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COERCIVE ESTIMATE FOR NON-HOMOGENEOUS DIFFERENTIAL OPERATOR ON HEISENBERG GROUP

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Abstract: We constructed a linear non-homogeneous differential operator $\mathcal{Q}$ on the Heisenberg group, the kernel of which is interconnected with the Lie algebra of the group of conformal mappings. More precisely, the kernel of $\mathcal{Q}$ coincides with first two coordinate functions of mappings of the Lie algebra of the conformal mappings. We received integral representation formula and proved a coercive estimate for this operator.

Keywords: Heisenberg group, integral representation formula, conformal mapping, coercive estimate.

1 Introduction

Consider a deformation $F\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$. In linearized theory of elasticity, linear strain tensor $\varepsilon = \frac{1}{2}\big((DF) + (DF)^t\big)$ defines a deviatoric strain tensor ε' of deformation which is defined by

$$\varepsilon' = \varepsilon - \frac{\operatorname{tr}\varepsilon}{3} I.$$

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It measures the distortion independent of volume change.

Multidimensional generalization of deviatoric strain is defined as differential operator

$$\mathbb{Q}F=\frac{DF+(DF)^t}{2}-\frac{\operatorname{tr}DF}{n}I,\quad F\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}^n.$$

The operator $\mathbb{Q}$ linearizes the differential equation defining conformal mappings: $(DF)^tDF=(\det DF)^2I$. The latter means that DF is generalized orthogonal matrix. The kernel of operator $\mathbb{Q}$ coincides with the Lie algebra of the group of conformal mappings. Coercive estimate for $\mathbb{Q}$ states

$$\|F-\Pi F\mid W_p^1(\Omega,\mathbb{R}^n)\|\leqslant C\|\mathbb{Q}F\mid L_p(\Omega)\|,\quad F\in W_p^1(\Omega,\mathbb{R}^n),\ p>1,$$

where Π is a projection to kernel of $\mathbb{Q}$. This coercive estimate is the basis of Yu. G. Reshetnyak's proof of the stability of conformal mappings [18].

The aim of this paper is to construct the analog of deviatoric strain on the Heisenberg group $\mathbb{H}$ with the sub-Riemannian metric, and to prove the coercive estimate for it. Note that the analog of the linear strain tensor on $\mathbb{H}$ was constructed and studied in [10]. On Heisenberg groups $\mathbb{H}^n$ for $n>1$ the author have constructed the analog of the operator $\mathbb{Q}$ and proved the stability of conformal mappings with the help of coercive estimate for $\mathbb{Q}$ [8]. Therefore, in this paper we focus on the fisrt Heisenberg group $\mathbb{H}^1$.

Many papers are devoted to analysis on Heisenberg groups. H. M. Reimann and A. Koranyi wrote foundations for the theory of quasiconformal mappings on the Heisenberg group [14, 15]. The theory of mappings with bounded distortion on Heisenberg groups is developed in the works of S. K. Vodopyanov [20], L. Capogna [2], N. S. Dairbekov [4, 5]. Introduction to the Heisenberg group can be found in the book [3].

The Heisenberg group $\mathbb{H}$ is a Lie group diffeomorphic to $\mathbb{R}^3$ with the following group law:

$$(x_1,x_2,x_3)\cdot(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3-2x_1y_2+2x_2y_1).$$

Vector fields

$$X_1=\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_3},\quad X_2=\frac{\partial}{\partial x_2}-2x_1\frac{\partial}{\partial x_3},\quad X_3=\frac{\partial}{\partial x_3}=-\frac{1}{4}[X_1,X_2]$$

are left-invariant basis vector fields of the Lie algebra. Subbundle $H\mathbb{H}$ of the tangent bundle spanned by X_1,X_2 is said to be *horizontal*. Vector fields X_1,X_2 are supposed to be orthonormal. *Carnot–Carathéodory distance* d is the infimum of the lengths of all horizontal curves joining 2 points (absolutely continuous curve γ is *horizontal* if $\dot\gamma(t)\in H_{\gamma(t)}\mathbb{H}$ almost everywhere). In the sequel we will consider Heisenberg group $\mathbb{H}$ with metric d and Lebesgue measure in $\mathbb{R}^3$. The latter is the bi-invariant Haar measure on $\mathbb{H}$.

Consider a mapping $F=(f_1,f_2,f_3)\colon U\subset\mathbb{H}\to\mathbb{H}$. Differentiability conception on Heisenberg group means that F preserves horizontal subbundle: $X_1F(x),X_2F(x)\in H_{F(x)}\mathbb{H}$ [17], i. e. so called contact conditions

$$X_1f_3=2f_2X_1f_1-2f_1X_1f_2,\quad X_2f_3=2f_2X_2f_1-2f_1X_2f_2\tag{1.1}$$

are valid. In particular, f_1 and f_2 define vertical coordinate function f_3 up to a constant. Therefore, it is sufficient to consider the mappings $(f_1, f_2)\colon \Omega \subset \mathbb{H} \to \mathbb{R}^2$ instead of $(f_1, f_2, f_3)\colon \Omega \subset \mathbb{H} \to \mathbb{H}$.

We consider a mapping $w = (f_1, f_2) : \Omega \subset \mathbb{H} \to \mathbb{R}^2$ of the class $W_p^2(\Omega; \mathbb{R}^2)$ (for the definition of the Sobolev class, see the Section 2.4). Denote $D_h w = \begin{pmatrix} X_1 f_1 & X_2 f_1 \\ X_1 f_2 & X_2 f_2 \end{pmatrix}$. Introduce the analog of deviatoric strain:

$$\mathcal{Q}w = \begin{pmatrix} \mathcal{S}w \\ \mathcal{T}w \end{pmatrix}, \qquad \begin{aligned} \mathcal{S}w &= \frac{D_h w + (D_h w)^t}{2} - \frac{\operatorname{tr} D_h w}{2} I, \\ \mathcal{T}w &= \begin{pmatrix} X_2X_1f_2 - 2X_1X_2f_2 - X_1^2 f_1 \\ 2X_2X_1f_1 - X_1X_2f_1 + X_2^2 f_2 \end{pmatrix}. \end{aligned}$$

Operator $\mathcal{Q}$ consists of two parts: $\mathcal{S}$ and $\mathcal{T}$. Operator $\mathcal{S}$ generalizes deviatoric strain, it linearizes conformal equation $(D_h w)^t D_h w = (\det D_h w)^2 I$. Functions f_1 and f_2 determine subgradient $\nabla_h f_3 = (X_1 f_3, X_2 f_3)$ (1.1). Therefore, the analog of the equality of mixed partials should be valid, see [1, Theorem 2.9.8]. Operator $\mathcal{T}$ linearizes the analog of the equality of mixed partials. A detailed justification for the appearance of $\mathcal{T}$ is given in [10, §3.1].

The kernel of $\mathcal{S}$ is infinite-dimensional because it includes all the solutions to Cauchy – Riemann system (independent of x_3). Theorem 1.1 shows that kernel of $\mathcal{Q}$ is finite dimensional.

Theorem 1.1. *The kernel of the operator $\mathcal{Q}$ coincides with the horizontal coordinate functions of the Lie algebra of the group of conformal mapping on $\mathbb{H}$. In particular, $\ker \mathcal{Q}$ is finite dimensional.*

The Lie algebra of the group of conformal mapping is written in Lemma 3.1.

Next theorem establishes the integral representation formula of Sobolev functions in terms of the operator $\mathcal{Q}$.

Theorem 1.2. *Let $\varkappa = 2\sqrt{6} + \frac{3}{2}$. There is a projector $\mathbf{\Pi}\colon L_1(\mathrm{Box}(\mathbf{0}, 1); \mathbb{R}^2) \to \ker \mathcal{Q}$ such that for each function $w \in C^\infty(\mathrm{Box}(\mathbf{0}, \varkappa); \mathbb{R}^2)$ the integral representation formula*

$$w(\mathbf{x}) = \mathbf{\Pi} w(\mathbf{x}) + \int\limits_{\mathrm{Box}(\mathbf{0},\varkappa)} K(\mathbf{y}, \mathbf{x}) \mathcal{Q}w(\mathbf{y})\, d\mathbf{y}$$

holds for every $\mathbf{x} \in \mathrm{Box}(\mathbf{0}, 1)$, where

$$K(\mathbf{y}, \mathbf{x})\mathcal{Q}w(\mathbf{y}) = L(\mathbf{y}, \mathbf{x})\mathcal{Q}w(\mathbf{y}) + M(\mathbf{y}^{-1}\mathbf{x})\mathcal{S}w(\mathbf{y}) + N(\mathbf{y}^{-1}\mathbf{x})\mathcal{T}w(\mathbf{y})$$

with $L(\mathbf{y}, \mathbf{x}) \in C^\infty(\mathbb{H} \times \mathbb{H})$, $\operatorname{supp} L(\cdot, \mathbf{x}) \subseteq \overline{\mathrm{Box}(\mathbf{0}, \varkappa)}$ for $\mathbf{x} \in \mathrm{Box}(\mathbf{0}, 1)$; $M, N \in C^\infty(\mathbb{H} \setminus \{\mathbf{0}\})$, $\operatorname{supp} M, \operatorname{supp} N \subseteq \overline{\mathrm{Box}(\mathbf{0}, 1)}$, and

$$|X^J M(\mathbf{x})| \leqslant C d(\mathbf{x}, \mathbf{0})^{-k-3}, \quad |X^J N(\mathbf{x})| \leqslant C d(\mathbf{x}, \mathbf{0})^{-k-2}$$

for any multi-index $J = (i_1, \ldots, i_k) \in \{1, 2\}^k$. Here $X^J = X_{i_1} \ldots X_{i_k}$.

