# On the Two-sided Group Digraph with a normal adjacency matrix

**Farzaneh Nowroozi Larki**\*and **Shahram Rayat Pisheh**

Department of Mathematics, Faculty of Science,

Shahid Rajaee Teacher Training University,

Tehran, 16785-136, I. R. Iran

November 25, 2023

### Abstract

In this article, we explore the adjacency matrix of a two-sided group graph and its properties. We introduce the two-sided color group digraph as a generalization of the Cayley color graph and the two-sided group digraph. We also obtain the adjacency matrix of the latter digraph and provide a criterion for determining the normality of the adjacency matrix of a two-sided group graph. Moreover, we prove that if all the two-sided group digraphs of valency two for a certain group $G$ are normal, then $G$ is a Hamiltonian group. We also show that if a strongly connected two-sided group digraph of valency two is normal, the corresponding group is isomorphic to the product of two groups: a cyclic group with either $T_{k,n}$ or $H_{p,q}$, or an abelian group.

**Keywords:** Cayley digraph, Adjacency Matrix, Normal Matrix.

**Subject classification: 05C25**.

## 1 Introduction

A Cayley digraph is a directed graph that has a finite group, $G$, as its vertex set. An arc $(x, y)$ connects vertex $x$ to vertex $y$ if and only if $x^{-1}y \in S$, where $S$ is a subset of $G$ and the identity element, $e$, is not an element of $S$. The digraph is represented as

---

*Corresponding Author: fnowroozi@sru.ac.ir

$\overrightarrow{Cay}(G, S)$. If the subset $S$ is symmetric, meaning $S = S^{-1} = \{s^{-1} : s \in S\}$, then the digraph becomes a simple undirected graph called a Cayley graph. This graph is represented as $Cay(G, S)$. If the subset $S$ is empty, the related Cayley graph has no edges. The graph is connected if and only if $G$ is generated by $S$ [2]. There have been many generalizations of Cayley digraphs, including the recent introduction of a two-sided group digraph by Iradmusa and Praeger, denoted by $\overrightarrow{2S}(G; L, R)$ [6]. This digraph is defined on a group $G$ as the vertex set, with $L$ and $R$ as two non-empty subsets of $G$. An arc is formed between vertices $x$ and $y$ if and only if $y = l^{-1}xr$ for some $l \in L$ and $r \in R$. It is important to note that for distinct ordered pairs $(l_i, r_i)$ that occur in the relation $y = l_1 x r_1 = l_2 x r_2$, at most one arc from $x$ to $y$ is considered. If the digraph has no loops and the adjacency relation is symmetric, then it is a simple graph called a two-sided group graph, denoted by $2S(G; L, R)$. If a group is abelian, its associated two-sided group digraph will be a Cayley digraph. For the purposes of this paper, we will assume that all groups are non-abelian.

In a two-sided group digraph $\overrightarrow{2S}(G; L, R)$, the arcs that start from a vertex $x$ are referred to as the pairs $(x, y)$, where $y$ is known as the out-neighbor of $x$. The out-valency of a vertex $x$ is the number of distinct out-neighbors of $x$. Conversely, the in-valency of a vertex $x$ is the number of distinct in-neighbors of $x$, represented by the pairs $(y, x)$, where $y$ is called the in-neighbor of $x$. If all vertices in the digraph have the same out-valency and in-valency $c$, it is considered as regular of valency $c$. Let $x \in G$ be an arbitrary element; we define an equivalence relation $\sim_x$ on $L \times R$ as follows; $(l_1, r_1) \sim_x (l_2, r_2)$ if and only if $l_1^{-1} x r_1 = l_2^{-1} x r_2$, and the set $C_x(l, r) = \{(l', r')| \ l'^{-1} x r' = l^{-1} x r, l' \in L, r' \in R\}$ is the equivalence class containing $(l, r)$. Further, $C_x$ denotes the set of all equivalence classes of $\sim_x$. The out-valency of $x$ equals $|C_x|$, the cardinality of $C_x$. Similarly, we define another equivalence relation $\sim'_x$ on $L \times R$. By definition, $(l_1, r_1) \sim'_x (l_2, r_2)$ if and only if $l_1 x r_1^{-1} = l_2 x r_2^{-1}$. The set $C_x'(l, r) = \{(l', r')| \ l' x r'^{-1} = l x r^{-1}, l' \in L, r' \in R\}$ is the equivalence class including $(l, r)$ and $C_x'$ is the set of all equivalence classes of $\sim'_x$. Evidently, $|C_x'|$ is the in-valency of $x$. The following definition provides sufficient and necessary conditions by which a two-sided group digraph is simple with no loops and regular valency $|L||R|$.

