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THE TARSKI-LINDENBAUM ALGEBRA OF THE CLASS OF PRIME MODELS WITH INFINITE ALGORITHMIC DIMENSIONS HAVING OMEGA-STABLE THEORIES

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Abstract. We study the class of all prime strongly constructivizable models of infinite algorithmic dimensions having ω-stable theories in a fixed finite rich signature. It is proved that the Tarski-Lindenbaum algebra of this class considered together with a Gödel numbering of the sentences is a Boolean Π^0_4-algebra whose computable ultrafilters form a dense subset in the set of all ultrafilters; moreover, this algebra is universal with respect to the class of Boolean Σ^0_3-algebras. This gives a characterization to the Tarski-Lindenbaum algebra of the class of all prime strongly constructivizable models of infinite algorithmic dimensions having ω-stable theories.

Keywords: Tarski-Lindenbaum algebra, strongly constructive model, computable isomorphism, semantic class of models, ω-stable theory, prime model.

The problem of characterizing Tarski-Lindenbaum algebras of some semantic classes of models of a finite rich signature was considered in the works [10], [7], [11], [12], [13], and others. In this work, we describe the Tarski-Lindenbaum algebra of the class of all prime strongly constructivizable models of infinite algorithmic dimensions having ω-stable theories.

Preliminaries. We consider theories in first-order predicate logic *with equality* and use general concepts of model theory, algorithm theory, Boolean algebras and

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constructive models found in [6], [14], [15], [1], and [4]. We consider signatures admitting Gödel's numberings of the formulas. Generally, *incomplete theories* are considered.

A finite signature is called *rich*, if it contains at least one n-ary predicate or function symbol for $n>1$, or two unary function symbols. A theory F is called *finitely axiomatizable* if it is defined by a finite set of axioms, and its signature is finite. The following notations are used: $FL(\sigma)$ is the set of all formulas of signature σ, $FL_n(\sigma)$ is the set of all formulas of signature σ with n free variables $x_0, x_1, ..., x_{n-1}$, $SL(\sigma)$ is the set of all sentences (i.e., closed formulas) of signature σ. In the work, we use a finite rich signature σ, and consider a fixed Gödel numbering Φ_i, $i \in \mathbb{N}$, of the set $SL(\sigma)$. The set of all Godel numbers of formulas from a set $\Sigma \subseteq SL(\sigma)$ is denoted by $\mathrm{Nom}(\Sigma)$. We use notation $\mathcal{P}(A)$ for the power-set of A, and $|A|$ or $\mathrm{Card}(A)$ for cardinality of the set A. The set of all finite tuples α of the form $\alpha = \langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$, $\alpha_i \in \{0,1\}$, is denoted by $2^{<\omega}$. The empty tuple is denoted by $\varnothing$. The *canonical* (*Gödel*) *index* of a finite tuple of zeros and ones of the form $\varepsilon = \langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1} \rangle$, $\varepsilon_i \in \{0,1\}$, is the number $\mathrm{Nom}(\varepsilon) \ = 2^n + \varepsilon_0 2^{n-1} + \varepsilon_1 2^{n-2} + \ldots + \varepsilon_{n-1} - 1$. We often write shortly $\langle \varepsilon \rangle$ instead of $\mathrm{Nom}(\varepsilon)$.

A countable or finite model $\mathfrak{M}$ together with a numeration $\nu : \mathbb{N} \xrightarrow{onto} |\mathfrak{M}|$ is called a *numerated model* and is denoted by $(\mathfrak{M}, \nu)$. We denote by $\mathfrak{M}_\nu$ the model obtained by enriching signature of $\mathfrak{M}$ to $\sigma' = \sigma \cup \{c_i \mid i \in \mathbb{N}\}$ in such a way that each new constant c_i is interpreted by $\nu(n)$ in $|\mathfrak{M}|$. A numerated model $(\mathfrak{M}, \nu)$ is said to be *constructive* if the atomic diagram $AD(\mathfrak{M}_\nu)$ is computable, and $(\mathfrak{M}, \nu)$ is *strongly constructive* if the theory $\mathrm{Th}(\mathfrak{M}_\nu)$ is decidable. Two numerated models $(\mathfrak{N}, \nu)$ and $(\mathfrak{M}, \upsilon)$ are said to be *computably isomorphic*, denoted $(\mathfrak{N}, \nu) \cong (\mathfrak{M}, \upsilon)$, if there is an isomorphism μ between $\mathfrak{N}$ and $\mathfrak{M}$, and two computable functions $f(x)$ and $g(x)$, such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{N} & \overset{f}{\underset{g}{\rightleftarrows}} & \mathbb{N} \\ \nu \downarrow & & \downarrow \upsilon \\ \mathfrak{N} & \xrightarrow{\ \mu\ } & \mathfrak{M} \end{array}$$

In a parallel way, we also write $\mathfrak{N} \cong \mathfrak{M}$ to indicate that abstract models $\mathfrak{N}$ and $\mathfrak{M}$ (without numerations) are isomorphic.

We consider Boolean algebras in the signature $\sigma_{BA} = \{\cup, \cap, -, \mathbf{0}, \mathbf{1}\}$. Let $\mathcal{B}$ be a Boolean algebra, and $a \in \mathcal{B}$. By $\mathcal{B}[a]$, we denote the restriction of the Boolean algebra $\mathcal{B}$ on the set of all subelements of the element a counting that $\mathbf{1} = a$ and $-x$ is defined as $a \smallsetminus x$ in $\mathcal{B}[a]$. If b is an element in a Boolean algebra and $\alpha \in \{0,1\}$, then b^α means b for $\alpha = 1$ and $-b$ for $\alpha = 0$. Similarly, if Φ is a formula and $\alpha \in \{0,1\}$, then Φ^α means Φ for $\alpha = 1$ and $\neg\Phi$ for $\alpha = 0$.

Let T be a theory of signature σ. On the set of sentences $SL(\sigma)$, an equivalence relation $\sim_T$ is defined by the rule $\Phi \sim_T \Psi \Leftrightarrow T \vdash (\Phi \iff \Psi)$. The logical connectives $\vee$, $\&$, and $\neg$ generate Boolean operations $\cup$, $\cap$, and $-$ on the quotient set $SL(\sigma)/\sim_T$; One can easily check that, these operations are well-defined on the $\sim_T$-classes. Thereby, we obtain an algebra of the form

$$L(T) = \big(SL(\sigma)/\sim_T \,;\ \cup, \cap, -, \mathbf{0}, \mathbf{1}\big),$$

that, in fact, is a Boolean algebra. It is called the *Tarski-Lindenbaum algebra* of the theory T; alternatively, it is called the *Boolean sentence algebra* of T, cf. [3]. By $\mathcal{L}(T)$, we denote the Tarski-Lindenbaum algebra $L(T)$ considered together with a

Gödel numbering γ; thereby, the concept of a computable isomorphism is applicable to such objects. A similar construction with respect to formulas with n free variables gives a definition of the *Tarski-Lindenbaum algebra* $L_n(T)$ *of theory* T *with* n *free variables*, $n < \omega$. Such algebras are often considered in the case when the theory T is complete.

