On Projective Ricci Curvature of Cubic Metrics

M. K. Gupta and Suman Sharma

Department of Mathematics

Guru Ghasidas Vishwavidyalaya, Bilaspur (C.G.), India

E-mail: mkgiaps@gmail.com, sharma.suman209@gmail.com

Abstract

To study the projective Ricci curvature (**PRic** $-$ *curvature*) in Finsler geometry is interesting because it reflects the geometric properties that are invariant under the projective transformation. In this paper, we firstly derived an expression of S-curvature for the cubic Finsler metric and proved that this S-curvature vanishes if and only if β is a constant killing form. Next, we obtain an explicit expression of projective Ricci curvature for the cubic metric. We also proved that for the projective Ricci-flat Finsler space the 1-form β is closed and then the Riemannian metric of α is also Ricci-flat. Finally, we show that the cubic Finsler metric is of weak projective Ricci curvature if and only if it is a projectively Ricci-flat.

Keywords: Finsler space, Cubic metrics, S-curvature, Riemann curvature, Ricci curvature, Projective Ricci curvature.

1 Introduction

Finsler geometry extends the classical Riemannian geometry by considering more general metric structures. A very important class of Finsler metric is known as (α, β)-metric, which is introduced by M. Matsumoto [4] in 1972. An (α, β)-metric can be expressed as $F = \alpha\phi(s)$, where α is a Riemannian metric and $s = \frac{\beta}{\alpha}$, β is 1-form. Randers metric, Kropina metric, exponential metric, Matsumoto metric, and cubic metric are important classes of (α, β)-metric.

To study the curvature characteristics is a central problem in Finsler geometry. The Ricci curvature and S-curvature are very important non-Riemannian quantities in the Finslerian manifold. The Ricci curvature in Finsler geometry is a natural extension of the Ricci curvature in Riemannian geometry and is defined as the trace of the Riemann curvature. The S-curvature is a mathematical quantity and measures the rate of change of volume form of a Finsler space along the geodesics.

In Finsler geometry, the study of curvature involves understanding the deviation from flatness. The projective Ricci curvature is one aspect of this analysis. The concept of projective Ricci curvature in Finsler geometry is introduced by X. Cheng [1] in 2017. Projective geometry deals with the properties that are invariant under projective transformations. The projective Ricci curvature measures the deviation of the Finsler metric from being projectively flat. Projective Ricci curvature has applications in various areas of Mathematics and Physics. It plays a crucial role in understanding the geometry of Finsler manifolds and connects to the problems in the calculus of variations, differential equations, and geometric optics.

In 2020 H. Zhu [10] has given an expression of projective Ricci curvature for an (α, β)-metric. Later on many geometers [3,7,8] have studied the geometric properties of projective Ricci curvature. In this article, we obtain the geometric properties and flatness condition of projective Ricci curvature for the cubic Finsler metric which is defined as $F = \alpha\phi(s)$ with

$$\phi = (1+s)^3\,, \tag{1.1}$$

i.e., $F = \frac{(\alpha+\beta)^3}{\alpha^2}$. Cubic metric is Finsler metric for $|b^2| < \frac{1}{4}$ [9].

The following nptations will be used to state our main result

$$\begin{aligned} &2s_{jk} = b_{j;k} - b_{k;j}\,, \quad 2r_{jk} = b_{j;k} + b_{k;j}\,, \quad s^j_k = a^{jl}s_{kl}\,, \quad r^j_k = a^{jl}r_{kl}\,, \quad s_j = b^l s_{lj} = b_k s^k_j\,, \\ &r_j = b^l r_{lj} = b_k r^k_j\,, \quad r_{j0} = r_{jk}y^k\,, \quad r_{00} = r_{jk}y^j y^k\,, \quad r = r_{jk}b^j b^k = b^j r_j\,, \quad s_{j0} = s_{jk}y^k\,, \\ &s_0 = s_j y^j\,, \quad r_0 = r_j y^j\,, \quad b^j = a^{jk}b_k\,, \quad t_{jk} = s_{jm}s^m_k\,, \quad t_j = b^m t_{mj} = s^i s^i_j\,, \end{aligned} \tag{1.2}$$

where ";" denotes the covariant derivative with respect to the Levi-Civita connection of the Riemannian metric α.

A 1-form β is said to be a Killing form if $r_{ij} = 0$. The 1-form β is said to be a constant Killing form if it is a Killing form and constant length concerning α, equivalently $r_{ij} = 0$ and $s_i = 0$. We first prove the following result

Theorem 1.1 *For the cubic Finsler metric* $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ *on an n-dimensional Finsler manifold M, the S-curvature vanishes if and only if β is a constant Killing-form.*

Next, we obtain the flatness condition for the projective Ricci curvature as

Theorem 1.2 *If the cubic Finsler metric* $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ *is projective Ricci-flat ($PRic = 0$) then β is parallel with respect to the Riemannian metric α.*

In view of the above result we get

Corollary 1.3 *If the cubic Finsler metric* $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ *is projective Ricci flat then it vanishes the S-curvature. Therefore the Riemannian metric of* α *is Ricci flat* ($Ric^\alpha = 0$).