We must remark that if $g \in G$ and $L \subseteq G$, then $L^g = g^{-1}Lg$.

**Definition 1** *[6] Let $G$ be a group, $L$ and $R$ be two non-empty subsets of $G$. A pair $(L, R)$ has the 2S-graph-property if for all $g \in G$ the following conditions hold:*

$(i)$ $L^{-1}gR = LgR^{-1}$,
$(ii)$ $L^g \cap R = \emptyset$,
$(iii)$ $(LL^{-1})^g \cap (RR^{-1}) = \{e\}$.

Note that, unlike Cayley graphs, the condition $G = \langle L \cup R \rangle$ is insufficient $\overrightarrow{2S}(G; L, R)$ graphs to be connected.

**Lemma 1** *[6] Let $L, R$ be two non-empty subsets of a group $G$. If $\overrightarrow{2S}(G; L, R)$ is connected, then $G = \langle L \rangle \langle R \rangle$ and there exist words $w$ in $L \cup L^{-1}$ and $w'$ in $R \cup R^{-1}$, with lengths having opposite parity, such that $ww'$ evaluates to the identity element $e$ in $G$.*

In this discussion, we have focused on color graphs that involve labeling the arcs of digraphs with complex numbers. A color graph is defined as a pair consisting of a vertex set and a mapping that associates a color to each arc. Non-edges are represented by the color 0. The adjacency matrix of a color graph is obtained by evaluating the color mapping on each pair of vertices. Cayley color graphs, a specific type of color graph, are defined using a group and a function that maps group elements to complex numbers.

If the matrix $N$ satisfies the relation $N^T N = N N^T$ [10], then it is considered normal. The relationship between the eigenvalues of these matrices and their entries is significant, as stated in Theorem 2.5.3 [5]. Basically, an $n \times n$ matrix $A = [a_{ij}]$ is considered normal if and only if $\sum_{i,j} |a_{ij}|^2 = \sum_i |\lambda_i|^2$, where $\lambda_i$ are the eigenvalues of the matrix and $1 \leq i \leq n$. This relation shows a connection between the normality of the adjacency matrix of a digraph and its spectrum. Several authors have recently studied the computation of eigenvalues of digraphs [3, 7, 11]; moreover, this issue justifies how this topic is essential.

We emphasize that a digraph is normal only if its adjacency matrix is normal. It is clear that whenever $N = N^T$, $N$ is a normal matrix; therefore, if all arcs of a digraph are double and both $(x, y)$ and $(y, x)$ are arcs, for all vertices $x, y$, the adjacency matrix of the digraph is symmetric, and so the digraph is normal. A proper normal digraph (PND) is a normal digraph where all arcs are not double and without loops [10] . According to above definition, all arcs of a two-sided digraph $\overrightarrow{2S}(G; L, R)$ are double if and only if $L^{-1}gR = LgR^{-1}$, for each $g \in G$, making the related digraph an undirected graph and a normal digraph (graph). Further, the relation $L^g \cap R = \emptyset$, for all $g \in G$, indicates that $\overrightarrow{2S}(G; L, R)$ is a digraph without a loop.

**Corollary 1** *Let $G$ be a group with two non-empty subsets $L$ and $R$. Assume that the two-sided group digraph $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal. Then it is a PND if and only if $L^{-1}gR \neq LgR^{-1}$, for some $g \in G$, and $L^g \cap R = \emptyset$, for all $g \in G$.*

It is evident that the following result is true by citing Proposition 4 from reference [10].

