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APPROXIMATION OF DISCRETE FUNCTIONS USING SPECIAL SERIES BY MODIFIED MEIXNER POLYNOMIALS

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ABSTRACT. This article is devoted to the study of approximative properties of the special series by modified Meixner polynomials $M_{n,N}^\alpha(x)$ ($n = 0, 1, \dots$). For $\alpha > -1$ these polynomials form an orthogonal system on the grid $\Omega_\delta = \{0, \delta, 2\delta, \dots\}$ with respect to the weight function $w(x) = e^{-x} \frac{\Gamma(Nx+\alpha+1)}{\Gamma(Nx+1)}$, where $\delta = \frac{1}{N}$, $N > 0$. We obtained upper estimate on $[\frac{\theta_n}{2}, \infty)$ for the Lebesgue function of partial sums of a special series, where $\theta_n = 4n + 2\alpha + 2$.

Keywords: Meixner polynomials, Fourier series, special series, Lebesgue function.

1. INTRODUCTION

Suppose $\Omega_\delta = \{0, \delta, 2\delta, \dots\}$, where $\delta = \frac{1}{N}$, $N > 0$. We denote by $M_{n,N}^\alpha(x) = M_n^\alpha(Nx, e^{-\delta})$ ($n = 0, 1, \dots$) the modified Meixner polynomials that form for $\alpha > -1$ an orthogonal system on the grid Ω_δ with respect to the weight function $w(x) = e^{-x} \frac{\Gamma(Nx+\alpha+1)}{\Gamma(Nx+1)}$, i.e.

$$\sum_{x \in \Omega_\delta} M_{n,N}^\alpha(x) M_{k,N}^\alpha(x) w(x) = (1 - e^{-\delta})^{-\alpha-1} h_{n,N}^\alpha \delta_{nk}, \quad \alpha > -1,$$

where δ_{nk} is the Kronecker symbol and $h_{n,N}^\alpha = \binom{n+\alpha}{n} e^{n\delta} \Gamma(\alpha+1)$. The corresponding orthonormal polynomials with the weight function $\rho_N(x) = (1 - e^{-\delta})^{\alpha+1} w(x)$ we denote by $m_{n,N}^\alpha(x) = (h_{n,N}^\alpha)^{-1/2} M_{n,N}^\alpha(x)$ ($n = 0, 1, \dots$). In this paper, we continue the study of the approximative properties of partial sums of special series by the modified Meixner polynomials $m_{n,N}^\alpha(x)$. The special series by the system

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of polynomials $m_{n,N}^\alpha(x)$ is a natural (and alternative to Fourier–Meixner series) apparatus for simultaneous approximation of functions and their finite differences. Moreover, partial sums of special series interpolate the original function at the points $0, \delta, \dots, (r - 1)\delta$. Note that partial sums of Fourier–Meixner series do not have this property. However, the approximative properties of partial sums of the special series are not sufficiently studied. In particular, the problem of estimating on the interval $[\frac{\theta_n}{2}, \infty)$ the Lebesgue function $l_{n,N}^{\alpha,r}(x)$ of partial sums of a special series was not studied. For the estimate of the function $l_{n,N}^{\alpha,r}(x)$, the Christoffel–Darboux formula (1) and weighted estimates (3), (5) play an important role. These estimates hold for any $x \in [0, \infty)$ and depend on the location of the variable x on the semi-axis. Therefore, to estimate the Lebesgue function $l_{n,N}^{\alpha,r}(x)$ on $[r\delta, \infty)$, it is necessary to split the semi-axis into sets $G_1 = [r\delta, \frac{3\lambda}{\theta_n}]$, $G_2 = [\frac{3\lambda}{\theta_n}, \frac{\theta_n}{2}]$, $G_3 = [\frac{\theta_n}{2}, \frac{3\theta_n}{2}]$ and $G_4 = [\frac{3\theta_n}{2}, \infty)$, where $\theta_n = 4n + 2\alpha + 2$. For $x \in G_1 \cup G_2$ the function $l_{n,N}^{\alpha,r}(x)$ was estimated in [1]. The main result of this paper is Theorem 1. In this theorem, we obtained upper estimates for the function $l_{n,N}^{\alpha,r}(x)$ on the sets G_3 and G_4 .

2. SOME PROPERTIES OF THE MODIFIED MEIXNER POLYNOMIALS

To study the approximative properties of partial sums of a special series, we need some properties of the polynomials $M_{n,N}^\alpha(x)$. Let $N > 0$, $\delta = \frac{1}{N}$, $\Omega_\delta = \{0, \delta, 2\delta, \dots\}$, and let α be an arbitrary real number. Then for the polynomials $M_{n,N}^\alpha(x)$ we have [2]:

- the Rodrigues formula

$$M_{n,N}^\alpha(x) = \frac{\Gamma(Nx + 1)e^{n\delta+x}}{n!\Gamma(Nx + \alpha + 1)} \Delta_\delta^n \left\{ \frac{\Gamma(Nx + \alpha + 1)}{\Gamma(Nx - n + 1)} e^{-x} \right\},$$

where $\Delta_\delta^0 f(x) = f(x)$, $\Delta_\delta f(x) = f(x + \delta) - f(x)$, $\Delta_\delta^n f(x) = \Delta_\delta(\Delta_\delta^{n-1} f(x))$;

- the orthogonality relation

$$\sum_{x \in \Omega_\delta} M_{n,N}^\alpha(x) M_{k,N}^\alpha(x) \rho_N(x) = h_{n,N}^\alpha \delta_{nk}, \quad \alpha > -1;$$