We also prove the following result

Theorem 1.4 *The cubic Finsler metric* $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ *is weak* ***PRic****-curvature if and only if it is a* ***PRic****-flat metric.*

2 Preliminaries

Let F be an n-dimensional Finsler manifold and G^j be the geodesic coefficients of F, which is defined as

$$G^j = \frac{1}{4}\, g^{jl}\Big[\frac{\partial^2(F^2)}{\partial x^k \partial y^l} y^k - \frac{\partial(F^2)}{\partial x^l}\Big], \qquad y \,\epsilon\, T_x M.$$

The geodesic coefficients of an (α, β)-metrics is given as [2]

$$G^j = G^j_\alpha + \alpha Q s^j_0 + (r_{00} - 2\alpha Q s_0)(\Psi b^j + \frac{\Theta y^j}{\alpha}), \tag{2.1}$$

where G^i_α denotes the geodesic coefficients of the Riemannian metric α and

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \psi = \frac{\phi''}{2\phi(\phi - s\phi' + (B - s^2)\phi'')}, \quad \Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi(\phi - s\phi' + (B - s^2)\phi'')}. \tag{2.2}$$

For any $x \,\epsilon\, M$ and $y \,\epsilon\, T_x M \backslash \{0\}$, the Riemann curvature R_y is defined as

$$R_y(v) = R^j_k(y) v^k \frac{\partial}{\partial x^j}, v = v^j \frac{\partial}{\partial x^j},$$

where,

$$R^j_k = 2\frac{\partial G^j}{\partial x^k} + 2G^i \frac{\partial^2 G^j}{\partial y^i \partial y^k} - \frac{\partial^2 G^j}{\partial x^i \partial y^k} y^i - \frac{\partial G^j}{\partial y^i}\frac{\partial G^i}{\partial y^k},$$

The trace of Riemann curvature is called Ricci curvature $Ric = R^m_m$, which is a mathematical object that regulates the rate at which a metric ball's volume in a Manifold grows. A Finsler metric F is called Einstein metric if Ricci curvature satisfies the equation $Ric(x, y) = (n-1)\gamma F^2$, where $\gamma = \gamma(x)$ is a scalar function.

In 1997, Z. Shen [6] has discussed S-curvature which measures the average rate of change of $(T_x M; F_x)$ in the direction $y \,\epsilon\, T_x M$ and is defined as

$$S(x, y) = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial(\log \sigma_F)}{\partial x^m}$$

where σ_F is defined as

$$\sigma_F \;=\; \frac{Vol(B^n)}{Vol\{y^i \,\epsilon\, \mathbb{R}^n | F(x,y) < 1\}}$$

and Vol denotes the Euclidean volume and $B^n(1)$ denotes the unit ball in R^n.
The expression of S-curvature for an (α, β)-metrics is given as [5]

$$S \;=\; (s_0 + r_0)\,(2\psi - \Pi) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0), \tag{2.3}$$

where

$$\begin{aligned} \Pi &= \frac{f'(b)}{bf(b)}\,, \quad \Delta = 1 + sQ + (B - s^2)Q_s\,, \quad B = b^2 \\ \Phi &= -(Q - sQ_s)(n\Delta + 1 + sQ) - (B - s^2)(1 + sQ)Q_{ss}\,. \end{aligned} \tag{2.4}$$

The projective Ricci curvature is first defined by X. Cheng [1] as

$$PRic = Ric + \frac{n-1}{n+1}S_{|m}y^m + \frac{n-1}{(n+1)^2}S^2\,, \tag{2.5}$$

where "|" denotes the horizontal covariant derivative with respect to the Berwald connections of F. A Finsler space F is called weak projective Ricci curvature if

$$PRic = (n-1)\Big[\frac{3\theta}{F} + \gamma\Big]F^2\,, \tag{2.6}$$

where $\gamma = \gamma(x)$ is a scalar function and $\theta = \theta_i(x)y^i$ is a 1-form. If γ = constant, then F is called constant projective Ricci curvature. If $\theta = 0$, then F is called isotropic projective Ricci curvature **PRic**$=(n-1)\gamma F^2$.