Here

$$\mathrm{Box}(\mathbf{0}, r) = \{\mathbf{x} \in \mathbb{H} : |x_1| < r,\ |x_2| < r,\ |x_3| < r^2\}, \quad r > 0.$$

In this paper we obtain also coercive estimates for the operator $\mathcal{Q}$ on John domains. In 1961 F. John [12] introduced the notion of a twisted interior cone condition that nowadays is referred to as John domains. The class of John domains is much larger than the class of domains with the interior cone condition. It includes smooth domains, Lipschitz domains and certain fractal domains (for example the snowflake domain).

A domain $\Omega \subset \mathbb{H}$ is a *John domain* $J(\alpha, \beta)$, $0 < \alpha \leq \beta$ [12], if there exists a point $\mathbf{x}_0 \in \Omega$ such that every $\mathbf{x} \in \Omega$ can be joined in Ω with $\mathbf{x}_0$ by a rectifiable curve γ parameterized by the arc length, such that

$$\gamma(0) = \mathbf{x},\ \gamma(l) = \mathbf{x}_0,\ l \leq \beta, \text{ and } \mathrm{dist}(\gamma(s), \partial\Omega) \geq \frac{\alpha s}{l} \text{ for all } s \in [0, l].$$

It is obvious that $B(\mathbf{x}_0, \alpha) \subset \Omega \subset B(\mathbf{x}_0, \beta)$.

Denote by $\|\cdot\|_{q,\Omega}$ the L_q-norm of a measurable vector-valued function on a set Ω. L_q-norm is taken with respect to the Lebesgue measure on $\mathbb{R}^3$.

Next theorem is the main result of the paper. It establishes the coercive estimate for $\mathcal{Q}$.

Theorem 1.3. *Let $1 < p < \infty$, Ω be a John domain $J(\alpha, \beta)$, mapping $w : \Omega \subset \mathbb{H} \to \mathbb{R}^2$ be of the Sobolev class $W_p^2(\Omega; \mathbb{R}^2)$. Then the coercive estimates*

$$\|w - \mathbf{\Pi} w\|_{q,\Omega} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^{\theta} (\mathrm{diam}\,\Omega)^{1-\frac{4}{p}+\frac{4}{q}} \Big[\|\mathcal{S}w\|_{p,\Omega} + \mathrm{diam}\,\Omega \|\mathcal{T}w\|_{p,\Omega}\Big],$$

$$\|D_h(w - \mathbf{\Pi} w)\|_{p,\Omega} \leqslant D\Big(\frac{\beta}{\alpha}\Big)^{6} \Big[\|\mathcal{S}w\|_{p,\Omega} + \mathrm{diam}\,\Omega \|\mathcal{T}w\|_{p,\Omega}\Big]$$

are fulfilled where
1) $\mathbf{\Pi} \colon L_1(\Omega; \mathbb{R}^2) \to \ker \mathcal{Q}$ *is the projection to the kernel of* $\mathcal{Q}$;
2) $\theta = \begin{cases} 7 & \textit{if } q \neq \infty, \\ 7 + 4/p & \textit{if } q = \infty, \end{cases}$
3) q *satisfies*

$$\begin{cases} p \leqslant q \leqslant \frac{4p}{4-p} \textit{ for } 1 < p < 4; \\ p \leqslant q < \infty \textit{ for } p = 4; \\ p \leqslant q \leqslant \infty \textit{ for } p > 4. \end{cases} \tag{1.2}$$

Constants C, D are independent of w and Ω.

Corollary 1.1. *Let $1 < p < \infty$, Ω be a bounded domain in $\mathbb{H}$, mapping $w : \Omega \subset \mathbb{H} \to \mathbb{R}^2$ be of the Sobolev class $W_{p,O}^2(\Omega; \mathbb{R}^2)$. Then the coercive estimates*

$$\|w\|_{q,\Omega} \leqslant C(\mathrm{diam}\,\Omega)^{1-\frac{4}{p}+\frac{4}{q}} \Big[\|\mathcal{S}w\|_{p,\Omega} + \mathrm{diam}\,\Omega \|\mathcal{T}w\|_{p,\Omega}\Big],$$

$$\|D_h w\|_{p,\Omega} \leqslant D\Big[\|\mathcal{S}w\|_{p,\Omega} + \mathrm{diam}\,\Omega \|\mathcal{T}w\|_{p,\Omega}\Big]$$

are fulfilled where q is a number satisfying (1.2). *Constants C, D are independent of w and Ω.*

The paper is organized as follows. In Section 2 we give definitions and auxiliary assertions, which are necessary for the formulations and proofs of Theorems 1.1 – 1.3. It includes definitions of convolution, polynomials and Sobolev spaces on Heisenberg group. In Section 3, we calculate the kernel of $\mathcal{Q}$. Section 4 is devoted to the proof of Sobolev-type integral representation theorem. Coercive estimate is proved in Section 5. We base on the results of the work [9] for proving Theorems 1.2 and 1.3. Theorems of [9] are valid for the homogeneous differential operators, therefore we can not directly use them. In Section 6, we write down the operator $\mathcal{Q}$ on groups $\mathbb{H}^n$ for $n > 1$ and show why it does not suit to the first Heisenberg group.

The statement of the results of the present article and a brief scheme of proofs are given in the short communication [11].

2 Definitions and Auxiliary Results

2.1. Heisenberg group. Points in Heisenberg group we denote by bold symbols: $\mathbf{x} \in \mathbb{H}$. It is convenient to use the following notations:

$$\mathbf{x} = (x, y, t), \qquad X = X_1, \quad Y = X_2, \quad T = X_3.$$

$$\mathbf{x} = (z, t), \quad z = x + iy, \qquad Z = \frac{1}{2}(X - iY), \quad \overline{Z} = \frac{1}{2}(X + iY).$$

Dilation $\delta_s(x, y, t) = (sx, sy, s^2 t)$, $s > 0$, is the group homomorphism. Metric d is homogeneous: $d(\delta_s \mathbf{x}, \delta_s \mathbf{y}) = s^4 d(\mathbf{x}, \mathbf{y})$. Hausdorff dimension of $\mathbb{H}$ with respect to Carnot – Carathéodory metric d equals 4.

Set $U \subset \mathbb{H}$. By $|U|$ we denote the Lebesgue measure of U in $\mathbb{R}^3$. Lebesgue measure is biinvariant Haar measure.

2.2. Box quasimetric on $\mathbb{H}$. Consider the norm

$$d_\infty(\mathbf{x}) = \sup\{|x_1|, |x_2|, \sqrt{|x_3|}\}$$

for any point $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{H}$. The norm d_∞ defines the quasimetric $d_\infty(\mathbf{x}, \mathbf{y}) = d_\infty(\mathbf{x}^{-1} \cdot \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{H}$.

The function $d_\infty(\cdot, \cdot)$ is a quasimetric with constant $\sqrt{\frac{3}{2}}$ [10, Remark 3]. Quasimetric d_∞ is homogeneous, left-invariant and equivalent to Carnot – Carathéodory metric d [16].

Denote a ball in quasimetric d_∞ by $\operatorname{Box}(\mathbf{a}, r) = \{\mathbf{x} : d_\infty(\mathbf{a}, \mathbf{x}) < r\}$. It follows $|\operatorname{Box}(\mathbf{a}, r)| = r^4 |\operatorname{Box}(\mathbf{0}, 1)|$, where $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^3$.

2.3. Convolution. For any integrable functions f and g, the convolution $f * g$ is a function defined as follows:

$$f * g(\mathbf{x}) = \int_{\mathbb{H}} f(\mathbf{x}^{-1}\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{H}} f(\mathbf{y}) g(\mathbf{y}^{-1}\mathbf{x})\, d\mathbf{y}.$$

(Here integration is taken over the Lebesgue measure in $\mathbb{R}^3$.)

It is easy to verify that the following properties holds for any smooth integrable functions f, g, and a left-invariant vector-field X_i, $i = 1, 2, 3$:

1) $f * g \neq g * f$;
2) $X_i(f * g) = f * X_i g$;
3) $(X_i f) * g = -f * (X_i^R g)$, where

$$X_1^R = -\frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \quad X_2^R = -\frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}, \quad X_3^R = -\frac{\partial}{\partial x_3}$$

are the right-invariant vector fields: $X_i^R f = R X_i R f$ with $Rf(\mathbf{x}) = f(\mathbf{x}^{-1})$.

2.4. $C^k_{\mathbb{H}}$-smooth functions and Sobolev functions. Let $\sigma_1 = \sigma_2 = 1$, $\sigma_3 = 2$ be the weights of vector fields X_1, X_2, X_3.

To a multi-index $I = (i_1, \dots, i_k) \in \{1, 2, 3\}^k$, it corresponds the differential operator $X^I = X_{i_1} \dots X_{i_k}$ and the weight $d(I) = \sum_{j=1}^k \sigma_{i_j}$. By multi-index with subindex h we shall always denote the horizontal multi-index $I_h = (i_1, \dots, i_k) \in \{1, 2\}^k$. Obviously, the length of the horizontal multi-index coincides with its weight: $d(I_h) = k$.