- the Christoffel–Darboux formula

$$(1) \quad \mathcal{K}_{n,N}^\alpha(t, x) = \sum_{k=0}^n m_{k,N}^\alpha(t) m_{k,N}^\alpha(x) = \frac{\delta \sqrt{(n+1)(n+\alpha+1)} m_{n+1,N}^\alpha(t) m_{n,N}^\alpha(x) - m_{n,N}^\alpha(t) m_{n+1,N}^\alpha(x)}{e^{\frac{\delta}{2}} - e^{-\frac{\delta}{2}}} \frac{1}{x-t},$$

which can be written [3, 4] as follows:

$$\mathcal{K}_{n,N}^\alpha(t, x) = \frac{\alpha_n}{(\alpha_n + \alpha_{n-1})} m_{n,N}^\alpha(t) m_{n,N}^\alpha(x) + \frac{\alpha_n \alpha_{n-1}}{(\alpha_n + \alpha_{n-1})} \frac{\delta}{(e^{\frac{\delta}{2}} - e^{-\frac{\delta}{2}})} \frac{1}{(x-t)} \times$$

$$(2) \quad [m_{n,N}^\alpha(x) (m_{n+1,N}^\alpha(t) - m_{n-1,N}^\alpha(t)) - m_{n,N}^\alpha(t) (m_{n+1,N}^\alpha(x) - m_{n-1,N}^\alpha(x))],$$

where $\alpha_n = \sqrt{(n+1)(n+\alpha+1)}$, $m_{-1,N}^\alpha(x) = 0$.

For $0 < \delta \leq 1$, $N = \frac{1}{\delta}$, $\lambda > 0$, $1 \leq n \leq \lambda N$, $\alpha > -1$, $0 \leq x < \infty$, $s \geq 0$, $\theta_n = 4n + 2\alpha + 2$ the following weighted estimates hold [2, 5]:

$$(3) \quad e^{-\frac{x}{2}} |m_{n,N}^\alpha(x \pm s\delta)| \leq c(\alpha, \lambda, s) \theta_n^{-\frac{\alpha}{2}} A_n^\alpha(x),$$

$$(4) \quad A_n^\alpha(x) = \begin{cases} \theta_n^\alpha, & 0 \leq x \leq \frac{1}{\theta_n}, \\ \theta_n^{\frac{\alpha}{2}-\frac{1}{4}} x^{-\frac{\alpha}{2}-\frac{1}{4}}, & \frac{1}{\theta_n} < x \leq \frac{\theta_n}{2}, \\ \left[\theta_n(\theta_n^{\frac{3}{2}} + |x - \theta_n|)\right]^{-\frac{1}{4}}, & \frac{\theta_n}{2} < x \leq \frac{3\theta_n}{2}, \\ e^{-\frac{3x}{8}}, & \frac{3\theta_n}{2} < x < \infty, \end{cases}$$

$$(5) \quad c(\alpha, \lambda, s) \begin{cases} \theta_n^{\frac{\alpha}{2}-1}, & 0 \leq x \leq \frac{1}{\theta_n}, \\ \theta_n^{-\frac{3}{4}} x^{-\frac{\alpha}{2}+\frac{1}{4}}, & \frac{1}{\theta_n} < x \leq \frac{\theta_n}{2}, \\ x^{-\frac{\alpha}{2}} \theta_n^{-\frac{3}{4}} \left[\theta_n^{\frac{3}{2}} + |x - \theta_n|\right]^{\frac{1}{4}}, & \frac{\theta_n}{2} < x \leq \frac{3\theta_n}{2}, \\ e^{-\frac{3x}{8}}, & \frac{3\theta_n}{2} < x < \infty. \end{cases}$$

Hereinafter $c, c(\alpha), c(\alpha, \lambda, s)$ are positive numbers which can be different in different places of the text.

3. LEBESGUE'S INEQUALITY FOR THE PARTIAL SUMS OF SPECIAL SERIES BY MODIFIED MEIXNER POLYNOMIALS

We denote by $l^2_{\rho_N}(\Omega_\delta)$ the space of functions f defined on the grid Ω_δ and such that $\sum_{x \in \Omega_\delta} f^2(x) \rho_N(x) < \infty$. For $f \in l^2_{\rho_N}(\Omega_\delta)$ and $x \in \Omega_{r,\delta} = \{r\delta, (r+1)\delta, \dots\}$, we can define a discrete analog of the Taylor polynomial of the following form $P_{r-1,N}(x) = \sum_{\nu=0}^{r-1} \frac{\Delta_\delta^\nu f(0)}{\nu!} (Nx)^{[\nu]}$. It is easy to show that the function $f_r(x) = \frac{f(x) - P_{r-1,N}(x)}{N^{-r}(Nx)^{[r]}}$ belongs to the space $l^2_{\rho_{N,r}}(\Omega_{r,\delta})$, where $\rho_{N,r}(x) = \rho_N(x - r\delta)$. Since the modified Meixner polynomials $m_{k,N,r}^\alpha(x) = m_{k,N}^\alpha(x - r\delta)$ ($k = 0, 1, \dots$) for $\alpha > -1$ form an orthonormal basis in $l^2_{\rho_{N,r}}(\Omega_{r,\delta})$, then we can define the Fourier-Meixner coefficients of f_r

$$\hat{f}_{r,k}^\alpha = \sum_{t \in \Omega_{r,\delta}} f_r(t) \rho_{N,r}(t) m_{k,N,r}^\alpha(t) = \sum_{t \in \Omega_{r,\delta}} \frac{f(t) - P_{r-1,N}(t)}{N^{-r}(Nt)^{[r]}} \rho_{N,r}(t) m_{k,N,r}^\alpha(t)$$

and the Fourier-Meixner series of f_r

$$(6) \quad f_r(x) = \sum_{k=0}^{\infty} \hat{f}_{r,k}^\alpha m_{k,N,r}^\alpha(x).$$