In 2020 H.Zhu [10] has given an expression of the projective Ricci curvature for the (α, β)-metrics as

$$\begin{aligned} PRic = Ric^{\alpha} &+ \frac{1}{n+1}\Big[\frac{r_{00}^2}{\alpha^2}V_1 - \frac{r_{00}s_0}{\alpha}V_2 - \frac{r_{00}r_0}{\alpha}V_3 + \frac{r_{00|0}}{\alpha}V_4 + s_0^2V_5 + r_{00}rV_6 - 4r_0^2V_7 + 2r_0s_0V_8\Big] \\ &+ (r_{00}r_i^i + r_{00|b} - b^ir_{i0|0} - r_{0i}r_0^i)V_9 + r_{0i}s_0^iV_{10} + s_{0|0}V_{11} + s_{0i}s_0^iV_{12} + \alpha r s_0V_{13} + \alpha s_js_0^jV_{14} \\ &+ \alpha\Big[\frac{4}{n+1}r_is_0^i - 2s_{0|b} - 2r_i^is_0 + 3r_{0i}s^i + b^is_{|0}^i\Big]V_{15} + \alpha s_{0|i}^iV_{16} + \alpha^2s_is^iV_{17} + \alpha^2s_j^is_i^jV_{18} \\ &+ r_{0|0}V_{19} + \frac{2(n-1)}{n+1}\Big[\frac{\Psi_s}{\alpha}(r_{00} - 2\alpha Qs_0)(B - s^2) + 2\Psi(r_0 + s_0)\Big]\rho_0 + (n-1)\Big[-2\Psi(r_{00} \\ &- 2\alpha Qs_0)\rho_b - 2\alpha Q\rho_ks_0^k + \rho_0^2 + \rho_{0|0}\Big]\,, \end{aligned} \tag{2.7}$$

where

$$V_1 = 4s\Psi_s + (4\Psi\Psi_{ss} - \Psi_s^2)(B - s^2)^2 - 2(\Psi_{ss} + 6s\Psi\Psi_s)(B - s^2) - \frac{n^2+1}{n+1}\Psi_s^2(B - s^2)^2\,.$$

$$
\begin{aligned}
V_2 &= 4\big[2\Psi(\Psi Q_{ss} + 2Q\Psi_{ss} + Q_s\Psi_s) - Q(\Psi_s)^2\big](B-s^2)^2 + 4\big[2(Q-sQ_s)(\Psi_s)^2 - (1+10sQ)\Psi\Psi_s \\
&\quad - 2\Psi Q_{ss} - 2Q\Psi_{ss} - Q_s\Psi_s + \Psi_{Bs}\big](B-s^2) + 2Q_{ss} + 8\Psi_s - 4Q\Psi + 4s\Psi Q_s + 20sQ\Psi_s \\
&\quad + (n-1)\big[4((\Psi)^2 Q_{ss} - Q(\Psi_s)^2)(B-s^2)^2 + 4((Q-sQ_s)(\Psi)^2) + \Psi(\Psi_s - Q_{ss})(B-s^2) \\
&\quad + 2s(Q_s\Psi + Q\Psi_s) - 2Q\Psi + Q_{ss}\big] + \frac{8}{n+1}\Psi_s\big[\Psi - Q\Psi_s(B-s^2)\big](B-s^2)\,, \\
V_3 &= 2\Psi_s - 2(3\Psi\Psi_s - \Psi_{Bs})(B-s^2) - (n-1)(1-2\Psi(B-s^2))\Psi_s + \frac{4}{n+1}\Psi\Psi_s(B-s^2)\,, \\
V_4 &= -2\Psi_s(B-s^2)\,, \\
V_5 &= (n-1)\Big\{4\big[2Q\Psi^2 Q_{ss} - \Psi^2(Q_s)^2 - Q^2(\Psi_s)^2\big](B-s^2)^2 + 4[2Q\Psi(Q\Psi - 2sQ_s\Psi - Q_{ss} + \Psi_s) \\
&\quad + \Psi(Q_s)^2 + Q_s\Psi_B](B-s^2) - 4Q\Psi(s^2Q\Psi - 3sQ_s + 2Q) + 4sQ(Q\Psi_s + \Psi_B) + 8Q_s\Psi \\
&\quad + 2QQ_{ss} - (Q_s)^2 + 4\Psi_B\Big\} + 4\big[4Q\Psi(\Psi Q_{ss} + Q\Psi_{ss} + Q_s\Psi_s) - 2(Q_s)^2\Psi^2 - Q^2(\Psi_s)^2\big](B-s^2)^2 \\
&\quad + 8\big[Q\Psi(-4sQ_s\Psi - 4sQ\Psi_s + 2Q\Psi - 2Q_{ss} - \Psi_s) + (Q_s)^2\Psi - Q^2\Psi_{ss} - QQ_s\Psi_s + Q\Psi_{Bs} \\
&\quad + Q_s\Psi_B\big](B-s^2) + 24sQ^2\Psi_s - 8Q\Psi(s^2Q\Psi - 3sQ_s + 2Q) + 8sQ\Psi_B + 4\Psi(\Psi + 4Q_s) + 4QQ_{ss} \\
&\quad + 16Q\Psi_s - 2(Q_s)^2 - \frac{8}{n+1}\big[\Psi - Q\Psi_s(B-s^2)\big]^2\,, \\
V_6 &= 4\big[(n+1)\Psi_B + 2(\Psi)^2\big]\,, \qquad V_7 = \frac{n^2+n+2}{n+1}\Psi^2 + 2\Psi_B\,, \\
V_8 &= (n-1)\Big[2(2Q_s\Psi^2 + 2Q\Psi\Psi_s + Q_s\Psi_B)(B-s^2) + 2sQ(2\Psi^2 + \Psi_B) - 2Q\Psi_s + 2\Psi_B\Big] \\
&\quad + 4\big[2Q_s\Psi^2 - 3Q\Psi\Psi_s + Q\Psi_{Bs} + Q_s\Psi_B\big](B-s^2) + 4sQ(2\Psi^2 + \Psi_B) + 4\Psi^2 + 4Q\Psi_s \\
&\quad - 4\Psi_B - \frac{8}{n+1}\Psi\big[\Psi - Q\Psi_s(B-s^2)\big]\,, \\
V_9 &= 2\Psi\,, \qquad V_{10} = 2\big[-2Q_s\Psi(B-s^2) - 2sQ\Psi - \Psi + Q_s + \frac{4}{n+1}Q\Psi_s(B-s^2)\big]\,, \\
V_{11} &= 2Q_s\Psi(B-s^2) + 2\Psi(1+2Q) - Q_s + \frac{4}{n+1}\big[Q\Psi_s(B-s^2) - \Psi\big]\,, \\
V_{12} &= 2Q_s - 2Q(Q-sQ_s)\,, \qquad V_{13} = -8Q[\Psi_B + \frac{2}{n+1}\Psi^2]\,, \\
V_{14} &= \frac{-2}{n+1}Q\big[(n-3)\Psi + 4Q\Psi_s(B-s^2)\big]\,, \qquad V_{15} = 2Q\Psi\,, \qquad V_{16} = 2Q\,, \\
V_{17} &= -4Q^2\Psi\,, \qquad V_{18} = -Q^2\,, \qquad V_{19} = \frac{2(n-1)}{n+1}\Psi\,,
\end{aligned}
\tag{2.8}
$$