By $\nabla_h^k = \{X^{I_h} : d(I_h) = k\}$ we will denote all horizontal derivatives of order k. That is

$$\begin{aligned} &\nabla_h f = (X_1 f, X_2 f) \text{ is a subgradient of a function } f, \\ &\nabla_h^2 f = \begin{pmatrix} X_1^2 f & X_1 X_2 f & X_2 X_1 f & X_2^2 f \end{pmatrix}, \\ &\nabla_h^3 f = \begin{pmatrix} X_1^3 f & X_1^2 X_2 f & X_2 X_1^2 f & X_1 X_2^2 f & X_2^2 X_1 f & X_2^3 f \end{pmatrix}. \end{aligned}$$

Suppose $\Omega \subset \mathbb{H}$ is an open set. Denote by $C^k_{\mathbb{H}}(\Omega)$ the vector space of continuous functions $f\colon \Omega \to \mathbb{R}$ so that $\nabla_h f, \dots, \nabla_h^k f$ are continuous.

We note that $C^k_{\mathbb{H}}$ is a proper subclass of C^k.

Let Ω be a domain in $\mathbb{H}$, $s, l \in \mathbb{N}$, $1 \leq q \leq \infty$. The *Sobolev space* $W^l_q(\Omega, \mathbb{R}^s)$ consists of the functions $f = (f_1, \dots, f_s)\colon \Omega \to \mathbb{R}^s$ having the weak derivatives $X^{I_h} f_j$ for $d(I_h) = k$, $k = 1, \dots, l$, $j = 1, \dots, s$, and a finite norm

$$\|f\|_{W^l_q(\Omega, \mathbb{R}^s)} = \|f\|_{q,\Omega} + \sum_{0 < d(I_h) \leqslant l} \|X^{I_h} f\|_{q,\Omega}$$

where $\| \cdot \|_{q,\Omega}$ is L_q-norm of a measurable vector-valued function on Ω. L_q-norm is taken with respect to the Lebesgue measure on $\mathbb{R}^3$.

As usual, $W^l_{q,O}(\Omega, \mathbb{R}^s)$ states for Sobolev functions supported in Ω.

Analogously to the Euclidean case, smooth functions are dense in the space of Sobolev functions [6].

2.5. Polynomials on $\mathbb{H}$. For a multi-index $I = (i_1, \dots, i_k) \in \{1, 2, 3\}^k$, set $\mathbf{x}^I = x_{i_1} \cdot \ldots \cdot x_{i_k}$. Clearly, $\mathbf{x}^I$ is homogeneous of degree $d(I)$, that is $(\delta_t \mathbf{x})^I = t^{d(I)} \mathbf{x}^I$.

A function P is said to be a *polynomial* on $\mathbb{H}$ if $P(\mathbf{x}) = \sum_I a_I \mathbf{x}^I$ where all but finitely many of the coefficients a_I vanish. For the polynomial P, the (*homogeneous*) *degree* is said to be $\max\{d(I)\colon a_I \neq 0\}$. Denote by $\mathcal{P}_k$ the linear space of polynomials on $\mathbb{H}$ of homogeneous degree $\leqslant k$.

Following [9, p. 70] define basis homogeneous polynomials Q_{I_h}. Let a number $k \in \mathbb{N}$ and a function $v \in C^\infty(\mathbb{H})$. We have

$$\frac{d^k f(0)}{ds^k} = \sum_{d(I_h)=k} Q_{I_h}(\mathbf{y}) X^{I_h} v(\mathbf{x}_0) \quad \text{where } f(s) = v(\mathbf{x}_0 \delta_s \mathbf{y}).$$

Write down basis polynomials for $k = 1, 2, 3$.

$k = 1$:

$$Q_1(\mathbf{x}) = x_1, \quad Q_2(\mathbf{x}) = x_2;$$

$k = 2$:

$$Q_{11}(\mathbf{x}) = x_1^2, \quad Q_{12}(\mathbf{x}) = x_1x_2 - \frac{x_3}{2}, \quad Q_{21}(\mathbf{x}) = x_1x_2 + \frac{x_3}{2}, \quad Q_{22}(\mathbf{x}) = x_2^2;$$

$k = 3$:

$$\begin{aligned}
&Q_{111}(\mathbf{x}) = x_1^3,\\
&Q_{112}(\mathbf{x}) = \frac{3}{4}(2x_1^2x_2 - x_1x_3), \quad Q_{121}(\mathbf{x}) = 0, \quad Q_{211}(\mathbf{x}) = \frac{3}{4}(2x_1^2x_2 + x_1x_3),\\
&Q_{221}(\mathbf{x}) = \frac{3}{4}(2x_1x_2^2 + x_2x_3), \quad Q_{212}(\mathbf{x}) = 0, \quad Q_{122}(\mathbf{x}) = \frac{3}{4}(2x_1x_2^2 - x_2x_3),\\
&Q_{222}(\mathbf{x}) = x_2^3.
\end{aligned}$$

3 Kernel of $\mathcal{Q}$

3.1. Lie algebra of the conformal group. The group $SU(1,2)$ acts in a natural way as a group $\mathbf{M}^+$ of conformal mappings on the one-point compactification $\overline{\mathbb{H}}$ of $\mathbb{H}$. The group $\mathbf{M}^+$ is generated by the subgroups [15, p. 35]:

1) left translations $\pi_{\mathbf{a}}(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$, $\mathbf{a} \in \mathbb{H}$;
2) dilations $\delta_s \mathbf{x} = (sz, s^2 t)$, $s > 0$, $\mathbf{x} = (z, t)$, $z \in \mathbb{C}$, $t \in \mathbb{R}$;
3) rotations $R_A(\mathbf{x}) = (Az, t)$, $A \in \mathbb{C}$, $|A| = 1$;
4) inversion $j(\mathbf{x}) = \left(\dfrac{z}{|z|^2 - it}, \dfrac{-t}{|z|^4 + t^2}\right)$.

Together with reflection $\iota(\mathbf{x}) = (\overline{z}, -t)$ they form the full group of conformal mappings: $\mathbf{M}^+ \cup \iota\mathbf{M}^+$.

In the following lemma we write vector fields in the basis $\{Z, \overline{Z}, T\}$, that is, the notion $(u + iv, p)$ stands for the vector field

$$V = uX + vY + pT = (u + iv)Z + (u - iv)\overline{Z} + pT.$$

The vector field $V \in C^1(\mathbb{H}, T\mathbb{H})$ is called an *infinitesimal generator of the one-parameter group* $F_s(\mathbf{x}) = F(s, \mathbf{x})$ if

$$\frac{d}{ds}F = V(F).$$

Lemma 3.1 ([7, Lemma 6]). *Let a vector field generates a conformal flow. Then it is a linear combination of the following fields*:

1) $\big(a, b+4\operatorname{Im}(a\overline{z})\big)$, $a\in\mathbb{C}$, $b\in\mathbb{R}$, $F_s=\pi_{(sa,sb)}$.
2) $(\alpha z, 2\alpha t)$, $\alpha\in\mathbb{R}$, $F_s=\delta_{e^{\alpha s}}$.
3) $\big(i\mu z, 2\mu|z|^2\big)$, $\mu\in\mathbb{R}$, $F_s=R_{e^{i\mu s}}$.
4) $\big((|z|^2-it)c-2z^2\overline{c}, -4|z|^2\operatorname{Im}(z\overline{c})-4t\operatorname{Re}(z\overline{c})\big)$, $c\in\mathbb{C}$, $F_s=j\circ\pi_{(sc,0)}\circ j$.
5) $\big(idz(|z|^2-it), d(|z|^4+t^2)\big)$, $d\in\mathbb{R}$, $F_s=j\circ\pi_{(0,sd)}\circ j$.

3.2. Proof of Theorem 1.1. Consider a mapping $w=(u,v)$. Recall $\mathcal{Q}w=\begin{pmatrix}\mathcal{S}w\\ \mathcal{T}w\end{pmatrix}$, where

$$\mathcal{S}w=\frac{1}{2}\big(D_hw+(D_hw)^t-\operatorname{tr}(D_hw)I\big)=\frac{1}{2}\begin{pmatrix}Xu-Yv & Yu+Xv\\ Xv+Yu & Yv-Xu\end{pmatrix}$$

and

$$\mathcal{T}w=\begin{pmatrix}YXv-2XYv-X^2u\\ 2YXu-XYu+Y^2v\end{pmatrix}.$$

Set

$$q_1=\frac{1}{2}(Xu-Yv),\qquad q_3=YXv-2XYv-X^2u,$$
$$q_2=\frac{1}{2}(Yu+Xv),\qquad q_4=2YXu-XYu+Y^2v.$$

Obvious calculations give us

$$XYu=3YXu-2Yq_1-q_4, \tag{3.1}$$
$$Y^2u=-3X^2u+4Xq_1+2Yq_2-q_3, \tag{3.2}$$
$$YXv=3XYv+2Xq_1+q_3, \tag{3.3}$$
$$X^2v=-3Y^2v-4Yq_1+2Xq_2+q_4. \tag{3.4}$$

Notice, that

$$X^2Y-2XYX+YX^2=-4[X,T]=0, \tag{3.5}$$
$$Y^2X-2YXY+XY^2=-4[Y,T]=0. \tag{3.6}$$

Applying conditions (3.5) and (3.6) we obtain

$$X^2Yu+YX^2u=2XYXu\overset{(3.1)}{=}\frac{2}{3}X(XYu+2Yq_1+q_4),$$
$$Y^2Xu+XY^2u=2YXYu\overset{(3.1)}{=}2Y(3YXu-2Yq_1-q_4),$$
$$X^2Yv+YX^2v=2XYXv\overset{(3.3)}{=}2X(3XYv+2Xq_1+q_3),$$
$$Y^2Xv+XY^2v=2YXYv\overset{(3.3)}{=}\frac{2}{3}Y(YXv-2Xq_1-q_3).$$