**Corollary 2** *There is no non-abelian group $G$ with two non-empty subsets $L$, and $R$, such that the two-sided group digraph $\Gamma = \overrightarrow{2S}(G; L, R)$ be a proper normal connected tree or a proper normal connected unicyclic digraph with at least one vertex of valency one.*

In this paper, we establish certain conditions that ensure the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal. A Hamiltonian group is a non-abelian group where all subgroups are normal. We prove that if all two-sided group digraphs of valency two for a certain group $G$ are normal, then $G$ is a Hamiltonian group. This outcome is similar to the previously researched Cayley digraphs of valency two [8]. Furthermore, we indicate that if a strongly connected two-sided group digraph of valency two is normal, then $G = K.\langle z \rangle$ in which $K$ is an abelian subgroup of $G$ and $\langle z \rangle$ is a cyclic group either $G \cong T_{k,n}.\langle z \rangle$ or $G \cong H_{p,q}.\langle z \rangle$ in which $T_{k,n}, H_{p,q}$ are as follows:

$$T_{m,n} = \langle x, y | \ x^{2m} = y^n = e, \ x^{-1}yx = y^{-1} \rangle, \text{ where } m \geq 1 \text{ and } n \geq 3; \text{ and}$$

$$H_{p,q} = \langle a, c | \ x^{4p} = e, \ y^{2q} = x^{2p}, \ x^{-1}yx = y^{-1} \rangle, \text{ where } p \geq 1 \text{ and } q \geq 1.$$

## 2 Two-sided Color Group Digraphs and their adjacency matrices

Now, we generalize a Cayley color graph, a digraph defined in [1], and we name it a two-sided color group digraph. Let $G$ be a finite group, $\alpha : G \rightarrow \mathbb{C}$, $\beta : G \rightarrow \mathbb{C}$ be two functions, and for $g, h \in G$, we define the function $c_\Gamma : G \times G \rightarrow \mathbb{C}$ as follows:

$$c_\Gamma(g, h) = \frac{1}{t_{g,h}} \sum_{x \in G} \alpha(g^{-1}xh)\beta(x), \tag{1}$$

where $T_{g,h} = \{x \in G : \alpha(g^{-1}xh) = \beta(x) \neq 0\}$ and

$$t_{g,h} = \begin{cases} |T_{g,h}| & \text{if } T_{g,h} \neq \emptyset \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

4

We introduce a two-sided color group digraph $\Gamma = \Gamma(G; \alpha, \beta)$ with the vertex set $V(\Gamma) = G$ and $c_\Gamma(g, h)$, where $c_\Gamma : G \times G \to \mathbb{C}$ defined by relation (1).

In the following theorem, we show that $\Gamma = \Gamma(G; \alpha, \beta)$ is a two-sided group digraph for two specific functions $\alpha$, and $\beta$.

**Proposition 1** *Let $G$ be a group. If a two-sided color group digraph $\Gamma = \Gamma(G; \alpha, \beta)$, in which two functions $\alpha, \beta$ are from $G$ to $\mathbb{C}$, have the values in $\{0, 1\}$, then $\Gamma$ is a two-sided group digraph.*

*Proof.* It suffices to have $\alpha, \beta : G \to \mathbb{C}$ such that $\alpha(g), \beta(g) \in \{0, 1\}$ for all $g \in G$, $R = \{r \in G : \alpha(r) = 1\}$, and $L = \{l \in G : \beta(l) = 1\}$, then $\Gamma$ is a two-sided group digraph denoted by $\Gamma = \overrightarrow{2S}(G; L, R)$ .

Let $g, h \in G$, for all $x \in G$, if $\alpha(g^{-1}xh) = 0$ or $\beta(x) = 0$, then $g^{-1}xh \notin R$ or $x \notin L$, therefore $(g, h)$ is not an arc, and according to relation (1), we have $c_\Gamma(g, h) = 0$. Now, if there is at least one $x \in G$ such that $\alpha(g^{-1}xh) = 1$ and $\beta(x) = 1$, then $g^{-1}xh = r$ and $x = l$ for some $r \in R$, $l \in L$, so $g^{-1}lh = r$, hence $(g, h)$ forms an arc and, in this case, $c_\Gamma(g, h) = 1$. $\square$

According to the proposition 1, the adjacency matrix of $\Gamma = (G; \alpha, \beta)$ is as follows:

$$A = \frac{1}{|G|} \sum_{g,h \in G} \sum_{x \in G} \frac{1}{t_{g,h}} \alpha(g^{-1}xh)\beta(x)L_g.R_h,$$

where $L_g = [l_{xy}(g)]_{x,y \in G}$, $R_g = [r_{xy}(g)]_{x,y \in G}$, $l_{xy}(g) = \begin{cases} 1 & if \ g^{-1}x = y \\ 0 & otherwise \end{cases}$ , and

$r_{xy}(g) = \begin{cases} 1 & if \ xg = y \\ 0 & otherwise \end{cases}$ .