3 S-curvature of cubic metrics

For equation (1.1), we obtain the following values

$$Q = \frac{3}{1-2s}\,, \quad Q_s = \frac{6}{(1-2s)^2}\,, \quad Q_{ss} = \frac{24}{(1-2s)^3}\,, \quad \psi = \frac{3}{1+6B-s-8s^2}$$

$$\psi_s = \frac{3+48s}{(1+6B-s-8s^2)^2}\,, \quad \psi_{ss} = \frac{18(3+16B+8s+64s^2)}{(1+6B-s-8s^2)^3}\,, \quad \psi_B = \frac{-18}{(1+6B-s-8s^2)^2}\,,$$

$$\psi_{Bs} = \frac{36+576s}{(1+6B-s-8s^2)^2}\,, \quad \Theta = \frac{3(1-4s))}{2(1+6B-s-8s^2)}\,, \quad \Theta_s = -\frac{9+72B-48s+96s^2}{2(1+6B-s-8\quad s^2)^2}\,,$$

$$\Theta_B = \frac{9(-1+4s)}{2(1+6B-s-8s^2)^2}\,, \quad \Delta = \frac{1+6B-s-8s^2}{(1-2s)^2}\,,$$

$$\Phi = \frac{-(3(1-5s-6s^2+B(8+6n+8s-24ns)+n(1-5s-4s^2+32s^3)}{(1-2s)^4}\,, \tag{3.1}$$

By using equations (2.1) and (3.1) we get the spray coefficient G^j for the cubic metric as

$$\begin{aligned} G^j = G^j_\alpha + \frac{1}{2\alpha(1-2s)(1+6B-s-8s^2)}\Big[&(6+36B-6s-48s^2)\alpha^2 s^j_0 + [18\alpha s_0(4s-1) \\ &+ 3r_{00}(1-6s+8s^2)]y^j - 6\alpha b^j[6s_0+(2s-1)r_{00}]\Big] \end{aligned} \tag{3.2}$$

In view of equations (2.5), (3.1) and using mathematica programme, we obtain the S-curvature for the cubic Finsler metric as

$$\begin{aligned} S = \frac{1}{2(\alpha-2\beta)(\alpha^2+6B\alpha^2-\alpha\beta-8\beta^2)^2}\Big[&-2r_0(\alpha-2\beta)((1+6B)\alpha^2-\alpha\beta-8\beta^2)[-\alpha\beta\Pi-8\beta^2\Pi \\ &+\alpha^2(-6+\Pi+6B\Pi)] - 2s_0[3\alpha^2((1+3n+6B(2+3n))\alpha^3-3(3+5n+8B(-2+3n))\alpha^2\beta \\ &-6(1+2n)\alpha\beta^2+32(-1+3n)\beta^3)+(\alpha-2\beta)(-(1+6B)\alpha^2+\alpha\beta+8\beta^2)^2\Pi]+3r_{00}(\alpha-2\beta)[(1+n \\ &+B(8+6n))\alpha^3-(5-8B+5n+24Bn)\alpha^2\beta-2(3+2n)\alpha\beta^2+32n\beta^3]\Big]\,, \end{aligned} \tag{3.3}$$

Now, we are in the position to prove theorem 1.1.

