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ROTATIONALLY SYMMETRIC SOLUTIONS TO ISOTHERMAL COMPRESSIBLE NAVIER-STOKES EQUATIONS

P.I. PLOTNIKOV

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Abstract: We consider a boundary value problem for compressible Navier-Stokes equations with the pressure function $p = \varrho$, where ϱ is the density of fluid. It is assumed the given data and a flow domain are invariant with respect to rotations around the vertical axis. The existence of weak rotationally symmetric solutions is proved.

Keywords: viscous gas, concentrations problem, isothermal fluid.

1 Problem formulation. Results

The paper is devoted to the study of boundary value problems for 3D compressible Navier-Stokes equations. Suppose a viscous compressible fluid occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with C^∞ boundary $\partial\Omega$. The state of the fluid is characterized by the macroscopic quantities: the density $\varrho(x,t)$ and the velocity $\mathbf{u}(x,t)$. For isentropic fluid, the pressure $p(\varrho)$ and internal energy $e(\varrho)$ are given by the formulae

$$p = \varrho^\gamma, \quad e = \frac{1}{\gamma - 1}\varrho^\gamma, \tag{1.1}$$

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where $\gamma > 1$ is the adiabatic constant. In the end-point isothermal case the pressure and internal energy are defined by the equalities

$$p = \varrho, \quad e = \varrho \ln \varrho - \varrho. \tag{1.2}$$

The problem is to find $\mathbf{u}(x,t)$ and $\varrho(x,t)$ satisfying the following equations and boundary conditions in the cylinder $Q_T = \Omega \times (0,T)$.

$$\begin{gathered}
\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\varrho) = \operatorname{div}\mathbb{S}(\mathbf{u}) + \varrho\mathbf{f} \quad \text{in} \quad Q_T,\\
\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0, \quad \varrho \geq 0 \ \text{ in } \ Q_T,\\
\mathbf{u} = 0 \quad \text{on } \partial\Omega\times(0,T),\\
\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \varrho(x,0) = \varrho_0(x) > 0 \quad \text{in } \Omega.
\end{gathered} \tag{1.3}$$

Here, the vector field $\mathbf{f} \in L^\infty(Q_T)$ denotes the density of external mass forces, the viscous stress tensor $\mathbb{S}(\mathbf{u})$ is defined by the equality

$$\mathbb{S}(\mathbf{u}) = \big(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\big) - \frac{2}{3}\operatorname{div}\mathbf{u}\,\mathbf{I}. \tag{1.4}$$

This is a typical form of the viscous stress tensor with shear viscosity equals 1 and bulk viscosity equals 0. We assume that given initial data satisfy the conditions

$$\mathbf{u}_0 \in L^2(\Omega), \quad \varrho_0 \in L^\infty(\Omega), \quad \rho_0 > 0.$$

Regular solutions to problem (1.3) satisfy the energy diffusion equality

$$\begin{aligned}
\int_\Omega \Big(\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2 + e(t)\Big)\,dx + \int_0^t\int_\Omega \mathbb{S}(\mathbf{u}(s)) : \nabla\mathbf{u}(s)\,dxds\\
= \int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + e(\varrho_0)\Big)\,dx \ \text{ for } \ t\in(0,T).
\end{aligned} \tag{1.5}$$

In particular, they admit the energy estimates:

$$\begin{gathered}
\|\varrho|\mathbf{u}|^2 + p(\varrho)\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq E \text{ for } \gamma > 1,\\
\|\varrho|\mathbf{u}|^2 + |\varrho\ln\varrho|\,\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq E \text{ for } \gamma = 1,
\end{gathered} \tag{1.6}$$

where the constant E depends only on Ω, T, $\|\varrho_0\|_{L^\infty(\Omega)}$, $\|\mathbf{u}_0\|_{L^2(\Omega)}$, and $\|f\|_{L^\infty(Q_T)}$. For the macroscopic variables satisfying estimates (1.6), we can define weak solutions to problem (1.3).