Now, we have the matrix $\Lambda_{g,h} = [\lambda_{xy}(g, h)]_{x,y \in G}$, where

$$\lambda_{xy}(g, h) = \begin{cases} 1 & if \ y = g^{-1}xh \\ 0 & otherwise. \end{cases}$$

Further, it is clear that $L_g R_h = \Lambda_{g,h}$, so the adjacency matrix of a two-sided color group digraph is represented by the following relation.

$$A = \frac{1}{|G|} \sum_{g,h \in G} \sum_{x \in G} \frac{1}{t_{g,h}} \alpha(g^{-1}xh)\beta(x)\Lambda_{g,h}.$$

**Example** Let $G = D_6 = \langle a, b | a^3 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ be the dihedral group of order 6 and $L = \{a, a^2\}$, $R = \{ab, a^2b\}$. Here, corresponding permutation matrices are $L_1 = R_1 = I_6$, the identity matrix of order 6, and $L_a = \left( \begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix} \right)$, $L_{a^2} = \left( \begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix} \right)$, $L_b = \left( \begin{smallmatrix} 0 & C \\ C & 0 \end{smallmatrix} \right)$, $L_{ab} = \left( \begin{smallmatrix} 0 & D \\ D & 0 \end{smallmatrix} \right)$, $L_{a^2b} = \left( \begin{smallmatrix} 0 & E \\ E & 0 \end{smallmatrix} \right)$, $R_a = \left( \begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix} \right)$, $R_{a^2} = \left( \begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix} \right)$, $R_b = \left( \begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix} \right)$, $R_{ab} = \left( \begin{smallmatrix} 0 & B \\ B & 0 \end{smallmatrix} \right)$, $R_{a^2b} = \left( \begin{smallmatrix} 0 & E \\ E & 0 \end{smallmatrix} \right)$ where 0, $A$, $B$, $C$, $D$, $E$ are matrices of order 3 as follows:
$A = \left( \begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix} \right)$, $B = \left( \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix} \right)$, $C = \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix} \right)$, $D = \left( \begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$, $E = \left( \begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix} \right)$, and 0 represents zero matrix.

With simple computations and mentioning $\alpha(g), \beta(g) \in \{0, 1\}$, we have

$$\sum_{g,h \in G} \sum_{x \in G} \frac{1}{t_{g,h}} \alpha(g^{-1}xh)\beta(x)L_g.R_h = 6 \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = 6T.$$

Here, $T$ is the adjacency matrix of $\Gamma = 2S(D_6; L, R)$.

# 3 Two-sided Group Digraphs with Normal Adjacency Matrices

For the adjacency matrix of a digraph to be considered normal, it must satisfy the condition that the number of common in-neighbors between any two vertices is equal to the number of common out-neighbors. This is a well-known fact that can be found in Proposition 1 of [10]. To apply this concept to a two-sided group digraph $\Gamma = \overrightarrow{2S}(G; L, R)$, where $L$ and $R$ be two non-empty subsets of a group $G$, we can use the fact that $L^{-1}gR$, $LgR^{-1}$ represent the sets of out-neighbors and in-neighbors of $g$ ($g \in G$), respectively. Therefore, the adjacency matrix of $\Gamma$ will be normal if and only if the intersection of $L^{-1}gR$ and $L^{-1}hR$ equals to the intersection of $LgR^{-1}$ and $LhR^{-1}$, for all $g, h \in G$.

**Proposition 2** *Let $G = D_{2n} = \langle a, b | a^n = b^2 = e, \ b^{-1}ab = a^{-1} \rangle$, $L$ and $R$ be two non-empty subsets of $G$. If one of the following three cases holds, $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal :*
*(i) $L, R \subseteq \{a^i b | 0 \leq i \leq n - 1\}$;*
*(ii) $L, R \subseteq \langle a \rangle$;*
*(iii) One of the sets $L$ or $R$ is a subset of $\langle a \rangle$, and another is a subset of $\{a^i b | 0 \leq i \leq n - 1\}$.*