Following Rogers, we use the notation W_n for nth computably enumerable set in Post's numbering of the family of all c.e. sets, [14, Sec. 5.2]. Moreover, W_n^t is a finite part of the set W_n that can be computed in t steps. Furthermore, we denote by W_n^X the computably enumerable set with c.e. index n relative to computability with an oracle $X \subseteq \mathbb{N}$, [14, Sec. 9.3].

We give an example of a complete set in the hierarchy class Π_1^1.

Lemma 0.1. *There is a computable sequence* $(\mathcal{L}_i, \nu_i)$, $i < \omega$, *of constructive linear orderings such that*

$$\mathcal{W} = \{n \mid \mathcal{L}_n \text{ is a well-ordered set}\} \tag{0.1}$$

is an m*-complete in the class* Π_1^1 *set.*

PROOF. See a construction in [9, p. 44]. □

1. Boolean algebras over classes of hierarchies

Let Ξ be a class of a hierarchy, eg. Σ_n^0, Δ_n^0, Π_n^0, Σ_n^1, or Π_n^1 with $0<n<\omega$ (actually, we do not concern other classes of hierarchies in this paper). A numerated Boolean algebra $(\mathcal{B}, \nu)$ is called a Ξ-*algebra*, if all its signature operations are uniformly presentable by computable functions on ν-numbers, while the equality predicate in $\mathcal{B}$ is a Ξ-relation in the numeration ν; i.e., there exist computable functions $u(x,y)$, $v(x,y)$, $w(x)$, and a relation $E(x,y)$ which represent the Boolean algebra as follows, for any $x, y \in \mathbb{N}$:

$$\begin{aligned} &\nu(x) \cup \nu(y) = \nu(u(x,y)), \\ &\nu(x) \cap \nu(y) = \nu(v(x,y)), \\ &-\nu(x) = \nu(w(x)), \\ &\nu(x) = \nu(y) \Leftrightarrow E(x,y), \quad E \in \Xi. \end{aligned} \tag{1.1}$$

A Boolean algebra $\mathcal{B}$ is said to be a Ξ-algebra, if there is a numeration ν such that $(\mathcal{B}, \nu)$ is a Boolean Ξ-algebra.

As it is accepted, cf. [14, Sec. 14.1], the class of all relations on $\mathbb{N}$ expressible by Σ_n^X-forms is denoted by Σ_1^X, while the class of all relations expressible by Π_n^X-forms is denoted by Π_1^X; we also denote $\Delta_n^X = \Sigma_n^X \cap \Pi_n^X$. In particular, Σ_1^X is the class of all X-enumerable relations on the set $\mathbb{N}$.

A numerated Boolean algebra $(\mathcal{B}, \nu)$ is said to be X-*enumerable* if all operations in $(\mathcal{B}, \nu)$ are presented by computable functions, while the equality relation is X-enumerable in the numeration ν. A Boolean algebra $\mathcal{B}$ is said to be an X-*enumerable* algebra if there is a numeration ν such that $(\mathcal{B}, \nu)$ is an X-enumerable Boolean algebra.

Lemma 1.1. *For an arbitrary numerated Boolean* Σ_1^X*-algebra* $(\mathcal{B}, \nu)$, *there is a numeration* υ *of* $\mathcal{B}$ *such that* $(\mathcal{B}, \upsilon)$ *is a Boolean* Σ_1^X*-algebra whose computable ultrafilters form a dense set in the set of all ultrafilters of the algebra* $(\mathcal{B}, \upsilon)$.

PROOF. Cf. [12, Lem. 2]. □

2. An isomorphism criterion for Boolean sentence algebras

Generalizing constructions of computable isomorphisms in the works [11], [12], and [13], we obtain the following criterion for isomorphism of a given numerated Boolean algebra to a segment of the Tarski-Lindenbaum algebra of a class of models.

Theorem 2.1. *Given a numerated Boolean algebra $(\mathcal{B},\nu)$ together with a class K of models of a finite rich signature σ and a sentence Φ of the signature σ. Suppose that g_i, $i\in\mathbb{N}$, is a generating set for $\mathcal{B}$, and θ_i, $i\in\mathbb{N}$ is a generating set for $\mathcal{L}(\mathrm{Th}(K))$. A partial mapping λ^* defined by rule $\lambda^*(g_i)=\theta_i$, $i\in\mathbb{N}$, uniquely determines an isomorphism*

$$\lambda:\mathcal{B}\to\mathcal{L}(\mathrm{Th}(K\cap\mathrm{Mod}_\sigma(\Phi))) \tag{2.1}$$

if and only if for all α in $2^{<\omega}$, $\alpha=\langle\alpha_0,\alpha_1,...,\alpha_k\rangle$, we have:

$$g_0^{\alpha_0}\cap...\cap g_k^{\alpha_k}\neq\mathbf{0}\ \Leftrightarrow\ \Phi\,\&\,\theta_0^{\alpha_0}\,\&...\&\,\theta_k^{\alpha_k}\ \text{has a } K\text{-model.} \tag{2.2}$$

Moreover, the mapping λ represents a computable isomorphism

$$\lambda:(\mathcal{B},\nu)\to(\mathcal{L}(\mathrm{Th}(K\cap\mathrm{Mod}_\sigma(\Phi))),\gamma) \tag{2.3}$$

with respect to a Godel numbering γ of $SL(\sigma)$, whenever the sequence g_i, $i\in\mathbb{N}$, is computable in $(\mathcal{B},\nu)$, while θ_i, $i\in\mathbb{N}$, is computable in the Godel numbering γ.

Proof. For any tuple α in $2^{<\omega}$, $\alpha=\langle\alpha_0,...,\alpha_k\rangle$, consider elements:

$$\begin{aligned} b_\alpha &= g_0^{\alpha_0}\cap...\cap g_k^{\alpha_k},\\ \Theta_\alpha &= \theta_0^{\alpha_0}\,\&...\&\,\theta_k^{\alpha_k}.\end{aligned} \tag{2.4}$$

For an arbitrary finite set of tuples α, β, ... $\delta\in 2^{<\omega}$, we put

$$\lambda(b_\alpha\cup b_\beta\cup...\cup b_\delta)\ =\ \Theta_\alpha\vee\Theta_\beta\vee...\vee\Theta_\delta. \tag{2.5}$$

Based on (2.2), it is possible to show that λ is a bijective mapping from $\mathcal{B}$ to $\mathcal{L}\big(\mathrm{Th}(K\cap\mathrm{Mod}(\Phi))\big)$ because the expressions involved in (2.5) cover all elements of the algebras we are considering; moreover, μ is an isomorphism from $\mathcal{B}$ into $\mathcal{L}\big(\mathrm{Th}(K\cap\mathrm{Mod}(\Phi))\big)$. Obviously, μ is a computable isomorphism in numerations ν and γ provided that the generating sequences are computable in these numerations.

Theorem 2.1 is proved. □

3. Strong constructivizations of prime models

A model $\mathfrak{M}$ is called *prime* if it can be elementarily embedded in any other model of theory $T=\mathrm{Th}(\mathfrak{N})$. Vaught's criterion in [16] states that a countable complete theory T has a prime model if and only if the family of principal type in of this theory is dense in the family of all types (equivalently, all Tarski-Lindenbaum algebras $L_n(T)$, $n<\omega$, are atomic), while the model $\mathfrak{N}$ of theory T is prime if and only if any finite tuple $\bar{a}\in|\mathfrak{N}|$ realizes a principal type of this theory.