It yields immediately

$$YX^2u = -\frac{1}{3}X^2Yu + \frac{4}{3}XYq_1 + \frac{2}{3}Xq_4, \tag{3.7}$$

$$XY^2u = 5Y^2Xu - 4Y^2q_1 - 2Yq_4, \tag{3.8}$$

$$YX^2v = 5X^2Yv + 4X^2q_1 + 2Xq_3, \tag{3.9}$$

$$XY^2v = -\frac{1}{3}Y^2Xv - \frac{4}{3}YXq_1 - \frac{2}{3}Yq_3. \tag{3.10}$$

We derive

$$\begin{aligned}
XY^2u &= XY(2q_2 - Xv) \overset{(3.3)}{=} 2XYq_2 - X(3XYv + 2Xq_1 + q_3) \\
&\overset{(3.9)}{=} 2XYq_2 - 2X^2q_1 - Xq_3 - \frac{3}{5}(YX^2v - 4X^2q_1 - 2Xq_3) \\
&\overset{Xv=2q_2-Yu}{=} \frac{3}{5}YXYu + 2XYq_2 - \frac{6}{5}YXq_2 + \frac{2}{5}X^2q_1 + \frac{1}{5}Xq_3 \\
&\overset{(3.1)}{=} \frac{9}{5}Y^2Xu + 2XYq_2 - \frac{6}{5}YXq_2 + \frac{2}{5}X^2q_1 - \frac{6}{5}Y^2q_1 + \frac{1}{5}Xq_3 - \frac{3}{5}Yq_4 \\
&\overset{(3.8)}{=} \frac{9}{25}XY^2u + 2XYq_2 - \frac{6}{5}YXq_2 + \frac{2}{5}X^2q_1 + \frac{6}{25}Y^2q_1 + \frac{1}{5}Xq_3 + \frac{3}{25}Yq_4
\end{aligned}$$

and

$$XY^2u = \frac{25}{8}XYq_2 - \frac{15}{8}YXq_2 + \frac{5}{8}X^2q_1 + \frac{3}{8}Y^2q_1 + \frac{5}{16}Xq_3 + \frac{3}{16}Yq_4. \tag{3.11}$$

Therefore, XY^2u is a linear function on $\nabla_h \mathcal{T} w$ and $\nabla_h^2 \mathcal{S}_2 w$. So, the same are $Y^2Xu, X^3u, YXYu, X^2Yv, YX^2v, Y^3v, XYXv$.

Set $\alpha = YX^2u$. Calculate $\nabla_h^3 w$ in terms of α and $q_1, \dots, q_4$. We have

$$\begin{aligned}
X^2Yu &\overset{(3.7)}{=} -3\alpha + 4XYq_1 + 2Xq_4, \\
XYXu &\overset{(3.5)}{=} \frac{1}{2}(X^2Yu + YX^2u) = -\alpha + 2XYq_1 + Xq_4, \\
Y^3u &\overset{(3.2)}{=} -3\alpha + 2Y^2q_2 - Yq_3 + 4YXq_1,
\end{aligned}$$

and

$$\begin{aligned}
YXYv &= YX(Xu - 2q_1) = \alpha - 2YXq_1, \\
Y^2Xv &\overset{(3.3)}{=} 3YXYv + 2YXq_1 + Yq_3 = 3\alpha + Yq_4 - 4YXq_1, \\
XY^2v &= XY(Xu - 2q_1) = -\alpha + Xq_4, \\
X^3v &\overset{(3.4)}{=} -3XY^2v + 2X^2q_2 + Xq_4 - 4XYq_1 \\
&= 3\alpha + 2X^2q_2 - 2Xq_4 - 4XYq_1.
\end{aligned}$$

Suppose, $w \in \ker \mathcal{Q}$. Then

$$XY^2u = Y^2Xu = X^3u = YXYu = X^2Yv = YX^2v = Y^3v = XYXv = 0$$

and

$$\alpha = YX^2u = -XYXu = YXYv = -XY^2v,$$
$$-3\alpha = X^2Yu = Y^3u = -Y^2Xv = -X^3v.$$

Functions u and v are polynomials of degree 3 if $\alpha = const$. It is sufficient to show that $X\alpha = Y\alpha = 0$. The latter is true since

$$X\alpha = XYX^2u \stackrel{(3.5)}{=} \frac{1}{2}(X^2Y + YX^2)Xu = \frac{1}{2}(-X\alpha)$$

and

$$Y\alpha = Y^2X^2\alpha \stackrel{(3.6)}{=} (2YXY - XY^2)Xu = -2Y\alpha.$$

Next we find u and v among all polynomials of degree 3.

Case 1. Suppose $v = a_{00} + a_{10}x + a_{11}y$ is a polynomial of degree 1. Then

$$Xu = Yv = a_{11}, \quad Yu = -Xv = -a_{10}$$

and $u = b_{00} + a_{11}x - a_{10}y$. In complex notation $w = u + iv = (a_{00} + ib_{00}) + a_{11}z + ia_{10}z$.

Case 2. Suppose $v = a_{20}x^2 + a_{21}xy + a_{22}y^2 + a_{23}t$ is a homogeneous polynomial of degree 2. Then

$$Xv = 2a_{20}x + a_{21}y + 2a_{23}y, \quad Yv = a_{21}x + 2a_{22}y - 2a_{23}x$$

and

$$YXv = a_{21} + 2a_{23} = 3YXv = 3(a_{21} - 2a_{23}) \quad \Rightarrow \quad a_{21} = 4a_{23},$$
$$X^2v = 2a_{20} = -3Y^2v = -3(2a_{22}) \quad \Rightarrow \quad a_{20} = -3a_{22}.$$

Therefore $v = a_{22}(y^2 - 3x^2) + a_{23}(t + 4xy)$. It is easy to find a function $u = a_{23}(x^2 - 3y^2) + a_{22}(4xy - t)$ and $w = u + iv = (|z|^2 - it)(-a_{23} - ia_{22}) - 2z^2(-a_{23} + ia_{22})$.

Case 3. Suppose $v = a_{30}x^3 + a_{31}x^2y + a_{32}xy^2 + a_{33}y^3 + a_{34}xt + a_{35}yt$ is a homogeneous polynomial of degree 3. Then

$$Xv = 3a_{30}x^2 + 2a_{31}xy + a_{32}y^2 + a_{34}t + 2a_{34}xy + 2a_{35}y^2,$$
$$Yv = a_{31}x^2 + 2a_{32}xy + 3a_{33}y^2 - 2a_{34}x^2 + a_{35}t - 2a_{35}xy,$$

and

$$YXv = 2a_{31}x + 2a_{32}y + 4a_{35}y = 3XYv = 3(2a_{31}x + 2a_{32}y - 4a_{34}x),$$
$$X^2v = 6a_{30}x + 2a_{31}y + 4a_{34}y = -3Y^2v = -3(2a_{32}x + 6a_{33}y - 4a_{35}x).$$

Hence,

$$a_{31} = 3a_{34}, \quad a_{32} = a_{35}, \quad a_{30} = a_{35}, \quad a_{33} = -\frac{5}{9}a_{34},$$

and $v = a_{35}(x^3 + xy^2 + yt) + a_{34}(3x^2y - 5y^3/9 + xt)$. Recall that $Y^3v = 0$. It implies $a_{34} = 0$.

Next we find u. We have

$$Xu = Yv = a_{35}t, \quad Yu = -Xv = -3a_{35}(x^2 + y^2), \quad Tu = a_{35}x.$$

Therefore $u = a_{35}(xt - x^2y - y^3)$ and $w = u + iv = ia_{35}z(|z|^2 - it)$.

Theorem 1.1 is proved.

4 Integral representation formula for operator $\mathcal{Q}$. Proof of Theorem 1.2

To prove integral representation formula for operator $\mathcal{Q}$ we use the Sobolev-type integral representation formula from paper [9] with $l = 4$ (Theorem 4.1). To do this we need to prove two assertions. First, we show that $P_3 w$ splits into two parts, one of them is a projection to the kernel of $\mathcal{Q}$, and another one is an integral over $\mathcal{Q}w$ (Lemma 4.1). Second, we show that $\nabla_h^4 w$ can be calculated via $\mathcal{Q}w$ (Lemma 4.2).