*Proof.* (i) In this case, since $L$ and $R$ are inverse-closed subsets, then $\Gamma$ is a simple undirected graph; so, the adjacency matrix of $\Gamma$ is symmetric, which is normal. (ii) Let $L, R \subseteq \langle a \rangle$, then $L = \{a^{i_1}, ..., a^{i_s}\}$ and $R = \{a^{j_1}, ..., a^{j_m}\}$. We must show that $|L^{-1}a^k b^w R \cap L^{-1}a^{k'} b^{w'} R| = |La^k b^w R^{-1} \cap La^{k'} b^{w'} R^{-1}|$, for all $k, k', w, w'$, where $0 \le k, k' \le n - 1$, $0 \le w, w' \le 1$. It is convenient to call these two recent subsets $A$ and $B$, respectively. We define the function $\phi : A \longrightarrow B$ as follows. If $x \in A$, then $x = a^{-i_p + k} b^w a^{j_q} = a^{-i_r + k'} b^{w'} a^{j_t}$, for some $i_p, i_r \in \{i_1, ..., i_s\}$ and $j_q, j_t \in \{j_1, ..., j_m\}$. Therefore, $a^{i_r + k} b^w a^{-j_t} = a^{i_p + k'} b^{w'} a^{-j_q}$. It implies that $y = a^{i_r + k} b^w a^{-j_t} = a^{i_p + k'} b^{w'} a^{-j_q}$ belongs to set $B$. Hence, it is sufficient to define $\phi(x) = y$. One can prove that $\phi$ is bijective; therefore, the adjacency matrix of $\Gamma$ is normal. (iii) It is similar to the case (ii). $\square$

By the Proposition 2, the Dihedral group does not have a PND in case (i), and if cases (ii) or (iii) occur, it likely has one. For example, if we consider the Dihedral group of order 12 with $L = \{a, a^2\}$, $R = \{a^3\}$, then $\Gamma = \overrightarrow{2S}(G; L, R)$ is a PND.

Now, we determine the necessary condition under which the adjacency matrix of the two-sided group digraph is normal.

**Lemma 2** *Let $G$ be a finite group and $L, R \subseteq G$ and $|L| \ge 2$. If the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal, then $(LL^{-1})^y RR^{-1} = (L^{-1}L)^y R^{-1}R$, for all $y \in G$.*

*Proof.* Suppose that $y \in G$ and $e \ne g \in (LL^{-1})^y RR^{-1}$, then $g = (l_1 l_2^{-1})^y r_2 r_1^{-1}$, for some $l_1, l_2 \in L$, $r_1, r_2 \in R$. Assume that $x = yg$, then $g = y^{-1}x$ and $x = (l_1 l_2^{-1})^y r_2 r_1^{-1}$. Hence, $l_1^{-1} x r_1 = l_2^{-1} y r_2 = z$ is a common out-neighbor of $x, y$, i.e., the set of common out-neighbors of $x$ and $y$ is non-empty. Since the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal, thus the number of common in-neighbors of $x, y$ equals the number of common out-neighbors of $x, y$. So, the set common in-neighbors of $x$ and $y$ is not empty. Therefore, $l_3 x r_3^{-1} = l_4 y r_4^{-1}$, for some $l_3, l_4 \in L$ and $r_3, r_4 \in R$. Thus $x = l_3^{-1} l_4 y r_4^{-1} r_3$, so $g = y^{-1}x = (l_3^{-1} l_4)^y r_4^{-1} r_3 \in (L^{-1}L)^y R^{-1}R$, and it implies $(LL^{-1})^y RR^{-1} \subseteq (L^{-1}L)^y R^{-1}R$. Similarly, the reverse is correct, so $(LL^{-1})^y RR^{-1} = (L^{-1}L)^y R^{-1}R$, for all $y \in G$. $\square$

Here, a sufficient condition of normality of $\Gamma = \overrightarrow{2S}(G; L, R)$ is presented.

**Lemma 3** *Let $G$ be a finite group and $L, R \subseteq G$. If one of the following conditions holds, then the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal.*
*(i) $l_1 l_2 = l_2 l_1$, $r_1 r_2 = r_2 r_1$, for all $l_1, l_2 \in L$ and $r_1, r_2 \in R$,*

*(ii)* $l_1^2 = l_2^2, r_1^2 = r_2^2$, for all $l_1, l_2 \in L$, and $r_1, r_2 \in R$.