Let us consider some technical facts about prime models.

Lemma 3.1. *Let T be a complete decidable theory. The set $\mathcal{A}'$ of all formulas of the Tarski-Lindenbaum algebras $\mathcal{L}_n(T)$, $n\in\mathbb{N}$ that are not atoms of theory T is computably enumerable.*

Proof. A formula $\varphi(\bar{x})$ is not an atom of theory T if either it is inconsistent with T or there exists a formula $\psi(\bar{x})$ such that both formulas $\varphi(\bar{x})\,\&\,\psi(\bar{x})$ and $\varphi(\bar{x})\,\&\,\neg\psi(\bar{x})$ are compatible with T. Based on this, it is simple to construct an effective enumeration of the set $\mathcal{A}'$. □

Lemma 3.2. *Let T be a complete decidable theory. The set $\mathcal{A}$ of atomic formulas of the Tarski-Lindenbaum algebras $\mathcal{L}_n(T)$, $n \in \mathbb{N}$ is computably enumerable if and only if $\mathcal{A}$ is computable, i.e. $\mathcal{A}$ is uniformly computable in the parameter n.*

PROOF. Implication $\Leftarrow$ is obvious. As for the implication $\Rightarrow$, this assertion is a simple consequence of Lemma 3.1. □

A model $\mathfrak{N}$ is said to be *strongly constructivizable* if it admits a strong constructivization. A model $\mathfrak{N}$ has algorithmic dimension 1, symbolically $\dim_{s.c.}(\mathfrak{N}) = 1$, if $\mathfrak{N}$ is strongly constructivizable, moreover, any two strong constructivizations ν_1 and ν_2 $\mathfrak{N}$ are equivalent, that is the numbered models $(\mathfrak{N}, \nu_1)$ and $(\mathfrak{N}, \nu_2)$ are constructively isomorphic, symbolically $(\mathfrak{N}, \nu_1) \cong (\mathfrak{N}, \nu_2)$. If $\dim_{s.c.}(\mathfrak{N}) = 1$, we say that the model $\mathfrak{N}$ is *autostable with respect to strong constructivizations* or simply *autostable*, since the case of weak constructivizations is not considered in this paper. A model $\mathfrak{N}$ has algorithmic dimension ω, symbolically $\dim_{s.c.}(\mathfrak{N}) = \omega$, if there is an infinite sequence of strong constructivizations ν_i, $i < \omega$ of this model such that any two numerated models in the sequence $(\mathfrak{N}, \nu_i)$, $i < \omega$ are not isomorphic.

It was proved in [8] that, for any strongly constructivizable model $\mathfrak{N}$ only cases $\dim_{s.c.}(\mathfrak{N}) = 1$ and $\dim_{s.c.}(\mathfrak{N}) = \omega$ are possible, while the cases of finite algorithmic dimensions $\dim_{s.c.}(\mathfrak{N}) = n$, $1 < n < \omega$, are impossible. Namely, the following assertion is proved.

Theorem 3.3. (Nurtazin criterion on autostability). *Let $\mathfrak{M}$ be a strongly constructivizable model of a complete decidable theory T. Then the following claims are equivalent*:

(a) *$\mathfrak{M}$ is autostable with respect to strong constructivizations,*

(b) *there is a finite sequence $\bar{a}$ of elements $|\mathfrak{M}|$ such that enrichment $(\mathfrak{M}, \bar{a})$ of the model $\mathfrak{M}$ with these constants is a prime model and the family of sets of atoms of Tarski-Lindenbaum algebras $\mathcal{L}_n(T')$, $n \in \mathbb{N}$, of theory $T' = \mathrm{Th}(\mathfrak{M}, \bar{a})$ is computable, i.e. , the indicated family is uniformly computable in the parameter n.*

Moreover, if conditions (a) and (b) are not satisfied, there exists an infinite sequence ν_i, $i < \omega$ of strong constructivizations of the model $\mathfrak{M}$ such that, for any $i, j < \omega$, $i \neq j$, numbered models $(\mathfrak{M}, \nu_i)$ and $(\mathfrak{M}, \nu_j)$ are not constructively embeddable in each other.

PROOF. See [8, Th. 1]. □

Now, we present two key statements for prime models.

Theorem 3.4. *Let T be a complete decidable theory with a prime model $\mathfrak{N}$. A model $\mathfrak{N}$ is strongly constructivizable if and only if the set of principal types of the theory T is computable.*

PROOF. See [2, Th. 1] or [5, Sec. 1]. □

Theorem 3.5. *Let T be a complete decidable theory with a strongly constructivizable prime model $\mathfrak{N}$. The following statements are equivalent to each other*:

(a) *the set of formulas $\mathcal{A}$ presenting atoms in the Tarski-Lindenbaum algebras $\mathcal{L}_n(T)$, $n \in \mathbb{N}$, is computably enumerable,*

(b) *the set of formulas $\mathcal{A}$ presenting atoms in the Tarski-Lindenbaum algebras $\mathcal{L}_n(T)$, $n \in \mathbb{N}$, is computable uniformly in n,*

(c) *the model $\mathfrak{N}$ is autostable with respect to strong constructivizations.*

PROOF. The proof is based on the Nurtazin criterion [8].

(a)$\Rightarrow$(b) This implication follows from Lemma 3.2.

(b)$\Rightarrow$(c) By assumption, the prime model $\mathfrak{N}$ is strongly constructivizable, while the theory T is complete and decidable. We also suppose that the set of atoms $\mathcal{A}$

of theory T is computable. Applying an effective version of Vaught's construction for an isomorphism between elementary equivalent prime models, [16], it is possible to construct a computable isomorphism $\mu : (\mathfrak{N}, \nu) \to (\mathfrak{N}, \nu')$ for any two strong constructivizations ν and ν' of the model $\mathfrak{N}$.

(c)⇒(a) Assume that the prime model $\mathfrak{N}$ is autostable with respect to strong constructivizations, i.e., Part (a) of Theorem 3.3 holds. Then, Part (b) of Theorem 3.3 must also hold, i.e., there is a finite sequence $\bar{a}$ of elements in $|\mathfrak{N}|$ such that the model $\mathfrak{N}^* = (\mathfrak{N}, \bar{a})$ of theory $T^* = \mathrm{Th}(\mathfrak{N}, \bar{a})$ of signature $\sigma^* = \sigma \cup \{a_i \mid a_i \in \bar{a}\}$ is a prime model and the family of atoms of the Tarski-Lindenbaum algebras $\mathcal{L}_n(\mathrm{Th}(\mathfrak{N}, \bar{a}))$, $n \in \mathbb{N}$, is computable uniformly in n. By condition, the model $\mathfrak{N}$ itself is prime. Since any finite tuple in a prime model realizes a principal type, there is a formula $\theta(\bar{z})$ that is an atom of theory T such that $\mathfrak{N} \models \theta(\bar{a})$ holds.