Series (6) converges to f_r uniformly with respect to $x \in \Omega_{r,\delta}$. This follows from the basis property of the system of polynomials $m_{k,N,r}^\alpha(x)$ ($k = 0, 1, \dots$) in the space $l^2_{\rho_{N,r}}(\Omega_{r,\delta})$. Then

$$(7) \quad f(x) = P_{r-1,N}(x) + N^{-r}(Nx)^{[r]} \sum_{k=0}^{\infty} \hat{f}_{r,k}^\alpha m_{k,N,r}^\alpha(x), \quad x \in \Omega_\delta.$$

Following [1, 6], we will call (7) the special series by the modified Meixner polynomials. Denote by $S_{n+r,N}^\alpha(f, x)$ the partial sum of the special series (7) of the form

$$S_{n+r,N}^\alpha(f, x) = P_{r-1,N}(x) + N^{-r}(Nx)^{[r]} \sum_{k=0}^n \hat{f}_{r,k}^\alpha m_{k,N,r}^\alpha(x).$$

We note that if $f(x) = p_{n+r}(x)$ is an algebraic polynomial of degree $n + r$, then $\hat{f}_{r,k}^\alpha = 0$ for $k \geq n + 1$. Therefore from (7) follows that $S_{n+r,N}^\alpha(p_{n+r}, x) \equiv p_{n+r}(x)$.

Further, denote by $q_{n+r}(x)$ an algebraic polynomial of degree $n + r$ such that

$$\Delta^i f(0) = \Delta^i q_{n+r}(0) \quad (i = \overline{0, r-1}).$$

Then for $x \in \Omega_{r,\delta}$ we have

$$\begin{aligned} e^{-\frac{\alpha}{2}x^{-\frac{r}{2}+\frac{1}{4}}} |f(x) - S_{n+r,N}^\alpha(f, x)| &\leq e^{-\frac{\alpha}{2}x^{-\frac{r}{2}+\frac{1}{4}}} |f(x) - q_{n+r}(x)| + \\ (8) \quad &e^{-\frac{\alpha}{2}x^{-\frac{r}{2}+\frac{1}{4}}} |S_{n+r,N}^\alpha(q_{n+r} - f, x)|. \end{aligned}$$

Since $P_{r-1,N}(q_{n+r} - f, x) = 0$,

$$\begin{aligned} e^{-\frac{\alpha}{2}x^{-\frac{r}{2}+\frac{1}{4}}} |S_{n+r,N}^\alpha(q_{n+r} - f, x)| &= \\ e^{-\frac{\alpha}{2}x^{-\frac{r}{2}+\frac{1}{4}}} N^{-r} (Nx)^{[r]} &\left| \sum_{k=0}^n (\widehat{q_{n+r} - f})_{r,k}^\alpha m_{k,N}^\alpha(x - r\delta) \right| \leq \\ e^{-\frac{\alpha}{2}x^{-\frac{r}{2}+\frac{1}{4}}} (Nx)^{[r]} \sum_{t \in \Omega_{r,\delta}} &\frac{|q_{n+r}(t) - f(t)|}{(Nt)^{[r]}} \rho_{N,r}(t) \left| \sum_{k=0}^n m_{k,N}^\alpha(t - r\delta) m_{k,N}^\alpha(x - r\delta) \right| = \\ (9) \quad e^{-\frac{\alpha}{2}x^{-\frac{r}{2}+\frac{1}{4}}} (Nx)^{[r]} \sum_{t \in \Omega_{r,\delta}} &\frac{|q_{n+r}(t) - f(t)|}{(Nt)^{[r]}} \rho_{N,r}(t) |\mathcal{K}_{n,N}^\alpha(t - r\delta, x - r\delta)|. \end{aligned}$$

Put

$$(10) \quad E_k^r(f, \delta) = \inf_{q_k} \sup_{x \in \Omega_{r,\delta}} e^{-\frac{\alpha}{2}x^{-\frac{r}{2}+\frac{1}{4}}} |f(x) - q_k(x)|,$$

where the infimum is taken over all algebraic polynomials $q_k(x)$ of degree k for which $\Delta^i f(0) = \Delta^i q_k(0)$ ($i = \overline{0, r-1}$). Then from (8) – (10) we have

$$(11) \quad e^{-\frac{\alpha}{2}x^{-\frac{r}{2}+\frac{1}{4}}} |f(x) - S_{n+r,N}^\alpha(f, x)| \leq E_{n+r}^r(f, \delta)(1 + l_{n,N}^{\alpha,r}(x)),$$

where

$$(12) \quad l_{n,N}^{\alpha,r}(x) = e^{-\frac{\alpha}{2}x^{-\frac{r}{2}+\frac{1}{4}}} (Nx)^{[r]} (1 - e^{-\delta})^{\alpha+1} \times \\ \sum_{t \in \Omega_{r,\delta}} \frac{e^{-\frac{\alpha}{2}+r\delta} t^{\frac{r}{2}-\frac{1}{4}} \Gamma(Nt - r + \alpha + 1)}{(Nt)^{[r]} \Gamma(Nt - r + 1)} |\mathcal{K}_{n,N}^\alpha(t - r\delta, x - r\delta)|.$$

Inequality (11) leads to the problem of estimating on $[r\delta, \infty)$ the Lebesgue function $l_{n,N}^{\alpha,r}(x)$ defined in (12). For $x \in G_1 \cup G_2$ this problem was solved in [1]. In this paper, we will estimate the function $l_{n,N}^{\alpha,r}(x)$ on the sets G_3 and G_4 .