*Proof.* (i) Let $x, y \in G$ and if $z_1, z_2$ are distinct common out-neighbors of $x, y$, then $z_1 = l_1^{-1}xr_1 = m_1^{-1}yn_1$ and $z_2 = l_2^{-1}xr_2 = m_2^{-1}yn_2$ for some $l_1, l_2, m_1, m_2 \in L$ and $r_1, r_2, n_1, n_2 \in R$. Since $l_1 l_2^{-1} = l_2^{-1}l_1$, $r_1 r_2^{-1} = r_2^{-1}r_1$ for all $l_1, l_2 \in L$ and $r_1, r_2 \in R$, we obtain distinct common in-neighbors $z_1' = l_1 y r_1^{-1} = m_1 x n_1^{-1}$, $z_2' = l_2 y r_2^{-1} = m_2 x n_2^{-1}$. Thus, in this case, we have a one-to-one mapping from $L^{-1}xR \cap L^{-1}yR$ to $LxR^{-1} \cap LyR^{-1}$. Therefore, the number of common in-neighbors $x, y$ equals the number of common out-neighbors of $x, y$.

(ii) The proof is similar to (i). $\square$

If $L$ and $R$ are two single-member subsets of $G$, then the sets of common out-neighbors and common in-neighbors of two distinct elements $x, y$ of $G$ are empty; then the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal. In additon, it can be the unique PND, up to isomorphism, the directed cycle $C_n$ [10].

**Corollary 3** *Let $G$ be a finite group with two subsets $L = \{x, y\}$ and $R$, in which $R$ is an arbitrary singleton. Then the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal, if and only if $x^2 = y^2$ or $xy = yx$.*

*Proof.* If we use Lemmas 2 and 3, the proof easily follows. $\square$

Since $\overrightarrow{2S}(G; L, R) \cong \overrightarrow{2S}(G; R, L)$, then Corollary 3 explains that if a two-sided group digraph of valency two is normal, then both elements of $L$ ( or $R$ if $|R| = 2$) are commuted or have the same square.

The following theorem shows that if $\Gamma = \overrightarrow{2S}(G; L, R)$ of valency two is normal, $G$ is a Hamiltonian group. This consequence is similar to Cayley digraphs of valency two with a normal adjacency matrix (see [8]). The next proposition presents a relationship between normality in matrices and group theory.

**Proposition 3** *For a finite group $G$; if the adjacency matrix of any two-sided group digraph $\Gamma = \overrightarrow{2S}(G; L, R)$ of valency two is normal, then any subgroup of $G$ is normal.*

*Proof.* Let $H$ be an arbitrary subgroup of $G$, $a \in H$, and $b \in G$. Let $L_1$, $L_2$ be $\{a, b\}$, $\{ba, a\}$ respectively, and $R$ be an arbitrary singleton of $G$. Since both $\Gamma = \overrightarrow{2S}(G; L_1, R)$ and $\Gamma = \overrightarrow{2S}(G; L_2, R)$ are normal if $ab \neq ba$, Corollary 3 concludes that $a^2 = b^2$ and $(ba)^2 = a^2$ and this implies $b^{-1}ab = a^{-1}$; therefore $H$ is a normal subgroup. $\square$

As a result, if both $\Gamma = \overrightarrow{2S}(G; L_1, R)$ and $\Gamma = \overrightarrow{2S}(G; L_2, R)$ are normal where

$L_1$, $L_2$ be $\{a, b\}$, $\{ba, a\}$ respectively, with an arbitrary singleton $R$, then $b^{-1}ab = a$ or $b^{-1}ab = a^{-1}$.

In the following, we explain a theorem that is similar to one pertaining to Cayley graphs of valency two as referenced in [8]. The methods employed in proving this theorem closely resemble those used in Theorem 1 of [8]. We should mention that $Q_8$ is the quaternion group of order 8, that is, $Q_8 = \langle x, y | \ x^4 = e,\ y^2 = x^2,\ y^{-1}xy = x^{-1} \rangle$. Further, Corollary 3 implies that two non-commuting elements of $L$, when $|L| = 2$, are the same square; therefore $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal, and obviously, any two elements of $Q_8 \times \mathbb{Z}_2^n$ has this condition while $\mathbb{Z}_2^n = \mathbb{Z}_2 \times ... \times \mathbb{Z}_2$ for some $n \in \mathbb{Z}_+$. Theorem 1 shows that $Q_8 \times \mathbb{Z}_2^n$, up to isomorphism, is the only example that satisfies the assumptions.