Proof of theorem 1.1: First we prove the converse part.

Let us assume that β is a constant killing form *i.e.* $s_0 = 0$ and $r_{00} = 0$, putting this in the equation (3.3) has vanishes the S-curvature.

For if part let us take $S = 0$, then equation (3.3) becomes

$$t_0 + t_1\alpha + t_2\alpha^2 + t_3\alpha^3 + t_4\alpha^4 + t_5\alpha^5 = 0\,. \tag{3.4}$$

where

$$t_0 = 64\beta^4(4\beta\Pi r_0 + 4\beta\Pi s_0 - 3nr_{00})\,, \qquad t_1 = 4\beta^3(-16\beta\Pi r_0 - 16\beta\Pi s_0 + 3(3+10n)r_{00})\,,$$
$$t_2 = (192\beta^3 - 92\beta^3\Pi - 384B\beta^3\Pi)r_0 + (192\beta^3 - 576n\beta^3 - 92384B\beta^3\Pi)s_0$$
$$\quad + (12\beta^2 - 48B\beta^2 + 18n\beta^2 + 144Bn\beta^2)r_{00}\,,$$
$$t_3 = (-72\beta^2 + 22\beta^2\Pi + 144B\beta^2\Pi)r_0 + (36\beta^2 + 72n\beta^2 + 22\beta^2\Pi + 144B\beta^2\Pi)s_0$$
$$\quad + (-21\beta - 24B\beta - 21n\beta - 108Bn\beta)r_{00}\,,$$
$$t_4 = (-36\beta - 144B\beta + 8\beta\Pi + 72B\beta\Pi + 144B^2\beta\Pi)r_0 + (54\beta - 288B\beta$$
$$\quad + 90n\beta + 432Bn\beta + 8\beta\Pi + 72B\beta\Pi + 144B^2\beta\Pi)s_0 + (3 + 24B + 3n + 18Bn)r_{00}\,,$$
$$t_5 = (12 + 72B - 2\Pi - 24B\Pi - 72B^2\Pi)r_0 + (-6 - 72B - 18n - 108Bn - 2\Pi - 24B\Pi - 72B^2\Pi)s_0$$

Taking rational and irrational part of the equation (3.4), we get

$$t_0 + \alpha^2(t_2 + \alpha^2 t_4) = 0\,, \tag{3.5}$$

$$t_1 + \alpha^2(t_3 + \alpha^2 t_5) = 0\,, \tag{3.6}$$

from equation (3.5) and (3.6). We can say that α^2 will divide t_0 as well as t_1. Since α^2 is coprime with β. Solving equations (3.5) and (3.6), we get respectively

$$4\beta\Pi(r_0 + s_0) - 3nr_{00} = \gamma_1\alpha^2\,, \quad \text{for} \quad \gamma_1 = \gamma_1(x)\,,$$
$$16\beta\Pi(r_0 + s_0) - 3(10n+3)r_{00} = \gamma_2\alpha^2\,, \quad \text{for} \quad \gamma_2 = \gamma_2(x)\,.$$

After simplification the above equations for some scalar function $c = c(x)$ on M, we get

$$r_{00} = c\alpha^2\,, \quad \text{then} \quad r_0 = c\beta\,, \tag{3.7}$$

putting the above values in equation (3.4) and simplifying, we get

$$256\Pi\beta^5(c\beta + s_0) = \alpha^2\ (....)\,,$$

where (...) denotes the polynomial term in α and β . Here also α^2 doesn't divide β^5 and $(c\beta + s_0)$. Therefore $c\beta + s_0 = 0$. Differentiating it with respect to y^i, we get $cb_i + s_i = 0$, which on contracting by b^i, gives $c = 0$, implies $s_0 = 0$ and $r_{00} = 0$. Which means β is a constant killing form.