Theorem 1. *Suppose that $r \in \mathbb{N}$, $r - \frac{1}{2} < \alpha < r + \frac{1}{2}$, $\theta_n = 4n + 2\alpha + 2$, $\lambda > 0$, $0 < \delta \leq 1$, $1 \leq n \leq \lambda N$. Then the following estimates hold:*

1) if $x \in G_3$, then

$$(13) \quad l_{n,N}^{\alpha,r}(x) \leq c(\alpha, \lambda, r) \left[\ln(n + 1) + \left(\frac{\theta_n}{\theta_n^{\frac{1}{3}} + |x - \theta_n|} \right)^{\frac{1}{4}} \right];$$

2) if $x \in G_4$, then

$$(14) \quad l_{n,N}^{\alpha,r}(x) \leq c(\alpha, \lambda, r) n^{-\frac{r}{2}+\frac{5}{4}} x^{\frac{r}{2}+\frac{1}{4}} e^{-\frac{\alpha}{4}}.$$

Proof. Assume that $x \in G_3$. We introduce the notations: $D_1 = [r\delta, x - \sqrt{x/\theta_n}] \cap \Omega_{r,\delta}$, $D_2 = [x - \sqrt{x/\theta_n}, x + \sqrt{x/\theta_n}] \cap \Omega_{r,\delta}$, $D_3 = [x + \sqrt{x/\theta_n}, \infty) \cap \Omega_{r,\delta}$. Using these notations, we can write

$$(15) \quad l_{n,N}^{\alpha,r}(x) = H_1 + H_2 + H_3,$$

where

$$(16) \quad H_i = e^{-\frac{x}{2}} x^{-\frac{r}{2} + \frac{1}{4}} (Nx)^{[r]} (1 - e^{-\delta})^{\alpha+1} \times \sum_{t \in D_i} \frac{e^{-\frac{t}{2} + r\delta} t^{\frac{r}{2} - \frac{1}{4}} \Gamma(Nt - r + \alpha + 1)}{(Nt)^{[r]} \Gamma(Nt - r + 1)} |\mathcal{K}_{n,N}^\alpha(t - r\delta, x - r\delta)|, \quad i = 1, 2, 3.$$

First, we estimate H_2 . Note that the Cauchy-Schwartz inequality yields

$$(17) \quad |\mathcal{K}_{n,N}^\alpha(t - r\delta, x - r\delta)| \leq |\mathcal{K}_{n,N}^\alpha(t - r\delta, t - r\delta)|^{1/2} |\mathcal{K}_{n,N}^\alpha(x - r\delta, x - r\delta)|^{1/2}.$$

If $x \in G_3$, then for $t \in D_2$ we have $c_1x \leq t \leq c_2x$. Therefore, from (16) and (17) we obtain the inequality

$$H_2 \leq c(\alpha, r) x^{\frac{r}{2} + \frac{1}{4}} |e^{-x} \mathcal{K}_{n,N}^\alpha(x - r\delta, x - r\delta)|^{1/2} \delta \times \sum_{t \in D_2} t^{\alpha - \frac{r}{2} - \frac{1}{4}} |e^{-t} \mathcal{K}_{n,N}^\alpha(t - r\delta, t - r\delta)|^{1/2}.$$

For the value $|e^{-t} \mathcal{K}_{n,N}^\alpha(t - r\delta, t - r\delta)|$ in [1] the following estimate was obtained:

$$(18) \quad |e^{-t} \mathcal{K}_{n,N}^\alpha(t - r\delta, t - r\delta)| \leq c(\alpha, \lambda, r) t^{-\alpha - \frac{1}{2}} n^{\frac{1}{2}},$$

where $\alpha > -1$, $\theta_n = 4n + 2\alpha + 2$, $t \geq 3/\theta_n$, $\lambda > 0$, $1 \leq n \leq \lambda N$. From this estimate it follows that

$$(19) \quad H_2 \leq c(\alpha, \lambda, r) x^{\frac{r}{2} + \frac{1}{4}} x^{-\frac{\alpha}{2} - \frac{1}{4}} n^{\frac{1}{4}} \delta \sum_{t \in D_2} t^{\alpha - \frac{r}{2} - \frac{1}{4}} t^{-\frac{\alpha}{2} - \frac{1}{4}} n^{\frac{1}{4}} = c(\alpha, \lambda, r) x^{\frac{r}{2} - \frac{\alpha}{2}} n^{\frac{1}{2}} \delta \sum_{t \in D_2} t^{\frac{\alpha}{2} - \frac{r}{2} - \frac{1}{2}} \leq c(\alpha, \lambda, r) x^{-\frac{1}{2}} n^{\frac{1}{2}} \sum_{x \in D_2} \delta \leq c(\alpha, \lambda, r).$$

We proceed to estimating H_3 . To this end, we use equality (2) and write

$$(20) \quad H_3 \leq c(\alpha, r)(H_{31} + H_{32} + H_{33}),$$

where

$$H_{31} = e^{-\frac{x}{2}} x^{\frac{r}{2} + \frac{1}{4}} |m_{n,N}^\alpha(x - r\delta)| \delta \sum_{t \in D_3} e^{-\frac{t}{2}} t^{-\frac{r}{2} - \frac{1}{4}} t^\alpha |m_{n,N}^\alpha(t - r\delta)|,$$

$$H_{32} = ne^{-\frac{x}{2}} x^{\frac{r}{2} + \frac{1}{4}} |m_{n+1,N}^\alpha(x - r\delta) - m_{n-1,N}^\alpha(x - r\delta)| \delta \times \sum_{t \in D_3} \frac{e^{-\frac{t}{2}} t^{\alpha - \frac{r}{2} - \frac{1}{4}}}{|t - x|} |m_{n,N}^\alpha(t - r\delta)|,$$

$$H_{33} = ne^{-\frac{x}{2}} x^{\frac{r}{2} + \frac{1}{4}} |m_{n,N}^\alpha(x - r\delta)| \delta \times \sum_{t \in D_3} \frac{e^{-\frac{t}{2}} t^{\alpha - \frac{r}{2} - \frac{1}{4}}}{|t - x|} |m_{n+1,N}^\alpha(t - r\delta) - m_{n-1,N}^\alpha(t - r\delta)|.$$