We consider a technical statement linking the theories T and T^*:

$$\begin{array}{l} \phi(\bar{x}) \in FL(\sigma) \textit{ is an atom of } T \Leftrightarrow \textit{ there is } \varphi(\bar{x}, \bar{z}) \in FL(\sigma) \textit{ s.t.} \\ \varphi(\bar{x}, \bar{a}) \textit{ is an atom of } T^*\!\textit{, and } T \vdash \phi(\bar{x}) \iff (\exists \bar{z})[\varphi(\bar{x}, \bar{z}) \,\&\, \theta(\bar{z})]. \end{array} \quad (3.1)$$

First, we suppose that formula $\phi(\bar{x})$ is an atom of theory T. Consider a tuple $\bar{c}$ in $\mathfrak{N}$ such that $\mathfrak{N} \models \phi(\bar{c})$. Since model $\mathfrak{N}$ is prime, all types realized in $\mathfrak{N}$ are principal. From this, it follows that there is a formula $\varphi(\bar{x}, \bar{z})$, which is an atom of theory T, and, in $\mathfrak{N}$, we have $\varphi(\bar{c}, \bar{a})$. Moreover, $\varphi(\bar{c}, \bar{a}) \,\&\, \theta(\bar{a})$ holds because, by assumption, we have $\mathfrak{N} \models \theta(\bar{a})$. Thus, formula $\varphi'(\bar{x}, \bar{z}) = \varphi(\bar{x}, \bar{z}) \,\&\, \theta(\bar{z})$ is satisfied in $\mathfrak{N}$ on the tuple $\xi = (\bar{c}, \bar{a})$. Therefore, the formula $\varphi'(\bar{x}, \bar{z})$ is equivalent in T to the formula $\varphi(\bar{x}, \bar{z})$ since the latter is an atom of theory T. As a result, we obtain that, in theory T, it is provable $\phi(\bar{x}) \sim (\exists \bar{z})[\varphi(\bar{x}, \bar{z}) \,\&\, \theta(\bar{z})]$.

Now, we suppose that $\varphi(\bar{x}, \bar{a})$ is an atom of theory T^*. Consider the formula

$$\phi(\bar{x}) = (\exists \bar{z})[\varphi(\bar{x}, \bar{z}) \,\&\, \theta(\bar{z})]. \quad (3.2)$$

Our goal is to show that this formula is an atom of the theory T. For this, it is sufficient to show that, for arbitrary tuples $\bar{c}_1$ and $\bar{c}_2$ in $\mathfrak{N}$ of appropriate lengths, it follows from $\mathfrak{N} \models \phi(\bar{c}_1)$ and $\mathfrak{N} \models \phi(\bar{c}_2)$ that there is an automorphism μ of the model $\mathfrak{N}$ such that $\mu(\bar{c}_1) = \bar{c}_2$. Since $\mathfrak{N} \models \phi(\bar{c}_1)$, the right-hand side expression in (3.2) ensures that there is a tuple $\bar{e}_1$ in $\mathfrak{N}$ such that $\mathfrak{N} \models \varphi(\bar{c}_1, \bar{e}_1) \,\&\, \theta(\bar{e}_1)$. Similarly, by virtue of $\mathfrak{N} \models \phi(\bar{c}_2)$, expression (3.2) provides that there is a tuple $\bar{e}_2$ in $\mathfrak{N}$ such that $\mathfrak{N} \models \varphi(\bar{c}_2, \bar{e}_2) \,\&\, \theta(\bar{e}_2)$. Since the formula $\theta(\bar{z})$ is an atom of theory T, and, in the model $\mathfrak{N}$ both $\theta(\bar{a})$ and $\theta(\bar{e}_1)$ hold, there must be an automorphism $\mu_1 : \mathfrak{N} \to \mathfrak{N}$ sending $\bar{e}_1$ to $\bar{a}$. Denote $\mu_1(\bar{c}_1)$ by $\bar{c}_1'$. Similarly, both $\theta(\bar{a})$ and $\theta(\bar{e}_2)$ are satisfied in the model $\mathfrak{N}$, so there must be an automorphism $\mu_2 : \mathfrak{N} \to \mathfrak{N}$ sending $\bar{e}_2$ to $\bar{a}$. Denote $\mu_2(\bar{c}_2)$ by $\bar{c}_2'$. In view of the properties of automorphisms μ_1 and μ_2, we have $\mathfrak{N} \models \varphi(\bar{c}_1', \bar{a}) \,\&\, \theta(\bar{a})$ and $\mathfrak{N} \models \varphi(\bar{c}_2', \bar{a}) \,\&\, \theta(\bar{a})$. By assumption, formula $\varphi(\bar{x}, \bar{a})$ is an atom of theory T^*. Therefore, there is an automorphism $\mu_0 : \mathfrak{N} \to \mathfrak{N}$ that is identical on tuple $\bar{a}$ and maps tuple $\bar{c}_1'$ into tuple $\bar{c}_2'$. It remains to verify that the composition of automorphisms $\mu = \mu_2^{-1}\mu_0\mu_1$ ensures the relations $\mu(\bar{c}_1) = \bar{c}_2$ and $\mu(\bar{a}) = \bar{a}$. This shows that the formula (3.2) is an atom of theory T.

Thereby, the relation (3.1) indeed takes place, thus allowing us to complete proof of the part (c)⇒(a). Based on the presentation (3.1), it is possible to organize an effective enumeration of the set $\mathcal{A}$ of atoms of theory $T = \mathrm{Th}(\mathfrak{N})$ that is exactly what is required.

Theorem 3.5 is proved.

4. Compact binary trees and the canonical construction

Definition of the notion of a compact binary tree can be found in [9, Sec, 2.1]. In this work, we use for such trees a more specialized term *compact binary trees* instead.

A *full compact binary tree* is a partially ordered set $\mathcal{D}_0 = \langle \mathbb{N}, \preccurlyeq \rangle$ of the form shown in Fig. 4.1(a). In particular, $1 \preccurlyeq 8$, $6 \preccurlyeq 6$, $\neg(6 \preccurlyeq 9)$, and $\neg(9 \preccurlyeq 6)$ is satisfied. There are two natural operations within the tree, $L(n) = 2n+1$ is the *left successor*, and $R(n) = 2n+2$ is the *right successor* of an element $n \in \mathbb{N}$. A nonempty set $\mathcal{D} \subseteq \mathbb{N}$ is said to be a *compact binary tree* whenever the following conditions are satisfied: $m \preccurlyeq n \,\&\, n \in \mathcal{D} \Rightarrow m \in \mathcal{D}$, and $L(n) \in \mathcal{D} \Leftrightarrow R(n) \in \mathcal{D}$, for all $m, n \in \mathbb{N}$. An element n in a compact binary tree $\mathcal{D}$ such that $L(n) \notin \mathcal{D}$ is said to be a *dead end* of the tree $\mathcal{D}$. The set of all dead-end elements in a tree $\mathcal{D}$ is denoted by $\mathrm{Dend}(\mathcal{D})$. A tree is called *atomic* if, above each of its elements, there is at least one dead-end element.

A few examples of compact binary trees are given in Fig. 4.1(a,b,c).