This complete the proof of theorem 1.1. □

4 Ricci curvature of cubic metric

In this section we obtain the the projective Ricci curvature for the aforesaid metric. For this we first obtain all the values of equation (2.8) by using equation (3.1) and mathematica programme as

$$V_1 = \frac{-1}{(1+n)(-1-6B+s+8s^2)^4}\big[3(6B(6(1+n)+8(1+n)s+(36-(-37+n)n)s^2 - 4(45+n(37+8n))s^3 - 256(4+n(3+n))s^4) + s(-4(1+n) - 92(1+n)s + 92(1+n)s^2 + (-238+n(-241+3n))s^3 + 32(23+n(20+3n))s^4 + 256(14+n(11+3n))s^5) + 3B^2(66+24s(1+32s)+(n+16ns)^2+n(65+8s(-1+64s))))\big],$$

$$V_2 = \frac{-1}{(1+n)(-1+2s)(-1-6B+s+8s^2)^4}\big[6(-5+192B+1224B^2-6n+186Bn+1206B^2n - n^2 - 18Bn^2 - 54B^2n^2 - 6(3(5+6n+n^2)+12B^2(-10+n+9n^2)+B(16+41n+51n^2))s + 3(-99-104n-n^2-36B(-12-5n+n^2)+384B^2(8+n+3n^2))s^2 + 2(508+675n+245n^2 + 12B(-244-125n+105n^2))s^3 - 6(330+183n-45n^2+64B(70+23n+15n^2))s^4 - 96(-36-11n+23n^2)s^5 + 2048(8+3n+n^2)s^6)\big],$$

$$V_3 = \frac{3(1+16s)(6B(3+5n)+n(-2+2s-26s^2)+3(-1+s-4s^2)-n^2(-1+s+2s^2))}{(1+n)(-1-6B+s+8s^2)^3},$$

$$V_4 = \frac{6(1+16s)(-B+s^2)}{(-1-6B+s+8s^2)^2},$$

$$V_5 = \frac{1}{(1+n)(-1+2s)^3(-1-6B+s+8s^2)^4}\big[36(-6-7n+n^2-(63+n(82+35n))s+15(8 + 3n(3+n))s^2 + (849+n(1354+785n))s^3 - (2562+n(2351+1123n))s^4 - 6(-398+495n + 771n^2)s^5 + 64(45+n(-7+100n))s^6 + 2048(-3+n(7+2n))s^7 - 216B^3(1+n)^2(-1+8s) + 9B^2(74+9n(9+n)-144s-6n(47+37n)s+48(1+n(-11+8n))s^2+160(2+3n(5+n))s^3) + 6B(13+13n+2n^2-(70+n(109+89n))s+(151+55n(1+2n))s^2+2(-178+n(317+505n))s^3 - 8(47+n(-136+199n))s^4 - 1024(-1+n(5+n))s^5))\big],$$

$$V_6 = \frac{-72n}{(1+6B-s-8s^2)^2}, \qquad V_7 = \frac{9(-2-3n+n^2)}{(1+n)(1+6B-s-8s^2)^2},$$

$$V_8 = \frac{1}{(1+n)(-1+2s)(1+6B-s-8s^2)^3}\big[18(-7+n(-8+3n)-33s+3n(-4+3n)s +6(10+(7-5n)n)s^2-256(1+2n)s^3 - 6B(1+n-2n^2+4(-14+(-23+n)n)s))\big],$$

$$V_9 = \frac{6}{1+6B-s-8s^2}\,,$$

$$V_{10} = \frac{-6(1+n+6B(3+n))+18(1+4B(-15+n)+n)s+36(3+n)s^2-96(-11+n)s^3}{(1+n)(-1+2s)(-1-6B+s+8s^2)^2}\,,$$

$$V_{11} = \frac{12(1+3B-3s-60Bs-3s^2+64s^3)}{(1+n)(-1+2s)(-1-6B+s+8s^2)^2}\,, \qquad V_{12} = \frac{6(1-8s)}{(-1+2s)^3}\,,$$

$$V_{13} = \frac{-432n}{(1+n)(-1+2s)(-1-6B+s+8s^2)^2}\,, \tag{4.1}$$

$$V_{14} = \frac{18(3+6B-n-6Bn+3(-3+4B(-19+n)+n)s+6(-1+n)s^2-16(-15+n)s^3)}{(1+n)(-1+2s)^2(-1-6B+s+8s^2)^2}\,,$$

$$V_{15} = \frac{18}{(-1+2s)(-1-6B+s+8s^2)}\,, \qquad V_{16} = \frac{6}{1-2s}\,,$$

$$V_{17} = \frac{108}{(-1+2s)^2(-1-6B+s+8s^2)}\,, \quad V_{18} = \frac{-9}{(-1+2s)^2}\,, \quad V_{19} = \frac{6(-1+n)}{(1+n)(1+6B-s-8s^2)}\,.$$