To estimate H_{31} , we represent it as

$$(21) \quad H_{31} = H_{31}^1 + H_{31}^2,$$

where

$$H_{31}^i = e^{-\frac{r}{2}} x^{\frac{r}{2} + \frac{1}{4}} |m_{n,N}^\alpha(x - r\delta)| \delta \sum_{t \in D_3^i} e^{-\frac{t}{2}} t^{-\frac{r}{2} - \frac{1}{4}} t^\alpha |m_{n,N}^\alpha(t - r\delta)|, \quad i = 1, 2,$$

$$D_3^1 = [x + \sqrt{x/\theta_n}, 3\theta_n/2 + \sqrt{x/\theta_n}] \cap \Omega_{r,\delta}, \quad D_3^2 = [3\theta_n/2 + \sqrt{x/\theta_n}, \infty) \cap \Omega_{r,\delta}.$$

From inequalities (3) and (4) we obtain

$$\begin{aligned} H_{31}^1 &\leq c(\alpha, \lambda, r) \frac{\theta_n^{-\alpha - \frac{1}{2}} x^{\frac{r}{2} + \frac{1}{4}}}{(\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}}} \delta \sum_{t \in D_3^1} \frac{t^{\alpha - \frac{r}{2} - \frac{1}{4}}}{(\theta_n^{\frac{1}{3}} + |t - \theta_n|)^{\frac{1}{4}}} \leq \\ &c(\alpha, \lambda, r) \frac{\theta_n^{-\frac{1}{2}}}{(\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}}} \delta \sum_{t \in D_3^1} \frac{1}{(\theta_n^{\frac{1}{3}} + |t - \theta_n|)^{\frac{1}{4}}} \leq \\ (22) \quad &c(\alpha, \lambda, r) \frac{\theta_n^{-\frac{1}{2}}}{(\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}}} \left(\delta \frac{2}{\theta_n^{\frac{1}{12}}} + \int_{x + \sqrt{x/\theta_n}}^{\theta_n} \frac{dt}{(\theta_n^{\frac{1}{3}} + \theta_n - t)^{\frac{1}{4}}} + \right. \\ &\left. \int_{\theta_n}^{3\theta_n/2 + \sqrt{x/\theta_n}} \frac{dt}{(\theta_n^{\frac{1}{3}} + t - \theta_n)^{\frac{1}{4}}} \right) \leq c(\alpha, \lambda, r) \left(\frac{\theta_n}{\theta_n^{\frac{1}{3}} + |x - \theta_n|} \right)^{\frac{1}{4}}, \end{aligned}$$

$$\begin{aligned} H_{31}^2 &\leq c(\alpha, \lambda, r) \frac{\theta_n^{-\alpha - \frac{1}{4}} x^{\frac{r}{2} + \frac{1}{4}}}{(\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}}} \delta \sum_{t \in D_3^2} t^{\alpha - \frac{r}{2} - \frac{1}{4}} e^{-\frac{3t}{8}} \leq \\ &c(\alpha, \lambda, r) \delta \sum_{t \in D_3^2} \left(\frac{t}{\theta_n} \right)^\alpha e^{-\frac{3t}{8}} \leq c(\alpha, \lambda, r) \int_{\frac{3\theta_n}{2}}^{\infty} \left(\frac{t}{\theta_n} \right)^\alpha e^{-\frac{3t}{8}} dt = \end{aligned}$$

$$(23) \quad c(\alpha, \lambda, r) \theta_n \int_{\frac{3}{2}}^{\infty} y^\alpha e^{-\frac{3\theta_n}{8} y} dt \leq c(\alpha, \lambda, r) \theta_n e^{-\frac{9\theta_n}{32}} \int_{\frac{3}{2}}^{\infty} y^\alpha e^{-\frac{3\theta_n}{16} y} dt \leq c(\alpha, \lambda, r).$$

From (21)–(23) we derive the following estimate

$$(24) \quad H_{31} \leq c(\alpha, \lambda, r) \left(\frac{\theta_n}{\theta_n^{\frac{1}{3}} + |x - \theta_n|} \right)^{\frac{1}{4}}.$$

Now we estimate H_{32} . To this end, we write it in the form

$$(25) \quad H_{32} = H_{32}^1 + H_{32}^2,$$

where

$$H_{32}^i = ne^{-\frac{\alpha}{2}} x^{\frac{\alpha}{2} + \frac{1}{4}} |m_{n+1,N}^\alpha(x - r\delta) - m_{n-1,N}^\alpha(x - r\delta)| \delta \times \sum_{t \in D_3^i} \frac{e^{-\frac{t}{2}} t^{\alpha - \frac{r}{2} - \frac{1}{4}}}{t - x} |m_{n,N}^\alpha(t - r\delta)|, \quad i = 1, 2.$$

Using the inequalities (3)–(5) we can write

$$H_{32}^1 \leq c(\alpha, \lambda, r) n x^{\frac{\alpha}{2} + \frac{1}{4}} x^{-\frac{\alpha}{2}} \theta_n^{-\frac{3}{4}} \left[\theta_n^{\frac{1}{3}} + |x - \theta_n| \right]^{\frac{1}{4}} \delta \times \sum_{t \in D_3^1} \frac{t^{\alpha - \frac{r}{2} - \frac{1}{4}} \theta_n^{-\frac{\alpha}{2}} \left[\theta_n (\theta_n^{\frac{1}{3}} + |t - \theta_n|) \right]^{-\frac{1}{4}}}{t - x} \leq c(\alpha, \lambda, r) \delta \sum_{t \in D_3^1} \frac{(\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}}}{(\theta_n^{\frac{1}{3}} + |t - \theta_n|)^{\frac{1}{4}} (t - x)}.$$