Theorem 4.1 ([9, Theorem 3]). *Let an integer $l > 0$, $\varkappa = 2\sqrt{6} + \frac{3}{2}$, and a function $u \in C^\infty(\mathrm{Box}(\mathbf{0}, \varkappa))$. Then for every $\mathbf{x} \in \mathrm{Box}(\mathbf{0}, 1)$ the integral representation formula*

$$u(\mathbf{x}) = P_{l-1}u(\mathbf{x}) + \sum_{d(I_h)=l} \int\limits_{\mathrm{Box}(\mathbf{0},\varkappa)} X^{I_h} u(\mathbf{y}) K_{I_h}(\mathbf{y}, \mathbf{x})\, d\mathbf{y}$$

holds where P_{l-1} is a projection on polynomials (4.1), $K_{I_h}(\mathbf{y}, \mathbf{x}) = \Psi_{I_h}(\mathbf{y}^{-1}\mathbf{x}) + \Phi_{I_h}(\mathbf{y}, \mathbf{x})$ with $\Phi_{I_h} \in C^\infty(\mathbb{H} \times \mathbb{H})$, $\operatorname{supp} \Phi_{I_h}(\cdot, \mathbf{x}) \subseteq \mathrm{Box}(\mathbf{0}, \varkappa)$ for $\mathbf{x} \in \mathrm{Box}(\mathbf{0}, 1)$; $\Psi_{I_h} \in C^\infty(\mathbb{H} \setminus \{\mathbf{0}\})$, $\operatorname{supp} \Psi_{I_h} \subseteq \overline{\mathrm{Box}(\mathbf{0}, 1)}$, and

$$|X^J \Psi_{I_h}(\mathbf{x})| \leqslant C_{d(J)} d_\infty(\mathbf{x}, \mathbf{0})^{l-d(J)-4} \quad \textit{for any multi-index } J.$$

Here $C_{d(J)} > 0$ is a constant independent of u and $\mathbf{x} \in \mathbb{H}$.

Remark. *Originally, $\varkappa = c + c^2 + 2c^3$ where c is the constant of the generalized triangle inequality of the quasimetric d_∞. On Heisenberg group Remark 3 in [10] gives the quantity $c = \sqrt{\frac{3}{2}}$. Therefore, $\varkappa = 2\sqrt{6} + \frac{3}{2}$.*

Following [9, p. 73] we define a projection of $L_1(\mathrm{Box}(\mathbf{0}, 1))$ to $\mathcal{P}_{l-1}$ as

$$P_{l-1}u(\mathbf{x}) = \int\limits_{\mathrm{Box}(\mathbf{0},1)} u(\mathbf{x}_0) \sum_{d(I_h)<l} \frac{(-1)^{d(I_h)}}{(d(I_h))!} X^{I_h}_{\mathbf{x}_0} \big(Q_{I_h}(\mathbf{x}_0^{-1}\mathbf{x})\varphi(\mathbf{x}_0)\big)\, d\mathbf{x}_0 \quad (4.1)$$

where

$$\varphi \in C^\infty(\mathbb{H}), \quad \operatorname{supp} \varphi \subseteq \overline{\mathrm{Box}(\mathbf{0}, 1)} \setminus \mathrm{Box}(\mathbf{0}, 1/2),$$

$$\int\limits_{\mathrm{Box}(\mathbf{0},1)} \varphi(\mathbf{x})\, d\mathbf{x} = 1, \qquad \int\limits_{\mathrm{Box}(\mathbf{0},1)} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \varphi(\mathbf{x})\, d\mathbf{x} = 0, \quad 0 < \alpha_1 + \alpha_2 + \alpha_3 < l.$$

Here Q_{I_h} is a polynomial defined in Section 2.5.

Notice, for C^∞-smooth functions u projection $P_{l-1}u(\mathbf{x})$ equals

$$\int\limits_{\mathrm{Box}(\mathbf{0},1)} \sum_{d(I_h)<l} \frac{X^{I_h} u(\mathbf{x}_0)}{(d(I_h))!} Q_{I_h}(\mathbf{x}_0^{-1}\mathbf{x})\varphi(\mathbf{x}_0)\, d\mathbf{x}_0.$$

Lemma 4.1. *There is a projection* $\mathbf{\Pi}$ *from* $L_1(\mathrm{Box}(\mathbf{0},1),\mathbb{R}^2)$ *to* $\ker\mathcal{Q}$ *such that*

$$P_3 w(\mathbf{x}) = \mathbf{\Pi} w(\mathbf{x}) + \int\limits_{\mathrm{Box}(\mathbf{0},1)} \Theta(\mathbf{y},\mathbf{x})\mathcal{Q}(\mathbf{y})\,d\mathbf{y}, \quad \mathbf{x}\in\mathrm{Box}(\mathbf{0},1),$$

for every $w\in L_1(\mathrm{Box}(\mathbf{0},1),\mathbb{R}^2)$ *where* Θ *is* C^∞*-smooth and* $\operatorname{supp}\Theta(\cdot,\mathbf{x})\subseteq\overline{\mathrm{Box}(\mathbf{0},1)}$ *for every* $\mathbf{x}\in\mathrm{Box}(\mathbf{0},1)$.

Proof. We construct a projection $\mathbf{\Pi}$ stepwise by degree of polynomials.

Step 0.

$$P_0 w(\mathbf{x}) = \int\limits_{\mathrm{Box}(\mathbf{0},1)} w(\mathbf{y})\varphi(\mathbf{y})\,d\mathbf{y}\in\ker\mathcal{Q}.$$

Step 1. We have

$$\nabla_h w = D_h w = \begin{pmatrix} Xu & Yu\\ Xv & Yv\end{pmatrix} = \mathcal{S}w+\mathcal{A}w$$

where

$$\mathcal{A}w = \frac{1}{2}\begin{pmatrix} Xu+Yv & Yu-Xv\\ Xv-Yu & Xu+Yv\end{pmatrix}.$$

Set

$$\Pi_1 w(\mathbf{x}) = \int\limits_{\mathrm{Box}(\mathbf{0},1)} \mathcal{A}w(\mathbf{y})\begin{pmatrix} x_1-y_1\\ x_2-y_2\end{pmatrix}\varphi(\mathbf{y})\,d\mathbf{y}$$

and show that $\Pi_1 w\in\ker\mathcal{Q}$. Obviously,

$$\begin{aligned}\Pi_1 w(\mathbf{x}) = &\int\limits_{\mathrm{Box}(\mathbf{0},1)} \frac{\big(Xu(\mathbf{y})+Yv(\mathbf{y})\big)\varphi(\mathbf{y})}{2}\begin{pmatrix} x_1-y_1\\ x_2-y_2\end{pmatrix}d\mathbf{y}\\ &+\int\limits_{\mathrm{Box}(\mathbf{0},1)} \frac{\big(Xv(\mathbf{y})-Yu(\mathbf{y})\big)\varphi(\mathbf{y})}{2}\begin{pmatrix} -x_2+y_2\\ x_1-y_1\end{pmatrix}d\mathbf{y}\\ &= \frac{\big(Xu+Yv\big)\varphi}{2}*\begin{pmatrix} Q_1\\ Q_2\end{pmatrix}+\frac{\big(Xv-Yu\big)\varphi}{2}*\begin{pmatrix} -Q_2\\ Q_1\end{pmatrix}.\end{aligned}$$

Since $\begin{pmatrix} Q_1\\ Q_2\end{pmatrix}, \begin{pmatrix} -Q_2\\ Q_1\end{pmatrix}\in\ker\mathcal{Q}$ we obtain $\Pi_1 w\in\ker\mathcal{Q}$.

Step 2.

$$\begin{aligned}\nabla_h^2 w = \begin{pmatrix} X^2u & XYu & YXu & Y^2u\\ X^2v & XYv & YXv & Y^2v\end{pmatrix} &= \big(X\nabla_h w \quad Y\nabla_h w\big)\\ &= \big(X\mathcal{S}w \quad Y\mathcal{S}w\big)+\mathcal{B}w\end{aligned}$$

where

$$\mathcal{B}w = \begin{pmatrix} X\mathcal{A}w & Y\mathcal{A}w \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} X^2u + XYv & XYu - X^2v & YXu + Y^2v & Y^2u - YXv \\ X^2v - XYu & X^2u + XYv & YXv - Y^2u & YXu + Y^2v \end{pmatrix}.$$

Denote as before

$$q_1 = \tfrac{1}{2}(Xu - Yv), \qquad q_2 = \tfrac{1}{2}(Xv + Yu),$$
$$q_3 = YXv - 2XYv - X^2u, \qquad q_4 = 2YXu - XYu + Y^2v. \tag{4.2}$$

Calculate elements of $\mathcal{B}w$ via $q_1, \ldots, q_4$. We have

$$X^2u + XYv = \frac{2}{3}X^2u + \frac{4}{3}XYv + \frac{1}{3}X(Xu - Yv) = \frac{2}{3}(YXv - q_3) + \frac{2}{3}Xq_1,$$
$$Y^2u - YXv = Y(Yu + Xv) - 2YXv = 2Yq_2 - 2YXv,$$
$$YXu + Y^2v = \frac{2}{3}Y^2v + \frac{4}{3}YXu + \frac{1}{3}Y(Yv - Xu) = \frac{2}{3}(q_4 + XYu) - \frac{2}{3}Yq_1,$$
$$X^2v - XYu = X(Yu + Xv) - 2XYu = 2Xq_2 - 2XYu.$$

Therefore, it follows

$$\mathcal{B}w = \frac{1}{3}\begin{pmatrix} Xq_1 - q_3 & -3Xq_2 & q_4 - Yq_1 & 3Yq_2 \\ 3Xq_2 & Xq_1 - q_3 & -3Yq_2 & q_4 - Yq_1 \end{pmatrix} + \mathcal{C}w,$$

where

$$\mathcal{C}w = \frac{1}{3}\begin{pmatrix} YXv & 3XYu & XYu & -3YXv \\ -3XYu & YXv & 3YXv & XYu \end{pmatrix}.$$

Put

$$\Pi_2 w(\mathbf{x}) = \frac{1}{2}\int\limits_{\mathrm{Box}(\mathbf{0},1)} \mathcal{C}w(\mathbf{y}) \begin{pmatrix} Q_{11} \\ Q_{12} \\ Q_{21} \\ Q_{22} \end{pmatrix} (\mathbf{y}^{-1}\mathbf{x})\varphi(\mathbf{y})\, d\mathbf{y}.$$