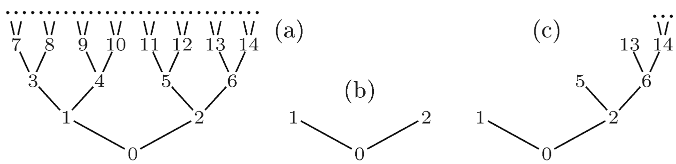

Fig. 4.1. Examples of compact binary trees

A natural operation of the *direct sum* of two given trees $\mathcal{D}_1$ and $\mathcal{D}_2$ is available. The tree $\mathcal{D}_1 \oplus \mathcal{D}_2$ is obtained by attaching isomorphic copies of the trees $\mathcal{D}_1$ and $\mathcal{D}_2$ to the two dead-end elements in the three-element tree given in Fig. 4.1(b) as it is schematically shown in Fig. 4.2.

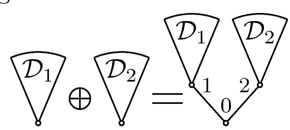

Fig. 4.2. Scheme of the operations $\oplus$ for compact binary trees

A set $\pi \subseteq \mathcal{D}$ is called a *chain* in the tree $\mathcal{D}$, if $m, n \in \pi \Rightarrow m \preccurlyeq n \vee n \preccurlyeq m$ and $m \preccurlyeq n \,\&\, n \in \pi \Rightarrow m \in \pi$, for all $m, n \in \mathbb{N}$. If $\mathcal{D}$ is a compact binary tree, we denote by $\Pi(\mathcal{D})$ the set of all *maximal chains*, while $\Pi^{fin}(\mathcal{D})$ is the set of all *maximal finite chains* in the tree $\mathcal{D}$. A tree $\mathcal{D}$ is said to be *superatomic*, if the set $\Pi(\mathcal{D})$ is at most countable.

See [9, Sec. 2.1] for definitions concerning the trees in detail.

Lemma 4.1. *There is a tree $\mathcal{D}^\star$ which is a superatomic (the more, atomic), computably enumerable, non-computable compact binary tree for which both families $\Pi^{fin}(\mathcal{D}^\star)$ and $\Pi(\mathcal{D}^\star)$ are computable.*

Proof. Let $\mathcal{D}$ be the tree shown in Fig. 4.1(c). Since $\mathcal{D}$ is computable, the set $\mathrm{Dend}(\mathcal{D})$ is also computable. Moreover, $\mathrm{Dend}(\mathcal{D})$ is infinite, thus there is a computable bijective mapping $f : \mathbb{N} \to \mathrm{Dend}(\mathcal{D})$. Let $K \subseteq \mathbb{N}$ be a computably enumerable non-computable set. Let $\mathcal{D}^\star = \mathcal{D} \cup \{L(f(i)) \mid i \in K\} \cup \{R(f(i)) \mid i \in K\}$. It is simple to check that the set $\mathcal{D}^\star$ is an atomic computably enumerable compact binary tree. Moreover, both families $\Pi(\mathcal{D}^\star)$ and $\Pi^{fin}(\mathcal{D}^\star)$ are computable. In particular, $\Pi(\mathcal{D}^\star)$ is countable ensuring that the tree $\mathcal{D}^\star$ is superatomic. By construction, reducibility $n \in K \Leftrightarrow L(f(n)) \in \mathcal{D}^\star$ takes place, thus the tree $\mathcal{D}^\star$ is not computable.

Lemma 4.1 is proved. □

Lemma 4.2. *There is an effective operator sending an arbitrary constructive linear ordering* $(\mathcal{L}, \nu)$ *into a tree* $\mathcal{D}(\mathcal{L}, \nu)$ *such that the following claims hold*:

(a) *for any constructive linear ordering* $(\mathcal{L}, \nu)$*, the tree* $\mathcal{D}(\mathcal{L}, \nu)$ *is computable,*

(b) *the tree* $\mathcal{D}(\mathcal{L}, \nu)$ *is superatomic if and only if* $\mathcal{L}$ *is well ordered,*

(c) *if* $\mathcal{L}$ *is well ordered, the family of chains* $\Pi(\mathcal{D}(\mathcal{L}, \nu))$ *is computable.*

PROOF. See [9, Lem. 2.2.1]. □

By $\mathcal{D}_n$, we denote the closure of the set W_n up to a compact binary tree, while $\mathcal{D}_n^X$ is the closure of the set W_n^X up to a compact binary tree. It can be checked that the tree $\mathcal{D}_n$ is computably enumerable, while the tree $\mathcal{D}_n^X$ is computably enumerable with oracle X. Moreover, each c.e. tree is presented in the sequence $\mathcal{D}_n$, $n \in \mathbb{N}$, and each c.e. tree in computation with oracle X is presented in the sequence $\mathcal{D}_n^X$, $n \in \mathbb{N}$. The number n is considered as a c.e. index for the tree $\mathcal{D}_n$, and n is considered as a c.e. index for the tree $\mathcal{D}_n^X$ considered in computability with an oracle X.

We denote by $\mathfrak{P}_n$ the truth-table condition with the Godel number n, cf [9, Sec. 3.1]. We write $A \models \mathfrak{P}_n$ to indicate that $\mathfrak{P}_n$ is satisfied in the set $A \subseteq \mathbb{N}$. The set of the form

$$\Omega(m) = \{A \subseteq \mathbb{N} \mid (\forall i \in W_m)\, A \models \mathfrak{P}_i\}, \tag{4.1}$$

is said to be the *parametric Stone space* with an index m, cf. definition of $\mathcal{R}_m$ in [9, Sec. 3.1].

Main statement on the *canonical construction* of finitely axiomatizable theories can be found in [9]. Its part involved in construction of this work states the following:

Theorem 4.3. [9, Th. 3.1.1] *Effectively in a pair of integers* (m, s) *and a finite rich signature* σ*, it is possible to construct a finitely axiomatizable theory* $F = \mathbb{F}\mathbb{C}(m, s, \sigma)$ *of signature* σ *together with an effective sequence* θ_n*,* $n \in \mathbb{N}$*, of sentences of the signature* σ *such that the family of extensions of* F *defined by*

$$F[A] = F \cup \{\theta_i | i \in A\} \cup \{\neg\theta_j | j \in \mathbb{N} \setminus A\},\ \ A \subseteq \mathbb{N}, \tag{4.2}$$

satisfies the following properties:

(A) *for any* $A \subseteq \mathbb{N}$*, the theory* $F[A]$ *is either complete or contradictory*;

(B) *the theory* $F[A]$*,* $A \subseteq \mathbb{N}$*, is consistent if and only if* $A \in \Omega(m)$;

(C) *for an arbitrary* $A \in \Omega(m)$*, theory* $F[A]$ *has a prime model if and only if the tree* $\mathcal{D}_s^A$ *is atomic; moreover,*

- *a prime model of the theory* $F[A]$*, if it exists, is strongly constructivizable if and only if the set* A *is computable and the family of chains* $\Pi^{fin}(\mathcal{D}_s^A)$ *is computable,*
- *a prime model of the theory* $F[A]$*, if it exists and is strongly constructivizable, has algorithmic dimension* 1 *if and only if the tree* $\mathcal{D}_s^A$ *is computable; otherwise, the model has algorithmic dimension* ω*.*

(D) *for an arbitrary* $A \in \Omega(m)$*, theory* $F[A]$ *is* ω*-stable if and only if the tree* $\mathcal{D}_s^A$ *is superatomic.*

Notice that, Part (A) and Part (B) of Theorem 4.3 state that the collection $F[A]$, $A \in \Omega(m)$, represents exactly the set of all possible complete extensions of the theory $F = \mathbb{F}\mathbb{C}(m, s, \sigma)$.