Plugging all the values of the above equation (4.1) into equation (2.7) and simplifying by using mathematica programme then we obtain the projective Ricci curvature for the aforesaid metric as

$$PRic = \frac{1}{(1+n)^2(\alpha-2\beta)^3(-(1+6B)\alpha^2+\alpha\beta+8\beta^2)^4}\sum_{i=0}^{i=13}\alpha^i t_i^{'}\,,$$

where

$$\begin{aligned} t_0^{'} &= -2048\beta^9(-3r_{00}^2(14+n(11+3n))+8(1+n)\beta(3r_{00|0}+2(1+n)Ric^{\alpha}\beta) \\ &\quad +8\beta(2(-1+n)(1+n)^2\beta(\rho_0^2+\rho_{0|0}))-3(-1+n^2)\rho_0 r_{00})\,, \\ t_1^{'} &= 256\beta^8(-3r_{00}^2(145+n(112+33n))+4(1+n)\beta(57r_{00|0}+32(1+n)Ric^{\alpha}\beta) \\ &\quad +4\beta(32(-1+n)(1+n)^2\beta(\rho_0^2+\rho_{0|0}))-57(-1+n^2)\rho_0 r_{00})\,, \\ &\vdots \\ t_{13}^{'} &= -9(1+6B)^3(12s_k s^k+s_k^i s_i^k+6Bs_k^i s_i^k)(1+n)^2\,. \end{aligned} \tag{4.2}$$

Next, we obtain the flatness condition under which the projective Ricci curvature vanishes. Let the projective Ricci curvature $PRic = 0$ which implies $U(\alpha,\beta) = 0$, where

$$U(\alpha,\beta) = t_0^{'}+\alpha t_1^{'}+\alpha^2 t_2^{'}+......+\alpha^{13}t_{13}^{'}\,, \tag{4.3}$$

Using Mathematica, we can see that

$$U(\alpha,\beta) = \frac{1}{4}(-6-7n+n^2)(\alpha-2\beta)^3(\alpha+\beta)^2(\alpha+16\beta)^2(r_{00}(\alpha-2\beta)-6s_0\alpha^2)^2 \quad \mathrm{mod}[(1+6B)\alpha^2-\alpha\beta-8\beta^2]\,.$$

Therefore

$$(...)[(1+6B)\alpha^2-\alpha\beta-8\beta^2])-\frac{1}{4}(-6-7n+n^2)(\alpha-2\beta)^3(\alpha+\beta)^2(\alpha+16\beta)^2(r_{00}(\alpha-2\beta)-6s_0\alpha^2)^2=0\,,$$

where (....) are polynomial in α and β. As $B, \frac{1}{4}$ therefore $((1+6B)\alpha^2-\alpha\beta-8\beta^2)$ doesn't divide $(\alpha-2\beta)^3$ or $(\alpha+\beta)^2$ or $(\alpha+16\beta)$. Therefore $((1+6B)\alpha^2-\alpha\beta-8\beta^2)$ will divide $(r_{00}(\alpha-2\beta)-6s_0\alpha^2)$, *i.e.*

$$(r_{00}(\alpha-2\beta)-6s_0\alpha^2)=(c_1+\alpha c_0)((1+6B)\alpha^2-\alpha\beta-8\beta^2)\,,$$

where c_1 is a $1-$form and c_0 is a scalar. Taking the rational and irrational part of the above equation we get

$$-2\beta r_{00}-6\alpha^2 s_0=c_1\alpha^2(1+6B)-8\beta^2c_1-c_0\alpha^2\beta\,, \tag{4.4}$$

$$r_{00}=c_0\alpha^2(1+6B)-\beta c_1-8c_0\beta^2\,, \tag{4.5}$$

solving the above equation we get $c_1=\frac{-8c_0\beta}{5}$, putting this value in (4.5), we get

$$r_{00}=c_0[\alpha^2(1+6B)-\frac{32}{5}\beta^2]. \tag{4.6}$$

substituting the above values into equation (4.4) and after simplification we get

$$(4B-1)c_0\beta+10s_0=0\,. \tag{4.7}$$

Differentiating the above equation with respect to y^i, gives $(4B-1)c_0b_i+s_i=0$, which on contracting by b^i, we obtain $c_0=0$. Then from equation (4.6) and (4.7), we get

$$r_{00}=0\,,\qquad s_0=0\,. \tag{4.8}$$

In view of (4.8), equation (4.3) becomes

$$\begin{aligned}&3\alpha^2(-2s_{0k}s^k_0(\alpha-8\beta)+(-3s^i_ks^k_i\alpha^2+2s^k_{0;k}(\alpha-2\beta))(\alpha-2\beta))+Ric^\alpha(\alpha-2\beta)^3\\&-(n-1)(\alpha-2\beta)^2(-(\alpha-2\beta)\rho_0^2+6\alpha^2s^k_0\rho_k-(\alpha-2\beta)\rho_{0|0})=0\,.\end{aligned}$$

By using mathematica programme, we can see that

$$\begin{aligned}&3\alpha^2(-2s_{0k}s^k_0(\alpha-8\beta)+(-3s^i_ks^k_i\alpha^2+2s^k_{0;k}(\alpha-2\beta))(\alpha-2\beta))+Ric^\alpha(\alpha-2\beta)^3\\&-(n-1)(\alpha-2\beta)^2(-(\alpha-2\beta)\rho_0^2+6\alpha^2s^k_0\rho_k-(\alpha-2\beta)\rho_{0|0})=144s_{0k}s^k_0\beta^3\quad \mathrm{mod}(\alpha-2\beta)\,.\end{aligned}$$