**Theorem 1** *Let $G$ be a finite non-abelian group. If the adjacency matrix of every two-sided group digraph $\Gamma = \overrightarrow{2S}(G; L, R)$ of valency two is normal, then $G \cong Q_8 \times \mathbb{Z}_2^n$ for some $n \in \mathbb{Z}_+$.*

*Proof.* Let $G$ be a non-abelian group with elements $x, y \in G$ such that $xy \neq yx$. Using Corollary 3, we can show that $x^2 = y^2$. Furthermore, the adjacency matrix of the group digraph $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal when $L = \{x, y\}$ or $L = \{yx, x\}$. This implies that $y^{-1}xy = x^{-1}$, and so $x^{-2} = x^2$. Thus, we have $x^4 = e$, and the subgroup $\langle x, y \rangle$ is isomorphic to $Q_8$ and that is a normal subgroup of $G$ by using Proposition 3.

Suppose that $\alpha \in G$ does not commute with $x$ and $y$. Using the same method as in the proof of Proposition 3, we can show that $\alpha$ commutes $xy$, so $\alpha$ commutes with at least one of $x$, $y$, $xy$. If $\alpha x = x\alpha$ and $y\alpha = \alpha y$, then we can use the adjacency matrix of the two-sided group digraph with $L = \{x\alpha, y\}$ to show that $\alpha^2 = e$ and thus $\alpha$ lies in the center of $G$. If $y\alpha \neq \alpha y$, then we have $\alpha^2 = y^2 = x^2$ and $(x\alpha)^2 = x^4 = e$, implying that $x\alpha$ lies in the center of $G$. Therefore, $G$ is generated by $Q$ and $Z(G)$, and since $Z(G)$ has exponent 2, we can choose a subgroup $X$ of index 2 of $Z(G)$ such that $X \cap \langle x, y \rangle = \{e\}$. This implies that $G = \langle x, y \rangle \times X$, and since $X$ is a subgroup of $Z(G)$, it is a group of exponent 2 and is isomorphic to $\mathbb{Z}_2^n$ (the other cases are similar to the considered one). $\square$

The next proposition and paragraph present more details about groups $T_{m,n}$, and $H_{p,q}$.

**Proposition 4** *[8] For the groups $T_{m,n}$ and $H_{p,q}$, the following three statements hold:*

*(1)* $T_{m,n} \cong T_{m',n'}$ *iff* $m = m'$ *and* $n = n'$;

*(2)* $H_{p,q} \cong H_{p',q'}$ *iff* $p = p'$ *and* $q = q'$;

*(3)* $H_{p,q} \not\cong T_{m,n}$ *for any* $p, q, m, n$.

According to research, both $T_{m,n}$ and $H_{p,q}$ are considered metacyclic groups. A metacyclic group $G$ is a group that has a cyclic normal subgroup $N$, and the quotient group $G/N$ is also cyclic [9]. In a specific case, $T_{1,n}$ is proven to be the dihedral group $D_{2n}$ of order $2n$, while $H_{1,q}$ is a dicyclic group of order $8q$ which is an extension of the cyclic group of order two by a cyclic group of order $4q$. Further, $T_{m,n}$ contains subgroup $\langle x^m, y \rangle \cong D_{2n}$ if $m$ is odd, and $\langle x^p, y \rangle$ is a subgroup of $H_{p,q}$ which is isomorphic to a dicyclic group of order $8q$ if $p$ is odd. There is a full classification of such groups in [4].

Using the definition, a digraph $\Gamma$ is strongly connected if, for any two vertices $v$ and $w$, there is a path from $v$ to $w$ in $\Gamma$. In the final theorem, we show that if $\Gamma = \overrightarrow{2S}(G; L, R)$ is a strongly connected two-sided group digraph of valency two with a normal adjacency matrix, $G$ has very particular conditions.

**Theorem 2** *Let $G$ be a finite group. Suppose that $\Gamma = \overrightarrow{2S}(G; L, R)$ is a strongly connected two-sided group digraph of valency two with $L = \{x, y\}$, $R = \{z\}$ whose adjacency matrix is normal. Then, $G$ is isomorphic to one of the following groups:*