Then

$$\Pi_2 w(\mathbf{x}) = \begin{pmatrix} \eta * \big(Q_{11} - 3Q_{22}\big) + \zeta * \big(3Q_{12} + Q_{21}\big) \\ \zeta * \big(-3Q_{11} + Q_{22}\big) + \eta * \big(Q_{12} + 3Q_{21}\big) \end{pmatrix}$$
$$= \eta * \begin{pmatrix} x_1^2 - 3x_2^2 \\ 4x_1x_2 + x_3 \end{pmatrix} + \zeta * \begin{pmatrix} 4x_1x_2 - x_3 \\ x_2^2 - 3x_1^2 \end{pmatrix} \in \ker \mathcal{Q}$$

with

$$\eta = \frac{1}{6}\varphi YXv, \quad \zeta = \frac{1}{6}\varphi XYu.$$

STEP 3. By proof of Theorem 1.1 we know that

$$\nabla_h^3 w = \begin{pmatrix} X^3u & X^2Yu & YX^2u & XY^2u & Y^2Xu & Y^3u \\ X^3v & X^2Yv & YX^2v & XY^2v & Y^2Xv & Y^3v \end{pmatrix}$$
$$= \Phi(\nabla_h \mathcal{T}w, \nabla_h^2 \mathcal{S}_2 w) + \begin{pmatrix} 0 & -3\alpha & \alpha & 0 & 0 & -3\alpha \\ 3\alpha & 0 & 0 & -\alpha & 3\alpha & 0 \end{pmatrix} \tag{4.3}$$

where $\alpha = YX^2u$ and $\Phi(\cdot,\cdot)$ is a linear function.

Define $\Pi_3 w(\mathbf{x})$ as

$$\int\limits_{\mathrm{Box}(\mathbf{0},1)} \frac{YX^2u(\mathbf{y})}{6} \begin{pmatrix} 0 & -3 & 1 & 0 & 0 & -3 \\ 3 & 0 & 0 & -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} Q_{111}(\mathbf{y}^{-1}\mathbf{x}) \\ Q_{112}(\mathbf{y}^{-1}\mathbf{x}) \\ Q_{211}(\mathbf{y}^{-1}\mathbf{x}) \\ Q_{122}(\mathbf{y}^{-1}\mathbf{x}) \\ Q_{221}(\mathbf{y}^{-1}\mathbf{x}) \\ Q_{222}(\mathbf{y}^{-1}\mathbf{x}) \end{pmatrix} \varphi(\mathbf{y})\, d\mathbf{y}$$
$$= \xi * \begin{pmatrix} x_1x_3 - x_2^3 - x_1^2x_2 \\ x_1^3 + x_1x_2^2 + x_2x_3 \end{pmatrix} \in \ker \mathcal{Q}$$

with $\xi = \frac{1}{2}\varphi YX^2u$.

Finally,

$$\mathbf{\Pi}w = P_0 w + \Pi_1 w + \Pi_2 w + \Pi_3 w \in \ker \mathcal{Q}.$$

It rests to show that Θ is smooth function. Indeed,

$$P_3 w = \mathbf{\Pi}w + \int\limits_{\mathrm{Box}(\mathbf{0},1)} \Xi(\mathcal{Q}w(\mathbf{y}), \nabla_h \mathcal{Q}w(\mathbf{y}), \nabla_h^2 \mathcal{S}w(\mathbf{y}))\psi(\mathbf{y},\mathbf{x})\, d\mathbf{y},$$

where $\Xi(\cdot,\cdot,\cdot)$ is some linear function, ψ is smooth function and $\operatorname{supp}\psi(\cdot,\mathbf{x}) \subseteq \overline{\mathrm{Box}(\mathbf{0},1)}$. Applying integrations by parts, we obtain the required equality:

$$P_3 w = \mathbf{\Pi}w + \int\limits_{\mathrm{Box}(\mathbf{0},1)} \Theta(\mathbf{y},\mathbf{x})\mathcal{Q}w(\mathbf{y})\, d\mathbf{y}$$

with smooth Θ. □

Lemma 4.2. *Let Ω be a domain in $\mathbb{H}$. Then $\nabla_h^4 w$ is a linear combination of $\nabla_h^3 \mathcal{S}w$ and $\nabla_h^2 \mathcal{T}w$ for every $w = (u,v) \in C^4_{\mathbb{H}}(\Omega, \mathbb{R}^2)$, $p \geqslant 1$.*

Proof. We show that all differentials of order 4 can be written down in terms of $q_1 - q_4$ defined in (4.2). In view of (4.3) it suffices to show that $X\alpha$ and $Y\alpha$ ($\alpha = YX^2u$) are linear combination of $q_1 - q_4$ and their derivatives.

Recall from the proof of Theorem 1.1 that

$$\begin{aligned} XYXu &= -\alpha + 2XYq_1 + Xq_4, \\ X^3u &\overset{(3.2)}{=} \frac{1}{3}X(4Xq_1 + 2Yq_2 - q_3 - Y^2u) \\ &= \frac{1}{8}\Big(9X^2q_1 - Y^2q_1 - 3XYq_2 + 5YXq_2 - \frac{7}{2}Xq_3 - \frac{1}{2}Yq_4\Big), \\ Y^2Xu &\overset{(3.8)}{=} \frac{1}{5}(XY^2u + 4Y^2q_1 + 2Yq_4) \\ &\overset{(3.11)}{=} \frac{5}{8}XYq_2 - \frac{3}{8}YXq_2 + \frac{1}{8}X^2q_1 + \frac{7}{8}Y^2q_1 + \frac{1}{16}Xq_3 + \frac{7}{16}Yq_4. \end{aligned}$$

Calculate now $X\alpha$ and $Y\alpha$. We have

$$X\alpha = XYX^2u = \frac{1}{2}(X^2Y + YX^2)Xu = -\frac{1}{2}X\alpha + X^2Yq_1 + \frac{1}{2}X^2q_4 + \frac{1}{16}\Big(9YX^2q_1 - Y^3q_1 - 3YXYq_2 + 5Y^2Xq_2 - \frac{7}{2}YXq_3 - \frac{1}{2}Y^2q_4\Big),$$

$$Y\alpha = Y^2X^2u = (2YXY - XY^2)Xu = 2Y(-\alpha + 2XYq_1 + Xq_4) - X\Big(\frac{5}{8}XYq_2 - \frac{3}{8}YXq_2 + \frac{1}{8}X^2q_1 + \frac{7}{8}Y^2q_1 + \frac{1}{16}Xq_3 + \frac{7}{16}Yq_4\Big).$$

Therefore,

$$3X\alpha = 2X^2Yq_1 + X^2q_4 + \frac{1}{16}\Big(18YX^2q_1 - 2Y^3q_1 - 6YXYq_2 + 10Y^2Xq_2 - 7YXq_3 - Y^2q_4\Big)$$

$$3Y\alpha = 4YXYq_1 + 2YXq_4 - \frac{1}{16}\Big(10X^2Yq_2 - 6XYXq_2 + 2X^3q_1 + 14XY^2q_1 + X^2q_3 + 7XYq_4\Big).$$

The lemma is proved. □

Proof of Theorem 1.2. Now we can apply Theorem 4.1 with $l = 4$. Integrating by parts we obtain

$$w(\mathbf{x}) - \mathbf{\Pi}w(\mathbf{x}) = \int\limits_{\mathrm{Box}(\mathbf{0},\varkappa)} K(\mathbf{y},\mathbf{x})\nabla_h^4 w(\mathbf{y})\,d\mathbf{y} + \int\limits_{\mathrm{Box}(\mathbf{0},1)} \Theta(\mathbf{y},\mathbf{x})\mathcal{Q}w(\mathbf{y})\,d\mathbf{y}$$

$$= \int\limits_{\mathrm{Box}(\mathbf{0},\varkappa)} \big(L(\mathbf{y},\mathbf{x})\mathcal{Q}w(\mathbf{y}) + M(\mathbf{y}^{-1}\cdot\mathbf{x})\mathcal{S}w(\mathbf{y}) + N(\mathbf{y}^{-1}\cdot\mathbf{x})\mathcal{T}w(\mathbf{y})\big)\,d\mathbf{y}$$

where $L \in C^\infty$, $\operatorname{supp} L(\cdot,\mathbf{x}) \subseteq \overline{\mathrm{Box}(\mathbf{0},1)}$ for every $x \in \mathrm{Box}(\mathbf{0},1)$, $M, N \in C^\infty(\mathbb{H}\setminus\{\mathbf{0}\})$, $\operatorname{supp} M, \operatorname{supp} N \subseteq \overline{\mathrm{Box}(\mathbf{0},1)}$,

$$|X^J M(\mathbf{x})| \leqslant C_{d(J)} d(\mathbf{x},\mathbf{0})^{-d(J)-3}, \quad |X^J N(\mathbf{x})| \leqslant C_{d(J)} d(\mathbf{x},\mathbf{0})^{-d(J)-2}$$

for any multi-index J. □

5 Coercive estimates for $\mathcal{Q}$. Proof of Theorem 1.3

To prove Theorem 1.3 we establish the local version of coercive estimate (Proposition 5.1). To pass from local to global result we apply Theorem 7 from [10].

In what follows, C denotes various positive constants. They may differ even in a same string of estimates. Set $\varkappa = 2\sqrt{6} + \frac{3}{2}$.