Therefore

$$(..)(\alpha-2\beta)=-144s_{0k}s^k_0\beta^3\,, \tag{4.9}$$

where the omitted terms are the polynomials in α and β. Again $(\alpha - 2\beta)$ doesn't divide β^3, therefore $(\alpha - 2\beta)$ will divide $s_{0i}s^i_0$. Thus

$$s_{0k}s^k_0 = (d_1 + \alpha d_0)(\alpha - 2\beta)\,,$$

where d_1 is 1−form and d_0 is a scalar. Taking rational and irrational part of the above equation and solving, we get

$$s_{0k}s^k_0 = d_0(\alpha^2 - 4\beta^3)\,, \tag{4.10}$$

by the above equation one can conclude that α is not positive-definite metric. Thus $d_0 = 0$. This implies that

$$s_{ik} = 0\,, \tag{4.11}$$

i.e. β is closed. In view of equations (4.8) and (4.11), we get $b_{i;k} = 0$, then 1-form β is parallel with respect to α.

This complete the proof of theorem 1.2. □

Now, we obtain the condition for the weak projective Ricci curvature of cubic Finsler metric.

Proof of Theorem 1.4. Let F be a cubic Finsler metric with weak projective Ricci curvature. Then from equation (2.6) we get

$$(n-1)\left[3\theta(\alpha+\beta)^3\alpha^2 + \gamma(\alpha+\beta)^6\right] = \frac{\alpha^4}{(1+n)^2(\alpha-2\beta)^3((1+6B)\alpha^2-\alpha\beta-8\beta^2)^4}\sum_{i=0}^{i=13}\alpha^i t'_i\,, \tag{4.12}$$

For the cubic metric, we have $|b^2| < \frac{1}{4}$, implies that α^4 doesn't divide $(\alpha+\beta)$ or $((1+6B)\alpha^2 - \alpha\beta - 8\beta^2)$ or $3\theta(\alpha+\beta)^3\alpha^2$. Consequently, it follows that α^4 must divide $\gamma(\alpha+\beta)^6$. However, such division is only possible if $\gamma = 0$. Combining this result with equation (4.12), then we deduce that $3\theta(\alpha+\beta)^3$ is divided by α^2. This is impossible unless $\theta = 0$. Then F reduces to a projective Ricci flat-metric.

The converse is obvious. This complete the proof. □

Example 4.1 *The Finsler metric* $\frac{\left(|y|+<a,y>\right)^3}{|y|^2}$ *for* a *=constant, is projectively Ricci flat.*

5 Conclusion

Projective Ricci curvature is a concept in differential geometry that generalizes the notion of Ricci curvature. It has various application in the fields of General Relativity, Optimal Transformation theory, Complex Geometry, Weyl Geometry, Einstein Metrics, and many more. In this article, we have proved that if the cubic metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ is projective Ricci flat ($PRic = 0$) then β is parallel with respect to Riemannian metric α. Then from equation

(2.3), the S-curvature vanishes. Therefore from equation (2.5) the Riemannian metric α is also Ricci-flat. Thus we get corollary 1.3.

References

[1] X. Cheng, Y Shen, and X. Ma. On a class of projective Ricci flat Finsler metrics. *Publ. Math. Debrecen*, 7528: 1–12, 2017.

[2] S. S Chern and Z. Shen. *Riemannian–Finsler Geometry.*World Scientific, Singapore, 2005.

[3] M. Gabrani, Rezaei and A Tayebi. On projective Ricci curvature pf Matsumoto metrics. *Acta Mathematica Universitatis Comenianae*, 90(1): 111-126, 2021.

[4] M. Matsumoto.On C-reducible Finsler spaces. *Tensor N.S.*, 24: 29–37, 1972.

[5] Z. Shen and X. Cheng. A class of Finsler metrics with isotropic S-curvature. *Israel J. Math*, 169(1): 317-340, 2009.

[6] Z. Shen. Volume comparison and its applications in Riemann–Finsler geometry. *Advance in Mathematics*, 128: 306-328,1997.

[7] Z. Shen and L Sun. On the projective Ricci curvature. *Science China Mathematics*, 64: 1629–1636, 2021.

[8] T. Tabatabaeifar *et al.*. Weighted projective Ricci curvature in Finsler geometry. *Mathematica Slovaca*, 71(1): 183–198, 2021.

[9] G. Yang.On a class of Einstein-reversible Finsler metrics. *Differential Geometry and its Applications*, 60: 80–103, 2018.

[10] H. Zhu. On a class of projectively Ricci-flat Finsler metrics. *Differential Geometry and its Applications*, 73: 101680, 2020.