Further, we consider two cases: 1) $\theta_n/2 \leq x \leq \theta_n - 2\theta_n^{\frac{1}{3}}$; 2) $\theta_n - 2\theta_n^{\frac{1}{3}} \leq x \leq 3\theta_n/2$. In the second case we have

$$\frac{\theta_n^{\frac{1}{3}} + |x - \theta_n|}{\theta_n^{\frac{1}{3}} + |t - \theta_n|} \leq 3,$$

then

$$(26) \quad H_{32}^1 \leq c(\alpha, \lambda, r) \delta \sum_{t \in D_3^1} \frac{1}{t - x} \leq c(\alpha, \lambda, r) \ln \frac{3\theta_n/2 + \sqrt{x/\theta_n} - x}{\sqrt{x/\theta_n}} \leq c(\alpha, \lambda, r) \ln(n + 1).$$

If $\theta_n/2 \leq x \leq \theta_n - 2\theta_n^{\frac{1}{3}}$, we can write

$$(27) \quad H_{32}^1 \leq c(\alpha, \lambda, r) (I_1 + I_2 + I_3),$$

where

$$I_k = \delta \sum_{t \in D_{3k}^1} \frac{(\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}}}{(\theta_n^{\frac{1}{3}} + |t - \theta_n|)^{\frac{1}{4}} (t - x)}, \quad k = 1, 2, 3,$$

$$D_{31}^1 = [x + \sqrt{x/\theta_n}, \theta_n - \theta_n^{\frac{1}{3}} + \sqrt{x/\theta_n}] \cap D_3^1, \quad D_{32}^1 = [\theta_n - \theta_n^{\frac{1}{3}} + \sqrt{x/\theta_n}, \theta_n + \theta_n^{\frac{1}{3}} + \sqrt{x/\theta_n}] \cap D_3^1,$$

$$D_{33}^1 = [\theta_n + \theta_n^{\frac{1}{3}} + \sqrt{x/\theta_n}, 3\theta_n/2 + \sqrt{x/\theta_n}] \cap D_3^1.$$

Since $2\theta_n^{\frac{1}{3}} \leq \theta_n - x$, we have

$$\begin{aligned}
I_1 &\leq \delta \sum_{t \in D_{31}^1} \frac{\left[\frac{3}{2}(\theta_n - x)\right]^{\frac{1}{4}}}{(\theta_n^{\frac{1}{3}} + |t - \theta_n|)^{\frac{1}{4}}(t - x)} \leq \delta \sum_{t \in D_{31}^1} \frac{\left[\frac{3}{2}(\theta_n - x)\right]^{\frac{1}{4}}}{(\theta_n - t)^{\frac{1}{4}}(t - x)} \leq \\
&\delta \frac{\left[\frac{3}{2}(\theta_n - x)\right]^{\frac{1}{4}}}{(\theta_n - x - \sqrt{x/\theta_n})^{\frac{1}{4}} \sqrt{x/\theta_n}} + \int_{x + \sqrt{x/\theta_n}}^{\theta_n - \theta_n^{\frac{1}{3}} + \sqrt{x/\theta_n}} \frac{\frac{1}{\theta_n - x} dt}{\left(1 - \frac{\theta_n - t}{\theta_n - x}\right) \left(\frac{\theta_n - t}{\theta_n - x}\right)^{1/4}} \leq \\
&c(\alpha) \int_{\frac{\theta_n^{\frac{1}{3}} - \sqrt{x/\theta_n}}{\theta_n - x}}^{\frac{\theta_n - x - \sqrt{x/\theta_n}}{\theta_n - x}} \frac{dy}{(1 - y)y^{1/4}} \leq c(\alpha) \ln(n + 1),
\end{aligned}$$

$$\begin{aligned}
I_2 &\leq \delta \sum_{t \in D_{32}^1} \frac{\left[\frac{3}{2}(\theta_n - x)\right]^{\frac{1}{4}}}{(\theta_n^{\frac{1}{3}} + |t - \theta_n|)^{\frac{1}{4}}(t - x)} \leq \left[\frac{3}{2}(\theta_n - x)\right]^{\frac{1}{4}} \theta_n^{-\frac{1}{12}} \delta \sum_{t \in D_{32}^1} \frac{1}{t - x} \leq \\
&c(\alpha) \left[\frac{3}{2}(\theta_n - x)\right]^{\frac{1}{4}} \theta_n^{-\frac{1}{12}} \ln \frac{\theta_n + \theta_n^{\frac{1}{3}} + \sqrt{x/\theta_n} - x}{\theta_n - \theta_n^{\frac{1}{3}} + \sqrt{x/\theta_n} - x} = \\
&c(\alpha) \left[\frac{3}{2}(\theta_n - x)\right]^{\frac{1}{4}} \theta_n^{-\frac{1}{12}} \ln \left(1 + \frac{2\theta_n^{\frac{1}{3}}}{\theta_n - \theta_n^{\frac{1}{3}} + \sqrt{x/\theta_n} - x}\right) \leq \\
&c(\alpha) \left[\frac{3}{2}(\theta_n - x)\right]^{\frac{1}{4}} \theta_n^{-\frac{1}{12}} \frac{2\theta_n^{\frac{1}{3}}}{\theta_n - \theta_n^{\frac{1}{3}} + \sqrt{x/\theta_n} - x} \leq \\
&c(\alpha)(\theta_n - x)^{\frac{1}{4}} \frac{2\theta_n^{\frac{1}{4}}}{\theta_n - x - \theta_n^{\frac{1}{3}}} \leq c(\alpha)(\theta_n - x)^{\frac{1}{4}} \frac{2\theta_n^{\frac{1}{4}}}{\theta_n - x - \frac{\theta_n - x}{2}} \leq \\
&c(\alpha) \left(\frac{4\theta_n}{(\theta_n - x)^3}\right)^{\frac{1}{4}} \leq c(\alpha) \left(\frac{4\theta_n}{(2\theta_n^{\frac{1}{3}})^3}\right)^{\frac{1}{4}} \leq c(\alpha),
\end{aligned}$$

$$\begin{aligned}
I_3 &\leq \delta \sum_{t \in D_{33}^1} \frac{\left[\frac{3}{2}(\theta_n - x)\right]^{\frac{1}{4}}}{(\theta_n^{\frac{1}{3}} + |t - \theta_n|)^{\frac{1}{4}}(t - x)} \leq c \sum_{t \in D_{33}^1} \frac{\delta(t - x)^{\frac{1}{4}}}{(t - \theta_n)^{\frac{1}{4}}(t - x)} \leq \\
&c \sum_{t \in D_{33}^1} \frac{\delta}{(t - \theta_n)^{\frac{1}{4}}(t - x)^{\frac{3}{4}}} \leq c \sum_{t \in D_{33}^1} \frac{\delta}{t - \theta_n} \leq c(\alpha) \ln(n + 1).
\end{aligned}$$