To prove coercive estimates we use estimates for singular integrals. Observe the following fractional integral operator (analog of the Riesz potential):

$$\mathcal{R}^\gamma v(x) = \int\limits_{\mathrm{Box}(\mathbf{0},\varkappa)} v(y)\,d_\infty(x,y)^{\gamma-4}\,dy, \quad 0 \leqslant v \in L_p(\mathrm{Box}(\mathbf{0},\varkappa)), \quad \gamma > 0.$$

Lemma 5.1. *Let $1 < p < \infty$ and $0 < \gamma < 4$. Then there exists a constant C such that for every nonnegative function $v \in L_p(\mathrm{Box}(\mathbf{0}, \varkappa))$ the following inequality holds:*

$$\|\mathcal{R}^\gamma v\|_{q,\mathrm{Box}(\mathbf{0},\varkappa)} \leqslant C\|v\|_{p,\mathrm{Box}(\mathbf{0},\varkappa)}$$

where

1) $p \leqslant q \leqslant \frac{4p}{4-\gamma p}$ for $\gamma p < 4$;

2) $p \leqslant q < \infty$ for $\gamma p = 4$;

3) $p \leqslant q \leqslant \infty$ for $\gamma p > 4$.

In the main case $\gamma p < 4$, Lemma 5.1 can be found in [19, Theorem 10]. The case $\gamma p \geqslant 4$ is given in [10, Lemma 3].

In the case $\gamma = 0$ we will use the following statement.

Lemma 5.2 ([9, Lemma 4])**.** *Let a function η satisfy the following conditions:*

(i) $\eta \in C^\infty(\mathbb{H})$, $\operatorname{supp}\eta \subseteq \overline{\mathrm{Box}(\mathbf{0}, 1)} \setminus \mathrm{Box}(\mathbf{0}, 1/4)$;

(ii) $\int_{\mathbb{H}} \eta(\mathbf{x})\, d\mathbf{x} = 0$.

For $v \in L_p(\mathbb{H})$, $1 < p < \infty$, set $K(\mathbf{x}) = \sum\limits_{k=0}^{\infty} T^k\eta(\mathbf{x})$, where $T\eta(\mathbf{x}) = 16\,\eta(\delta_2\mathbf{x})$, and

$$\mathcal{K}_\varepsilon v(\mathbf{x}) = \int\limits_{\mathbb{H}\setminus\mathrm{Box}(\mathbf{x},\varepsilon)} K(\mathbf{y}^{-1}\mathbf{x})\, v(\mathbf{y})\, d\mathbf{y}.$$

Then

$$\|\mathcal{K}_\varepsilon v\|_p \leqslant A_p\|v\|_p$$

where A_p is independent of v and ε. Moreover, for each function $v \in L_p(\mathbb{H})$, there exists $\lim\limits_{\varepsilon\to 0} \mathcal{K}_\varepsilon v \overset{L_p}{=} \mathcal{K}v$ and $\|\mathcal{K}v\|_p \leqslant A_p\|v\|_p$.

Now we formulate the local version of the coercive estimate.

Proposition 5.1. *Let $1 < p < \infty$ and $w \in W_p^2(\mathrm{Box}(\mathbf{0}, \varkappa); \mathbb{R}^2)$. Then*

$$\|w - \mathbf{\Pi} w\|_{q,\mathrm{Box}(\mathbf{0},1)} \leqslant C\|\mathcal{Q}w\|_{p,\mathrm{Box}(\mathbf{0},\varkappa)}$$

with q satisfying (1.2) and

$$\|\nabla_h(w - \mathbf{\Pi} w)\|_{p,\mathrm{Box}(\mathbf{0},1)} \leqslant C\|\mathcal{Q}w\|_{p,\mathrm{Box}(\mathbf{0},\varkappa)},$$

where $\mathbf{\Pi}$ is the projector from Theorem 1.2.

Proof. By Theorem 1.2 we have

$$\begin{aligned} w(\mathbf{x}) - \mathbf{\Pi} w(\mathbf{x}) &= \int\limits_{\mathrm{Box}(\mathbf{0},\varkappa)} \big(L(\mathbf{y},\mathbf{x})\mathcal{Q}w(\mathbf{y}) + M(\mathbf{y}^{-1}\mathbf{x})\mathcal{S}w(\mathbf{y}) + N(\mathbf{y}^{-1}\mathbf{x})\mathcal{T}w(\mathbf{y})\big)\, d\mathbf{y} \\ &= I_1 w(\mathbf{x}) + I_2 w(\mathbf{x}) + I_3 w(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathrm{Box}(\mathbf{0}, 1) \end{aligned}$$

with smooth L, $|M(\mathbf{y}^{-1}\mathbf{x})| \leqslant Cd(\mathbf{x}, \mathbf{y})^{-3}$, $|N(\mathbf{y}^{-1}\mathbf{x})| \leqslant Cd(\mathbf{x}, \mathbf{y})^{-2}$.

Since L is smooth we can write

$$\|I_1 w\|_{\infty,\mathrm{Box}(\mathbf{0},1)} \leqslant C \int\limits_{\mathrm{Box}(\mathbf{0},\varkappa)} |\mathcal{Q}w(\mathbf{y})|\,d\mathbf{y} \leqslant C\|\mathcal{Q}w\|_{p,\mathrm{Box}(\mathbf{0},\varkappa)}.$$

By Lemma 5.1 we have

$$\|I_2 w\|_{q,\mathrm{Box}(\mathbf{0},1)} \leqslant C\|\mathcal{S}w\|_{p,\mathrm{Box}(\mathbf{0},\varkappa)}, \quad \begin{cases} p \leqslant q \leqslant \frac{4p}{4-p},\ 1<p<4, \\ p\leqslant q<\infty,\ p=4, \\ p\leqslant q\leqslant\infty,\ p>4, \end{cases} \qquad (\gamma=1)$$

and

$$\|I_3 w\|_{q,\mathrm{Box}(\mathbf{0},1)} \leqslant C\|\mathcal{T}w\|_{p,\mathrm{Box}(\mathbf{0},\varkappa)}, \quad \begin{cases} p \leqslant q \leqslant \frac{4p}{4-2p},\ 1<p<2, \\ p\leqslant q<\infty,\ p=2, \\ p\leqslant q\leqslant\infty,\ p>2. \end{cases} \qquad (\gamma=2)$$

Combining conditions on pairs p, q we obtain the first part of the proposition.

Again by Theorem 1.2 we have

$$\begin{aligned} \nabla_h\big(w(\mathbf{x}) - \mathbf{\Pi}w(\mathbf{x})\big) = & \int\limits_{\mathrm{Box}(\mathbf{0},\varkappa)} \big(\nabla_{h,\mathbf{x}}L(\mathbf{y},\mathbf{x})\mathcal{Q}w(\mathbf{y}) \\ & + \nabla_{h,\mathbf{x}}M(\mathbf{y}^{-1}\mathbf{x})\mathcal{S}w(\mathbf{y}) + \nabla_{h,\mathbf{x}}N(\mathbf{y}^{-1}\mathbf{x})\mathcal{T}w(\mathbf{y})\big)\,d\mathbf{y} \\ = & J_1 w(\mathbf{x}) + J_2 w(\mathbf{x}) + J_3 w(\mathbf{x}) \quad \text{for all } \mathbf{x}\in\mathrm{Box}(\mathbf{0},1), \end{aligned}$$

with smooth $\nabla_{h,\mathbf{x}}L$,

$$|\nabla_{h,\mathbf{x}}M(\mathbf{y}^{-1}\mathbf{x})| \leqslant Cd(\mathbf{x},\mathbf{y})^{-4}, \quad |\nabla_{h,\mathbf{x}}N(\mathbf{y}^{-1}\mathbf{x})| \leqslant Cd(\mathbf{x},\mathbf{y})^{-3}.$$

As before, in view of smoothness of $\nabla_{h,\mathbf{x}}L$ we obtain

$$\|J_1 w\|_{\infty,\mathrm{Box}(\mathbf{0},1)} \leqslant C\|\mathcal{Q}w\|_{p,\mathrm{Box}(\mathbf{0},\varkappa)}.$$

Lemma 5.1 yields

$$\|J_3 w\|_{q,\mathrm{Box}(\mathbf{0},1)} \leqslant C\|\mathcal{T}w\|_{p,\mathrm{Box}(\mathbf{0},\varkappa)}, \quad \begin{cases} p \leqslant q \leqslant \frac{4p}{4-p},\ 1<p<4, \\ p\leqslant q<\infty,\ p=4, \\ p\leqslant q\leqslant\infty,\ p>4, \end{cases} \qquad (\gamma=1).$$

It rests to estimate $\|J_2 w\|_{p,\mathrm{Box}(\mathbf{0},1)}$.

All the elements in the kernel M are the linear combinations of the following elements:

$$(-1)^{l-1}X^{J_h,R}K_{I_h}, \quad d(J_h)=3,\ d(I_h)=4,$$

where K_{I_h} are kernels from Theorem 4.1. From [9, Proof of Theorem 2, p. 71] we know that

$$K_{I_h} = \sum_{k=0}^{\infty} 2^{-4k}T^k\zeta_{I_h^t}$$

with $\zeta_{I_h^t} \in C^\infty$ and $\operatorname{supp}\zeta_{I_h^t} \subseteq \overline{\mathrm{Box}(\mathbf{0},1)} \setminus \mathrm{Box}(\mathbf{0},1/4)$ (see [9, Lemma 1]). Here we have denoted $I_h^t = (i_4, i_3, i_2, , i_1)$ for the multi-index $I_h = (i_1, i_2, i_3, i_4) \in \{1,2\}^4$.

