J. Debnath *On statistical convergent sequence spaces of intuitionistic fuzzy numbers*, Bol. Soc. Parana. Mat., **36**(1), 235-242, 2018.

[4] S. Debnath and D. Rakshit, *On I-statistical convergence*, Iran. J. Math. Sci. Inform., **13**(2), 101-109, 2018.

[5] D. Dubois and H. Prade, *Gradual elements in a fuzzy set*, Soft Comput., **12**, 165-175, 2007.

[6] A. Esi, S. Debnath and S. Saha, *Asymptotically double lacunary statistical equivalent sequences of interval numbers*, Proyecciones, **35**(4), 469-479, 2016.

[7] M. Ettefagh, F. Y. Azari and S. Etemad, *On some topological properties in gradual normed spaces*, Facta Univ. Ser. Math. Inform., **35**(3), 549-559, 2020.

[8] M. Ettefagh, S. Etemad and F. Y. Azari, *Some properties of sequences in gradual normed spaces*, Asian-Eur. J. Math., **13**(4) 2050085, 2020.

[9] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2**, 241-244, 1951.

[10] J. Fortin, D. Dubois and H. Fargier, *Gradual numbers and their application to fuzzy interval analysis*, IEEE Trans. Fuzzy Syst., **16**, 388-402, 2008.

[11] J. A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc., **118**(4), 1187-1192, 1993.

[12] J. A. Fridy, *On statistical convergence*, Analysis, **5**(4), 301-313, 1985.

[13] J. A. Fridy and C. Orhan, *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc., **125**(12), 3625-3631, 1997.

[14] L. Lietard and D. Rocacher, *Conditions with aggregates evaluated using gradual numbers*, Control Cybernet., **38**, 395-417, 2009.

[15] M. Mursaleen, *$\lambda-$statistical convergence*, Math. Slovaca., **50(1)**, 111-115, 2000.

[16] I. Sadeqi and F. Y. Azari, *Gradual normed linear space*, Iran. J. Fuzzy Syst., **8**(5), 131-139, 2011.

[17] T. Salat, *On statistical convergent sequences of real numbers*, Math. Slovaca., **30**, 139-150, 1980.

[18] E. Savas and S. Debnath, *Lacunary statistical $\phi-$convergence*, Note Mat., **39**(2), 111-119, 2019.

[19] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2**, 73-74, 1951.

[20] E. A. Stock, *Gradual numbers and fuzzy optimization*, PhD. Thesis, University of Colorado Denver, Denver, America (2010).

[21] L. A. Zadeh, *Fuzzy sets*, Inf. Control, **8**(3), 338-353, 1965.

C. Choudhury* (Corresponding Author)
Space Central School
Department of Space
Indian Space Research Organisation (ISRO)
SDSC SHAR, Sriharikota,
Andhra Pradesh-524124, India.
Email: `chiranjibchoudhury123@gmail.com`

H. Debbarma
Department of Mathematics
Tripura University (A Central University)
Suryamaninagar-799022, Agartala, India.
Email: `debbarmah07@gmail.com`

S. Debnath
Department of Mathematics
Tripura University (A Central University)
Suryamaninagar-799022, Agartala, India.
Email: `shyamalnitamath@gmail.com`