From inequalities (26), (27) and the estimates for I_k , $k = 1, 2, 3$ it follows that

$$(28) \quad H_{32}^1 \leq c(\alpha, \lambda, r) \ln(n + 1).$$

To estimate H_{32}^2 , we use inequalities (3) and (5):

$$(29) \quad H_{32}^2 \leq c(\alpha, \lambda, r) n x^{\frac{r}{2} + \frac{1}{4}} x^{-\frac{\alpha}{2}} \theta_n^{-\frac{3}{4}} (\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}} \delta \times \\ \sum_{t \in D_{\frac{2}{3}}^2} \frac{t^{\alpha - \frac{r}{2} - \frac{1}{4}} \theta_n^{-\frac{\alpha}{2}} e^{-\frac{3t}{8}}}{t - x} \leq c(\alpha, \lambda, r) \theta_n^{-\alpha + \frac{1}{4}} (\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}} \delta \sum_{t \in D_{\frac{2}{3}}^2} \frac{t^\alpha e^{-\frac{3t}{8}}}{t - x} \leq \\ c(\alpha, \lambda, r) \theta_n^{-\alpha + \frac{1}{4}} (\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}} \int_{\frac{3\theta_n}{2} + \sqrt{\frac{3}{2}}}^{\infty} t^\alpha e^{-\frac{3t}{8}} dt \leq c(\alpha, \lambda, r).$$

From (25), (28) and (29) we obtain

$$(30) \quad H_{32} \leq c(\alpha, \lambda, r) \ln(n + 1).$$

Similarly, we get the estimate for H_{33} :

$$(31) \quad H_{33} = H_{33}^1 + H_{33}^2,$$

where

$$H_{33}^i = n e^{-\frac{\alpha}{2}} x^{\frac{r}{2} + \frac{1}{4}} |m_{n,N}^\alpha(x - r\delta)| \delta \times \\ \sum_{t \in D_{\frac{2}{3}}^i} \frac{e^{-\frac{1}{2}t} t^{\alpha - \frac{r}{2} - \frac{1}{4}}}{t - x} |m_{n+1,N}^\alpha(t - r\delta) - m_{n-1,N}^\alpha(t - r\delta)|, \quad i = 1, 2.$$

From (3)-(5) we have

$$(32) \quad H_{33}^1 \leq c(\alpha, \lambda, r) n x^{\frac{r}{2} + \frac{1}{4}} \frac{\theta_n^{-\frac{\alpha}{2} - \frac{1}{4}}}{(\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}}} \delta \times \\ \sum_{t \in D_{\frac{2}{3}}^1} \frac{t^{\alpha - \frac{r}{2} - \frac{1}{4}}}{t - x} t^{-\frac{\alpha}{2}} \theta_n^{-\frac{3}{4}} (\theta_n^{\frac{1}{3}} + |t - \theta_n|)^{\frac{1}{4}} \leq \\ c(\alpha, \lambda, r) \delta \sum_{t \in D_{\frac{2}{3}}^1} \frac{(\theta_n^{\frac{1}{3}} + |t - \theta_n|)^{\frac{1}{4}}}{(t - x)(\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}}} \leq c(\alpha, \lambda, r) \ln(n + 1),$$

$$(33) \quad H_{33}^2 \leq c(\alpha, \lambda, r) n x^{\frac{r}{2} + \frac{1}{4}} \frac{\theta_n^{-\frac{\alpha}{2} - \frac{1}{4}}}{(\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}}} \delta \sum_{t \in D_{\frac{2}{3}}^2} \frac{t^{\alpha - \frac{r}{2} - \frac{1}{4}} e^{-\frac{3t}{8}}}{t - x} \leq \\ c(\alpha, \lambda, r) \frac{\theta_n^{-\frac{\alpha}{2} + \frac{3}{4}}}{(\theta_n^{\frac{1}{3}} + |x - \theta_n|)^{\frac{1}{4}}} \int_{\frac{3\theta_n}{2} + \sqrt{\frac{3}{2}}}^{\infty} t^\alpha e^{-\frac{3t}{8}} dt \leq c(\alpha, \lambda, r) \left(\frac{\theta_n}{\theta_n^{\frac{1}{3}} + |x - \theta_n|} \right)^{\frac{1}{4}}.$$

From (31)-(33) it follows that

$$(34) \quad H_{33} \leq c(\alpha, \lambda, r) \left[\ln(n + 1) + \left(\frac{\theta_n}{\theta_n^{\frac{1}{3}} + |x - \theta_n|} \right)^{\frac{1}{4}} \right].$$

Comparing (24), (30) and (34) with (20) we have

$$(35) \quad H_3 \leq c(\alpha, \lambda, r) \left[\ln(n+1) + \left(\frac{\theta_n}{\theta_n^{\frac{1}{3}} + |x - \theta_n|} \right)^{\frac{1}{4}} \right].$$

Using the same scheme as for H_3 , we can obtain the following estimate

$$(36) \quad H_1 \leq c(\alpha, \lambda, r) \left[\ln(n+1) + \left(\frac{\theta_n}{\theta_n^{\frac{1}{3}} + |x - \theta_n|} \right)^{\frac{1}{4}} \right].$$

From estimates (19), (35) and (36) we deduce that

$$l_{n,N}^{\alpha,r}(x) \leq c(\alpha, \lambda, r) \left[\ln(n+1) + \left(\frac{\theta_n}{\theta_n^{\frac{1}{3}} + |x - \theta_n|} \right)^{\frac{1}{4}} \right].$$

Thus, (13) is proved.