Since $X_i T^k = 2^k T^k X_i$ and $X_i^R T^k = 2^k T^k X_i^R$, $i = 1, 2$, kernel $\nabla_h M$ is a linear combination of the following elements:

$$\sum_{k=0}^{\infty} T^k X_s X^{J_h,R} \zeta_{I_h^t}, \quad s = 1,2,\ d(J_h) = 3,\ d(I_h) = 4.$$

Hence, operator J_2 meets all the conditions of the Lemma 5.2 and

$$\|J_2 w\|_{p,\mathrm{Box}(\mathbf{0},1)} \leqslant C\|\mathcal{S}w\|_{p,\mathrm{Box}(\mathbf{0},\varkappa)}.$$

The proposition follows. □

To pass from local to global estimate we use Theorem 7 from [10]. It is formulated for metric spaces with doubling conditions but we rewrite it in Heisenberg group.

Let $\mathcal{P}$ be the vector space of $\mathbb{R}^m$-valued functions P on $\mathbb{H}$ with the following properties:

$$\sup_{\mathbf{x}\in sB} |P(\mathbf{x})| \leqslant Cs^l \sup_{x\in B} |P(\mathbf{x})|, \quad \sup_{\mathbf{x}\in B} |P(\mathbf{x})| \leqslant \frac{C}{|B|}\int_B |P(\mathbf{x})| d\mathbf{x} \qquad (5.1)$$

for all ball B and number $s \geqslant 1$, where $l \geqslant 0$ and the constant C do not depend on P and B.

Theorem 5.1 ([10, Theorem 7])**.** *Let $\mathcal{P}$ be a vector space of functions on $\mathbb{H}$ satisfying (5.1), Ω be a John domain $J(\alpha,\beta)$ with distinguished point x_0, and f, g and h be measurable functions defined on Ω. Suppose $\varkappa \geqslant 1$, $1 \leqslant p < \infty$, $p \leqslant q \leqslant \infty$, $\lambda, \xi \geqslant 0$, and, for each ball B with $\varkappa B \subset \Omega$, there exists a function $P(B) \in \mathcal{P}$ such that*

$$\|f - P(B)\|_{q,B} \leqslant r(B)^\lambda \|g\|_{p,\varkappa B} + r(B)^\xi \|h\|_{p,\varkappa B}.$$

Then

$$\|f - P(B_0)\|_{q,\Omega} \leqslant C\Big(\frac{\beta}{\alpha}\Big)^\theta \Big[(\operatorname{diam}\Omega)^\lambda \|g\|_{p,\Omega} + (\operatorname{diam}\Omega)^\xi \|h\|_{p,\Omega}\Big]$$

where $B_0 = B(x_0, \frac{d_\Omega(x_0)}{\varkappa})$ *and* $\theta = \begin{cases} l+4 & \text{if } q \neq \infty, \\ l+4+4/p & \text{if } q = \infty. \end{cases}$

Proof of Theorem 1.3. Kernel of operator $\mathcal{Q}$ consists of polynomials of degree $\leqslant 3$. Therefore, $\ker\mathcal{Q}$ satisfies (5.1) with $l = 3$ and $\nabla_h \ker\mathcal{Q}$ satisfies (5.1) with $l = 2$ (for example, see [13, Lemma 2.1]).

Proposition 5.1 is valid for functions defined on balls $B(\mathbf{0},\varkappa)$. Take a function $f \in W_p^2(\Omega;\mathbb{R}^2)$ and a ball $= B(\mathbf{a}, r)$ such that $\varkappa B = B(\mathbf{a}, \varkappa r) \subset \Omega$. Then $w = f \circ \pi_{\mathbf{a}} \circ \delta_r \in W_p^2(B(\mathbf{0},\varkappa);\mathbb{R}^2)$ and by Proposition 5.1 we have

$$\begin{aligned}\|f - P_B\|_{q,B} = r^{4/q}\|w - \mathbf{\Pi}w\|_{q,B(\mathbf{0},1)} \leqslant Cr^{4/q}\|\mathcal{Q}w\|_{p,B(\mathbf{0},\varkappa)} \\ \leqslant Cr^{4/q-4/p+1}\big(\|\mathcal{S}f\|_{p,\varkappa B} + r\|\mathcal{T}f\|_{p,\varkappa B}\big),\end{aligned}$$

$$\|D_h(f-P_B)\|_{p,B} = r^{4/p-1}\|D_h(w-\mathbf{\Pi}w)\|_{p,B(\mathbf{0},1)} \leqslant Cr^{4/p-1}\|\mathcal{Q}w\|_{p,B(\mathbf{0},\varkappa)} \\ \leqslant C\big(\|\mathcal{S}f\|_{p,\varkappa B} + r\|\mathcal{T}f\|_{p,\varkappa B}\big), \quad (5.2)$$

Here we set $P_B = \mathbf{\Pi}w \in \ker\mathcal{Q}$. Therefore we can apply Theorem 5.1 with $g = \mathcal{S}f$, $h = \mathcal{T}f$, $l = 3$ for $\mathcal{P} = \ker\mathcal{Q}$ and $l = 2$ for $\mathcal{P} = \nabla_h \ker\mathcal{Q}$. Theorem 1.3 follows. □

Proof of Corollary 1.1. We consider bounded domain $\Omega \subset \mathbb{H}$ and $u \in W^2_{p,O}(\Omega, \mathbb{R}^2)$. Take a big ball $B = B(\mathbf{a}, R)$ such that $\Omega \subset \varkappa B \setminus B$. Then $u \in W^2_{p,O}(\varkappa B, \mathbb{R}^2)$. Define projector on $\ker\mathcal{Q}$ as $\Pi u = \mathbf{\Pi}(u \circ \pi_{\mathbf{a}} \circ \delta_R)$ where $\mathbf{\Pi}$ is the projector from Proposition 5.1. We have $\Pi u = 0$ since $\Omega \cap B = \emptyset$. Similar to expression (5.2), we obtain the desired estimate:

$$\|D_h u\|_{p,\Omega} = \|D_h(u - \Pi u)\|_{p,B} \leqslant C\|\mathcal{Q}u\|_{p,\varkappa B} = C\|\mathcal{Q}u\|_{p,\Omega}. \qquad \square$$

6 Operator $\mathcal{Q}$ on Heisenberg groups $\mathbb{H}^n$, $n > 1$

We identify the points of $\mathbb{H}^n$ with the points of $\mathbb{R}^{2n+1}$ with the following group law:

$$(x_1, \dots, x_{2n+1}) \cdot (y_1, \dots, y_{2n+1}) \\ = (x_1 + y_1, \dots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} - 2\sum_{k=1}^{n} x_k y_{n+k} - x_{n+k} y_k).$$

The left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2x_{i+n}\frac{\partial}{\partial x_{2n+1}}, \quad X_{i+n} = \frac{\partial}{\partial x_{i+n}} - 2x_i\frac{\partial}{\partial x_{2n+1}}, \qquad i = 1, \dots, n,$$

constitute an orthonormal basis of the horizontal subbundle $H\mathbb{H}^n$.

Together with the vector field $X_{2n+1} = \frac{\partial}{\partial x_{2n+1}}$ they constitute the standard basis of the Lie algebra. The only nontrivial commutator relations are

$$[X_j, X_{j+n}] = -4X_{2n+1}, \quad j = 1, \dots, n.$$

In contrast to the Euclidean case, the horizontal differential of a contact mapping has some additional structure: up to a factor, $D_h F(x)$ is a symplectic matrix. Therefore, the operator $\mathcal{Q}$ consist of two parts: the first is responsible for generalized orthogonality, and the second, for symplecticity.

Given a domain U in $\mathbb{H}^n$, $n > 1$, denote by $\mathcal{Q}$ the homogeneous differential operator acting on a mapping $u\colon U \to \mathbb{R}^{2n}$ as

$$\mathcal{Q}u = \frac{1}{2}\begin{pmatrix} D_h u + (D_h u)^t - \frac{1}{n}\operatorname{tr}(D_h u) I \\ D_h u + J D_h u J \end{pmatrix}.$$

Here the $2n \times 2n$ matrix $D_h u$ equals $(X_i u_j)_{i,j=1,\dots,2n}$ and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

By [8, Lemma 2] kernel of $\mathcal{Q}$ coincides with horizontal coordinate functions of the Lie algebra of the group of conformal mappings. In particular, $\ker\mathcal{Q}$

is finite-dimensional and the coercive estimates is valid for $\mathcal{Q}$. The latter was used to establish the stability of conformal mappings [8].

In the case $n=1$ the group of symplectic matrices $Sp(2,\mathbb{R})$ coincides with the special linear group $SL(2,\mathbb{R})$ and the symplecticity condition does not play any essential role. Indeed, we have $\mathcal{Q}=\mathcal{S}$ since

$$D_h u + J D_h u J = \begin{pmatrix} X_1u_1 - X_2u_2 & X_2u_1 + X_1u_2 \\ X_1u_2 + X_2u_1 & X_2u_2 - X_1u_1 \end{pmatrix} = 2\mathcal{S}u.$$

The kernel of $\mathcal{Q}$ is infinite dimensional since it includes solutions to Cauchy – Riemann system (independent of x_3).

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