5. The sentence algebra of the class of prime models with infinite algorithmic dimensions having ω-stable theories

We fix a finite rich signature σ. We denote by $P(\sigma)$ the class of all prime models of signature σ, by $W(\sigma)$ the class of all models of signature σ with ω-stable theories, and by $M_{s.c.}^{\,\omega}(\sigma)$, the class of all strongly constructivizable models of signature σ having infinite algorithmic dimensions. Intersection of these classes $PW_{s.c.}^{\,\omega} = P(\sigma) \cap W(\sigma) \cap M_{s.c.}^{\,\omega}(\sigma)$ is the main object of our further study. We use a fixed Godel numbering Φ_i, $i \in \mathbb{N}$, for the set of all sentences of signature σ, and $\varphi_i(\bar{x}_i)$, $i \in \mathbb{N}$, for the set of all formulas of signature σ.

Theorem 5.1. *The following assertions hold*:

(a) $\big(\mathcal{L}(PW_{s.c.}^{\,\omega}), \gamma\big)$ *is a Boolean Σ_1^1-algebra, where γ is a Godel numbering of the set of sentences of signature σ,*

(b) *computable ultrafilters of $\big(\mathcal{L}(PW_{s.c.}^{\,\omega}), \gamma\big)$ form a dense set among arbitrary ultrafilters in the algebra,*

(c) *for an arbitrary numerated Boolean Σ_1^1-algebra $(\mathcal{B}, \nu)$ whose computable ultrafilters form a dense set among arbitrary ultrafilters, there is a sentence Θ of signature σ such that $(\mathcal{B}, \nu) \cong \big(\mathcal{L}\big(\mathrm{Th}(\mathrm{Mod}(\Theta) \cap PW_{s.c.}^{\,\omega})\big), \gamma\big)$.*

(d) *for an arbitrary Boolean Σ_1^1-algebra $\mathcal{B}$, there is a sentence Θ of signature σ such that $\mathcal{B} \cong \mathcal{L}\big(\mathrm{Th}(\mathrm{Mod}(\Theta) \cap PW_{s.c.}^{\,\omega})\big)$.*

Proof. (a) By the Goncharov-Nurtazin-Harrington criterion (Theorem 3.4), a prime model $\mathfrak{N}$ of a complete decidable theory T is strongly constructivizable if and only if the family of principal types realized in $\mathfrak{N}$ is computable. In addition, by a corollary of the Nurtazin criterion for autostability (Theorem 3.5) the strongly constructivizable prime model $\mathfrak{N}$ is autostable with respect to strong constructivizations if and only if the family $\mathcal{F}$ of formulas presenting atoms in the Tarski-Lindenbaum algebras $L_n(T)$, $n \in \mathbb{N}$, is computably enumerable. Otherwise, when the set $\mathcal{F}$ is not computably enumerable, by the Nurtazin criterion for autostability (Theorem 3.5), we have $\dim_{s.c.}(\mathfrak{N}) = \omega$. From this, we obtain that, a sentence Ψ of signature σ has a model in the class $PW_{s.c.}^{\,\omega}$ if and only if there are natural parameters m and n such that the following collection of conditions is satisfied

(a) $W_m \cap W_n = \varnothing \ \& \ W_m \cup W_n = \mathbb{N}$, $\quad\forall \,\&\, \forall\exists$

(b) $T = \{\Phi_i \mid i \in W_m\}$ is a complete theory, $\quad\forall\exists$

(c) $\Psi \in T$, $\quad\exists$

(d) $(\forall \varphi(\bar{x}))\,\big[\varphi(\bar{x})$ consistent $\Rightarrow \exists \theta(\bar{x})\,[\,(\,\theta(\bar{x}) \to \varphi(\bar{x})\,)\,\&\, \theta(\bar{x})$ is atomic$\,]\big]$, $\quad\forall\exists\forall$

(e) $(\forall q)\big[(\forall i \in W_q)\, \varphi_i(\bar{x}_i)$ is atomic $\Rightarrow (\exists j \notin W_q)\big(\varphi_j(\bar{x}_j)$ is atomic$\big)\big]$, $\quad\forall\exists\forall$

(f) $(\forall$ countable $\mathfrak{N} \in \mathrm{Mod}(T))(\forall n < \omega)\big[\mathcal{L}_n(\mathrm{Th}(\mathfrak{N}, |\mathfrak{N}|))$ is superatomic $\big]$. $\quad\forall^1$

Thus, we obtain a prefix $\forall^1$ for the condition as a whole. Finally, sentences Φ and Ψ are equivalent on the class $PW_{s.c.}^{\,\omega}$ if and only if $(\Phi \,\&\, \neg\Psi) \vee (\Psi \,\&\, \neg\Phi)$ does not have a model in this class. This gives the required prefix $\exists^1$ for (a).

(b) Let T be an arbitrary complete theory extending $\mathrm{Th}(PW_{s.c.}^{\,\omega})$, and Ψ be a sentence provable in T. Obviously, Ψ has a model $\mathfrak{N} \in PW_{s.c.}^{\,\omega}$. From this, we have that complete decidable theory $T' = \mathrm{Th}(\mathfrak{N})$ presenting a computable ultrafilter in $\mathrm{St}(\mathrm{Th}(PW_{s.c.}^{\,\omega}))$ is found in the neighborhood Ψ of the ultrafilter T in this Stone space. This gives the required density property posed in (b).

The proof of Part (c) is given in the forthcoming text. As for Part (d), it is a simple consequence of Part (c) and Lemma 1.1 with an m-complete in Σ^1_1 set X.

We begin the proof of Part (c) of Theorem 5.1.

We use the following m-complete in the class Π^1_1 set

$$A^1 = \{n \in \mathbb{N} \mid \mathcal{L}_n \textit{ is a well ordering}\}, \tag{5.1}$$

where $\mathcal{L}_n = (L_n, \nu_n)$, $n \in \mathbb{N}$, is an effective sequence of constructive linear orderings, cf. Lemma 0.1. Therefore, its complement $E^1 = \mathbb{N} \setminus A_1$ is an m-complete in Σ^1_1 set.

Given a numerated Boolean Σ^1_1-algebra $(\mathcal{B}, \nu)$ that satisfies

$$\textit{computable ultrafilters of } (\mathcal{B}, \nu) \textit{ form a dense set in } \mathrm{St}(\mathcal{B}). \tag{5.2}$$

It is possible to assume, that $\mathcal{B}$ is a nontrivial algebra. By definition, signature operations $\cup$, $\cap$ and $-$ in $\mathcal{B}$ are presentable by computable functions on ν-numbers, while the equality is a Σ^1_1-relation in the numeration ν, i.e., $\nu(x) = \nu(y) \Leftrightarrow H(x, y)$ is satisfied for a binary relation $H \in \Sigma^1_1$. Respectively, the inequality relation $H'(x, y) =_{dfn} (\nu(x) \neq \nu(y))$ is Π^1_1. Therefore, there is a unary relation H^* in Π^1_1 such that, for any finite tuple of zeros and ones $\alpha = \langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$, we have

$$\nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap \ldots \cap \nu(n)^{\alpha_n} \neq \mathbf{0} \ \Leftrightarrow \ \langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle \in H^*, \quad H^* \in \Pi^1_1.$$

Since H^* is Π^1_1, it is m-reducible to A^1, i.e., there is a general computable function $f(x)$ such that for arbitrary tuples $\alpha \in 2^{<\omega}$, $\alpha = \langle \alpha_0, ..., \alpha_n \rangle$, we have

$$\nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap \ldots \cap \nu(n)^{\alpha_n} \neq \mathbf{0} \ \Leftrightarrow \ \mathcal{L}_{f(\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle)} \textit{ is well ordering.} \tag{5.3}$$

Let us check that the object we built above represents some kind of a tree.