Now let us prove (14). Assume that $x \in G_4$. From (12) and (17) it follows that

$$(37) \quad l_{n,N}^{\alpha,r}(x) \leq c(\alpha, r) x^{\frac{r}{2} + \frac{1}{4}} (e^{-x} \mathcal{K}_{n,N}^{\alpha}(x - r\delta, x - r\delta))^{1/2} \delta \times \\ \sum_{t \in \Omega_{r,\delta}} t^{\alpha - \frac{r}{2} - \frac{1}{4}} (e^{-t} \mathcal{K}_{n,N}^{\alpha}(t - r\delta, t - r\delta))^{1/2} = \\ c(\alpha, r) x^{\frac{r}{2} + \frac{1}{4}} (e^{-x} \mathcal{K}_{n,N}^{\alpha}(x - r\delta, x - r\delta))^{1/2} (J_1 + J_2 + J_3),$$

where

$$J_k = \delta \sum_{t \in E_k} t^{\alpha - \frac{r}{2} - \frac{1}{4}} (e^{-t} \mathcal{K}_{n,N}^{\alpha}(t - r\delta, t - r\delta))^{1/2}, \quad k = 1, 2, 3,$$

$$E_1 = \left[r\delta, \frac{1}{\theta_n} \right] \cap \Omega_{r,\delta}, \quad E_2 = \left[\frac{1}{\theta_n}, \frac{3\theta_n}{2} \right] \cap \Omega_{r,\delta}, \quad E_3 = \left[\frac{3\theta_n}{2}, \infty \right) \cap \Omega_{r,\delta}.$$

First, estimate the value $(e^{-x} \mathcal{K}_{n,N}^{\alpha}(x - r\delta, x - r\delta))^{\frac{1}{2}}$. From (1), (3), and (4) we have

$$(38) \quad (e^{-x} \mathcal{K}_{n,N}^{\alpha}(x - r\delta, x - r\delta))^{\frac{1}{2}} = \left(\sum_{k=0}^n [m_{k,N}^{\alpha}(x - r\delta) e^{-\frac{x}{2}}]^2 \right)^{\frac{1}{2}} \leq \\ c(\alpha, \lambda, r) \left(\sum_{k=0}^n k^{-\alpha} e^{-\frac{3}{4}x} \right)^{\frac{1}{2}} \leq c(\alpha, \lambda, r) n^{\frac{1}{2}} e^{-\frac{x}{2}} e^{-\frac{x}{4}} \leq c(\alpha, \lambda, r) n^{\frac{1-\alpha}{2}} e^{-\frac{x}{4}}.$$

Let us estimate J_1 (we assume that $J_1 = 0$, if $r\delta > \frac{1}{\theta_n}$):

$$(39) \quad J_1 \leq c(\alpha, \lambda, r) \delta \sum_{t \in E_1} t^{\alpha - \frac{r}{2} - \frac{1}{4}} \left(\sum_{k=0}^n \theta_k^{\alpha} \right)^{\frac{1}{2}} \leq \\ c(\alpha, \lambda, r) \theta_n^{\frac{\alpha+1}{2}} \delta \sum_{t \in E_1} t^{\alpha - \frac{r}{2} - \frac{1}{4}} \leq c(\alpha, \lambda, r) \theta_n^{\frac{\alpha+1}{2}} \left(\frac{1}{\theta_n} \right)^{\alpha - \frac{r}{2} + \frac{3}{4}} \leq c(\alpha, \lambda, r) n^{\frac{r-\alpha}{2} - \frac{1}{4}}.$$

Further, estimate (18) yields

$$(40) \quad J_2 \leq c(\alpha, \lambda, r) \delta \sum_{t \in E_2} t^{\alpha - \frac{r}{2} - \frac{1}{4}} t^{-\frac{\alpha}{2} - \frac{1}{4}} n^{\frac{1}{4}} \leq c(\alpha, \lambda, r) n^{\frac{1}{4}} \left(\delta \left(\frac{1}{\theta_n} \right)^{\frac{\alpha}{2} - \frac{r}{2} - \frac{1}{2}} + \int_{\frac{1}{\theta_n}}^{\frac{3\theta_n}{2}} t^{\frac{\alpha}{2} - \frac{r}{2} - \frac{1}{2}} dt \right) \leq c(\alpha, \lambda, r) n^{\frac{\alpha - r}{2} + \frac{3}{4}}.$$

Finally, from (38) we have

$$(41) \quad J_3 = \delta \sum_{t \in E_3} t^{\alpha - \frac{r}{2} - \frac{1}{4}} (e^{-t} \mathcal{K}_{n,N}^{\alpha}(t - r\delta, t - r\delta))^{1/2} \leq c(\alpha, \lambda, r) n^{\frac{1-\alpha}{2}} \delta \sum_{t \in E_3} t^{\alpha - \frac{r}{2} - \frac{1}{4}} e^{-\frac{t}{4}} \leq c(\alpha, \lambda, r).$$

From inequalities (37)–(41) we deduce the following estimate

$$l_{n,N}^{\alpha,r}(x) \leq c(\alpha, \lambda, r) n^{-\frac{r}{2} + \frac{5}{4}} x^{\frac{r}{2} + \frac{1}{4}} e^{-\frac{x}{4}}.$$

This completes the proof of the theorem.

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