*(1) $K.\langle z \rangle$, where $K$ is an abelian group;*

*(2) $T_{m,n}.\langle z \rangle$, where $m \geqslant 1$ and $n \geqslant 3$;*

*(3) $H_{p,q}.\langle z \rangle$, where $p \geqslant 1$ and $q \geqslant 1$.*

*Proof.* Since $\Gamma$ is strongly connected, then it is connected, and using Lemma 1 implies $G = \langle L \rangle \langle R \rangle = \langle x, y \rangle \langle z \rangle$ and there exist words $w$ in $L \cup L^{-1}$ and $w'$ in $R \cup R^{-1}$, with lengths having opposite parity, such that $ww'$ evaluates to the identity element $e$ in G. Applying Corollary 3, we have $x^2 = y^2$ or $xy = yx$. If $xy = yx$, then the case (1) happens, otherwise $x^2 = y^2$ and suppose that $t = x^{-1}y$. It is clear that $\langle L \rangle = \langle x, t \rangle$ and we have $x^2 = y^2 = xtxt$ or, equivalently, $x^{-1}tx = t^{-1}$. If the order of $t$ is two, then $t^{-1} = t$. It implies $x^{-1}tx = t$, and thus $\langle L \rangle$ must be abelian and this is a contradiction. Therefore, the order of $t$ is at least three. Assume that $r$ is the order of $x$; since $x^2 \in Z(\langle L \rangle)$ and $x \notin Z(\langle L \rangle)$, we know that $r = 2m$ for some $m$. Let $n$ be the smallest natural number such that $t^n = x^s$ for some $s \in \{0, ..., 2m - 1\}$. We get $t^n = x^{-1}t^n x = (x^{-1}tx) \ldots (x^{-1}tx) = t^{-n}$ and therefore $x^{2s} = t^{2n} = e$. Hence, either $s = 0$ or $s = m$. If $s = 0$, then $\langle L \rangle = T_{m,n}$. In case $s = m$, then $\langle L \rangle$ has the form $\langle x, t | x^{2m} = e, t^n = x^m, x^{-1}tx = t^{-1} \rangle$. Since $x^2 \in Z(\langle L \rangle)$, $x^m \in Z(\langle L \rangle)$, and

$x \notin Z(\langle L \rangle)$, we have $m = 2p$ for some $p$. If $n$ is odd, then $(tx^m)^n = t^n x^m = t^{2n} = e$. Furthermore, $x^{-1}(tx^m)x = t^{-1}x^m = x^m t^{-1} = (tx^m)^{-1}$. So, we can conclude that $\langle L \rangle$ is isomorphic to $T_{m,n}$. Finally, if $n = 2q$ for some $q$, then $\langle L \rangle = H_{p,q}$. Therefore, $G \cong T_{m,n}.\langle z \rangle$ or $G \cong H_{p,q}.\langle z \rangle$. $\square$

# References

[1] L. Babai, *Spectra of Cayley Graphs*, J. Combin. Theory Ser. B 27(2)(1979), 180-189, https://doi.org/10.1016/0095-8956(79)90079-0.

[2] N. Biggs, *Algebraic Graph Theory*, Cambridge Univ. Press, Cambridge, 1974.

[3] M. Ghorbani & F. Nowroozi Larki, *On the spectrum of finite Cayley graphs*, J. Discrete Math. Sci. Cryptogr. 21(1)(2018) 83-112, https://doi.org/10.1080/09720529.2018.1449797.

[4] C. E. Hempel, *Metacyclic groups*, Comm. Algebra 28(8) (2007) 3865-3897, https://doi.org/10.1080/00927870008827063.

[5] *R*.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1986.

[6] M. N. Iradmusa, C. E. Praeger, *Two-sided Group Digraphs and Graphs* J. Graph Theory, 82(3) (2016), 279-295, https://doi.org/10.1002/jgt.21901.

[7] R. Krakovski and B. Mohar, *Spectrum of Cayley graphs on the symmetric group generated by transpositions*, Lin. Alg. Appl. 437(3) (2012), 1033-1039, https://doi.org/10.48550/arXiv.1201.2167 .

[8] D. S. Lyubshin, S. V. Savchenko, *Cayley Digraphs with Normal adjacency matrices*, Discrete Math. 309(13) (2009), 4343-4348, https://doi.org/10.1016/j.disc.2007.03.086.

[9] J. S. Rose, *A course on group theory*, Cambridge University Press, Cambridge, 1978.

[10] A. Torgasev, *The spectrum of a normal digraph*, Univ. Novi Sad, Coll. of papers Fac. Sci. (Ser. Math.) 17 (1) (1987) 187-200.

[11] Liu, Xiaogang, Sanming Zhou. *Eigenvalues of Cayley graphs.* arXiv:1809.09829
The Electronic Journal of Combinatorics, Volume 29, Issue 2 (2022), P2.9,

https://doi.org/10.48550/arXiv.1809.09829

.