Lemma 5.2. *The following assertions are true*:

(a) $f(\varnothing) \in A^1$,

(b) *for any* α *in* $2^{<\omega}$, $f(\alpha) \in A^1 \Leftrightarrow f(\alpha 0) \in A^1$ *or* $f(\alpha 1) \in A^1$,

(c) *for any* α *in* $2^{<\omega}$, $f(\alpha) \in E^1 \Leftrightarrow f(\alpha 0) \in E^1$ *and* $f(\alpha 1) \in E^1$.

Proof. Part (a) is a consequence of the fact that algebra $\mathcal{B}$ is nontrivial. Part (b) follows from the definition of the function $f(x)$ and relation $H^*(x)$, representing an ideal in the free Boolean algebra, while (c) is a corollary from (b). □

Now, our goal is to choose a pair (m, s) of integer parameters to be considered with the canonical construction, cf. Theorem 4.3.

Choice of m. We choose m such that $\Omega(m) = \mathcal{P}(\mathbb{N})$, cf. (4.1). For this, it is enough to get m such that $W_m = \varnothing$.

Choice of s. For this purpose, we describe a computable functional $\mathscr{F}$ from $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N})$, actually, yielding compact binary trees. For an arbitrary $A \subseteq \mathbb{N}$, witness of $\mathscr{F}$ with input A is found via the function $f(x)$ in (5.3) by scheme

$$\mathscr{F} : A \mapsto \mathrm{Tree}(L^*, \nu^*) \mapsto \mathrm{Tree}(L^*, \nu^*) \oplus \mathcal{D}^\star = \mathscr{F}(A) = \mathcal{D}^A_s, \quad \textit{where} \tag{5.4}$$

(a) $(L^*, \nu^*) = \mathcal{L}_{f(\varnothing)} + \mathcal{L}_{f(\langle \alpha_0 \rangle)} + \ldots + \mathcal{L}_{f(\langle \alpha_0, \ldots, \alpha_n \rangle)} + \ldots$,

(b) $\alpha_k = \begin{cases} 1, & \textit{if } k \in A, \\ 0, & \textit{if } k \notin A, \end{cases} \quad \textit{for all } k \in \mathbb{N};$

(c) *s is an index of the algorithm for* $A \mapsto \mathscr{F}(A)$ *with an oracle* A.

Here, $\mathcal{D}^\star$ is a fixed superatomic computably enumerable non-computable tree with computable $\Pi^{fin}(\mathcal{D})$, cf. Lemma 4.1, while $(L, \nu) \mapsto \mathrm{Tree}(L, \nu)$ is an effective transformation from linear orderings to binary trees, cf. Lemma 4.2. The transformation

$A \mapsto \mathscr{F}(A)$, $A \subseteq \mathbb{N}$, is realized via Turing's computation by an algorithm $\mathcal{M}$ with an oracle A. We put s to be a Godel number of the algorithm $\mathcal{M}$, cf. (5.4)(c).

Choice of the pair of parameters (m, s) is finished.

Let us study main properties of the transformation $\mathscr{F} : A \mapsto \mathcal{D}_s^A$.

Lemma 5.3. *The following assertions hold:*

(a) *For any $A \subseteq \mathbb{N}$, $\mathcal{D}_s^A$ is a tree.*

(b) *For any computable set $A \subseteq \mathbb{N}$, $\mathcal{D}_s^A$ is a computable enumerable non-computable tree.*

(c) *For any computable set $A \subseteq \mathbb{N}$, the family $\Pi^{fin}(\mathcal{D}_s^A)$ is computable.*

(d) *For any computable set $A \subseteq \mathbb{N}$, tree $\mathcal{D}_s^A$ is superatomic $\Leftrightarrow$ linear order L^* in (5.4)(a) is well ordered.*

PROOF. Given a set $A \subseteq \mathbb{N}$. Let (L^*, ν^*) be a linear order built by rule (5.4)(a) with using function $f(x)$ in (5.3). By construction, (L^*, ν^*) is a constructive linear order whenever A is a computable set.

We turn to prove separate parts of Lemma 5.3.

(a) Use Lemma 4.2 together with Lemma 4.1 describing properties of the tree $\mathcal{D}^\star$ in (5.4). Moreover, some elementary properties of the operation $\otimes$ for trees have to be used.

(b) Suppose that A is computable. Then, (L^*, ν^*) is a constructive linear ordering. By Lemma 4.2(a), the tree Tree(L^*, ν^*) is computable, however, by Lemma 4.1, another tree $\mathcal{D}^\star$ is a computably enumerable non-computable tree. From this, we obtain that their sum Tree$(L^*, \nu^*) \oplus \mathcal{D}^\star$ is a computably enumerable non-computable tree.

(c) Suppose that A is computable. Then, (L^*, ν^*) is a constructive linear ordering. By Lemma 4.2(a), tree Tree(L^*, ν^*) is computable, thus ensuring the family $\Pi^{fin}\big(\text{Tree}(L^*, \nu^*)\big)$ to be computable. By Lemma 4.1, another family $\Pi^{fin}(\mathcal{D}^\star)$ is also computable. Based on elementary properties of the operation $\otimes$, we obtain that the family $\Pi^{fin}\big(\text{Tree}(L^*, \nu^*) \oplus \mathcal{D}^\star\big)$ is computable.

(d) Suppose that A is computable. Then, (L^*, ν^*) is a constructive linear ordering. In the case when L^* is well ordering, by Lemma 4.2(b), the tree Tree(L^*, ν^*) is superatomic while by Lemma 4.1 the tree $\mathcal{D}^\star$ is also superatomic. Therefore, their sum Tree$(L^*, \nu^*) \otimes \mathcal{D}^\star$ is superatomic. In the other case, when the order L^* is not well-ordering, by Lemma 4.2(b), the tree Tree(L^*, ν^*) is not superatomic, thus, the whole tree Tree$(L^*, \nu^*) \otimes \mathcal{D}^\star$ is not superatomic as well.

Lemma 5.3 is proved. □

Now we turn to the final part of the proof.

In accordance with demands in Part (c) of Theorem 5.1, we have to point out a sentence Θ of a given finite rich signature σ. For this, we use the canonical construction, cf. Theorem 4.3. Let us apply Theorem 4.3 to the pair (m, s) specifying also signature σ for the target theory. As a result, we obtain an effective sequence θ_i, $i \in \mathbb{N}$, of sentences of signature σ together with a finitely axiomatizable theory $F = \mathbb{F}\mathbb{C}(m, s, \sigma)$ of signature σ. We put Θ to be the sentence that is a conjunction of axioms of the theory $F = \mathbb{F}\mathbb{C}(m, s, \sigma)$.

Notice that, Part (A) together with Part (B) of Theorem 4.3 ensure that, for an arbitrary set $A \in \Omega(m)$ and corresponding sequence $\alpha \in 2^{<\omega}$ linked with A via the rule (5.4)(b), it is satisfied

$$F[A] \vdash \theta_k^{\alpha_k}, \ \textit{for all } k \in \mathbb{N}. \tag{5.5}$$

Now, we are going to check that Θ satisfies all requirements in Theorem 5.1(c).

Consider an arbitrary finite tuple of zeros and ones $\alpha = \langle \alpha_0, ..., \alpha_k \rangle$. Let us construct an intersection of elements in $\mathcal{B}$ by the rule

$$b_\alpha = \nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap ... \cap \nu(k)^{\alpha_k}. \tag{5.6}$$

We also consider a conjunction of corresponding sentences θ_i $i \in \mathbb{N}$, by the rule

$$\beta_\alpha = \theta_0^{\alpha_0} \,\&\, \theta_1^{\alpha_1} \,\&\, ... \,\&\, \theta_k^{\alpha_k} \;. \tag{5.7}$$

The main idea behind the construction is to provide the following relation:

Lemma 5.4. *For any tuple $\alpha \in 2^{<\omega}$, $b_\alpha \neq \mathbf{0}$ if and only if formula $\Theta \,\&\, \beta_\alpha$ is satisfied in a model $\mathfrak{M} \in PW^{\,\omega}_{s.c.}$.*

Proof. First, we assume that $b_\alpha \neq \mathbf{0}$. Since computable ultrafilters form a dense set among arbitrary ultrafilters in the Boolean algebra $(\mathcal{B}, \nu)$, cf. (5.2), it is possible to find a sequence $\alpha^* = \langle \alpha_i \mid i < \omega \rangle$ extending α such that the set A linked with α^* by rule (5.4)(b) is computable, and

$$\nu(0)^{\alpha_0} \cap ... \cap \nu(i)^{\alpha_i} \neq \mathbf{0}, \quad \textit{for all} \;\; i \in \mathbb{N}. \tag{5.8}$$

By (5.3), we obtain that $\mathcal{L}_{f(\langle \alpha_0, ..., \alpha_i \rangle)}$ is a well ordered set for all $i \in \mathbb{N}$; thus, the sequence (5.4)(a) consists of well orderings only. Therefore, their sum (L^*, ν^*) is also a well ordering. By Lemma 5.3(d), the tree $\mathcal{D}_s^A$ is superatomic. The more, $\mathcal{D}_s^A$ is an atomic tree. We have, $A \in \Omega(m)$ because $\Omega(m) = \mathcal{P}(\mathbb{N})$ by choice of m. By Parts (A) and (B) of Theorem 4.3, theory $F[A]$ is consistent and complete. Furthermore, by Parts (C) and (D) of Theorem 4.3, theory $F[A]$ has a prime strongly constructivizable model $\mathfrak{M}$ of infinite algorithmic dimension whose theory is ω-stable. By virtue of (5.5), we have $F[A] \vdash \theta_i^{\alpha_i}$ for all $i \in \mathbb{N}$. From this, we obtain $\mathfrak{M} \models \beta_\alpha$. Thereby, the formula $\Theta \,\&\, \beta_\alpha$ is satisfied in the model $\mathfrak{M} \in PW^{\,\omega}_{s.c.}$.

Now, we assume that sentence $\Theta \,\&\, \beta_\alpha$ is satisfied in a model $\mathfrak{M} \in PW^{\,\omega}_{s.c.}$. Consider the set of integers

$$A = \{ i \in \mathbb{N} \mid \mathfrak{M} \models \theta_i \}, \tag{5.9}$$

which is obviously computable. Build an infinite sequence $\alpha^* = \langle \alpha_i \mid i < \omega \rangle$ linked with A by the rule (5.4)(b). Since $A \in \Omega(m) = \mathcal{P}(\mathbb{N})$, theory $F[A]$ is consistent and complete. By (5.9), all axioms of the theory $F[A]$ are satisfied in the model $\mathfrak{M}$. By Part (C) of Theorem 4.3, the tree $\mathcal{D}_s^A$ is superatomic, thus the tree $\mathrm{Tree}(L^*, \nu^*)$ in (5.4) is superatomic as well. By Lemma 5.3(c), linear order $\mathcal{L}^*$ in (5.4)(a) is well ordering. In view of (5.3), we obtain that $\mathcal{L}_{f(\langle \alpha_0, ..., \alpha_s \rangle)}$ is a well ordered set for all $s \in \mathbb{N}$. Applying relation (5.3) to the sequence α, we finally obtain $b_\alpha \neq \mathbf{0}$.

Lemma 5.4 is proved. □

Let us map elements $\nu(i)$, $i \in \mathbb{N}$, of Boolean algebra $\mathcal{B}$ to sentences θ_i, $i \in \mathbb{N}$, of signature σ by the rule $\lambda^*(\nu(k)) = \theta_k$, $k \in \mathbb{N}$. By Lemma 5.4 together with Theorem 2.1, it is possible to extend this mapping up to a computable isomorphism $\lambda : (\mathcal{B}, \nu) \to \big(\mathcal{L}\big(\mathrm{Th}\big(\mathrm{Mod}(\Theta) \cap PW^{\,\omega}_{s.c.}\big)\big), \gamma\big)$ that is exactly what is required.

Thereby, Part (c) of Theorem 5.1 is completely proved.

Theorem 5.1 is proved. □

6. Conclusion

Theorem 5.1 characterizes the Tarski-Lindenbaum algebra of the class of all prime strongly constructivizable models of infinite algorithmic dimensions having ω-stable theories. For comparison, the work [11] have been devoted to a similar problem for the class $P_{s.c.}$ of all strongly constructivizable prime models, while another work [13] concerned the same problem for the class $P^1_{s.c.}$ of all prime strongly constructivizable models of algorithmic dimension one.

Let us turn to a few open questions concerning statement of Theorem 5.1. We write $RM(T)$ for Morley Rank of a complete theory T. We use notation $R^{\leqslant\alpha}(\sigma)$ for the class of models $\mathfrak{M}$ of signature σ satisfying $RM(\mathrm{Th}(\mathfrak{M})) \leqslant \alpha$. Similar notation $R^{\geqslant\alpha}(\sigma)$ is also used.

By α, we denote an arbitrary constructive ordinal.

Question 6.1. *Find a characterization of the Tarski-Lindenbaum algebra of the class of models* $PW^{\ \omega}_{s.c.} \cap R^{\leqslant\alpha}$.

Question 6.2. *Find a characterization of the Tarski-Lindenbaum algebra of the class of models* $PW^{\ \omega}_{s.c.} \cap R^{\geqslant\alpha}$.

Hypothesis 6.3. *For any constructive ordinal* α*, the Tarski-Lindenbaum algebra of the class of models* $PW^{\ \omega}_{s.c.} \cap R^{\geqslant\alpha}$ *is characterized by a statement analogous to that presented in Theorem 5.1.*

Notice that, the canonical construction is actually applicable to the cases (involved in the questions we formulated above) with an arbitrary ordinal α satisfying $15 \leqslant \alpha < \omega_1^{CK}$, cf. [9, Th. 3.1.1].

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