Definition 1.1. *A weak solution to problem* (1.3) *is a couple* $(\varrho,\mathbf{u}) : Q_T \to \mathbb{R}^+\times\mathbb{R}^3$ *satisfying the following conditions*

- $$\varrho, p(\varrho), \varrho|\mathbf{u}|^2 \in L^\infty(0,T;L^1(\Omega)), \quad \mathbf{u}\in L^2(0,T;W_0^{1,2}(\Omega)). \tag{1.7}$$

- *The integral identity*

$$\int_{Q_T} \big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\xi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\xi} + p(\varrho)\,\mathrm{div}\ \boldsymbol{\xi} - \mathbb{S}(\mathbf{u}) : \nabla\boldsymbol{\xi}\big)\,dxdt$$
$$+ \int_{Q_T} \varrho\mathbf{f}\cdot\boldsymbol{\xi}\,dxdt + \int_\Omega \varrho_0\mathbf{u}_0(x)\cdot\boldsymbol{\xi}(x,0)\,dx = 0 \quad (1.8)$$

  holds for every vector fields $\boldsymbol{\xi}\in C^\infty(Q_T)$ vanishing in a neighborhood of the lateral boundary $\partial\Omega\times[0,T]$ and the lid $\Omega\times\{t=T\}$.
- *The integral identity*

$$\int_{Q_T} \big(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\big)\,dxdt + \int_\Omega \varrho_0(x)\psi(x,0)\,dx = 0 \quad (1.9)$$

  holds for every $\psi\in C^\infty(Q_T)$ vanishing in a neighborhood of $\Omega\times\{t=T\}$.

The first nonlocal results concerning the mathematical theory of compressible Navier-Stokes equations are due to P.-L. Lions. In monograph [12] he established the existence of a weak solution to nonstationary boundary value problem for the Navier-Stokes equations with the pressure function $p\sim\varrho^\gamma$ for all $\gamma>5/3$. More recently, Feireisl, Novotný, and Petzeltová, see [6], proved the existence result for all $\gamma>3/2$, see also monographs [7], [14] for references and details. For $\gamma\le 3/2$, the question on solvability of problem (1.3) is still open. The main difficulty is the so-called concentration problem, which was widely discussed in the mathematical literature in connection with vortex sheets dynamics.

In this work, we focus on the end-point case of the isothermal flows. Hereinafter we will assume that $\gamma=1$.

In this case the concentration problem can be formulated as follows. Let us consider a sequence ϱ_ε, $\mathbf{u}_\varepsilon$ of approximate solutions to problem (1.3) satisfying energy estimate (1.6). Here ε is the approximation parameter. Such solutions can be obtained by the regularization method.

We may assume, after passing to a subsequence, that

$$\mathbf{u}_\varepsilon\to\mathbf{u} \ \text{ weakly in } \ L^2(0,T;W^{1,2}_0(\Omega)),$$
$$\varrho_\varepsilon\to\varrho \ \text{ star weakly in } \ L^\infty(0,T;L^1(\Omega)),$$
$$\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\to\mathcal{M} \ \text{ star weakly in the space of Radon measures as } \ \varepsilon\to 0,$$

where $\mathcal{M}=(\mathcal{M}_{ij})_{3\times 3}$ denotes a matrix-valued Radon measure in Ω. In the general case the weak star defect measure

$$\boldsymbol{\mu} = \mathcal{M} - \varrho\mathbf{u}\otimes\mathbf{u}\neq 0.$$

In particular, the momentum balance equation for the limits ϱ and $\mathbf{u}$ reads

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}\ (\varrho\mathbf{u}\otimes\mathbf{u}) - \mathrm{div}\ \mathbf{S}(\mathbf{u}) + \nabla\varrho + \mathrm{div}\ \boldsymbol{\mu} = \varrho f \ \text{ in } \ Q_T.$$

The concentrations are canceled if div $\boldsymbol{\mu} = 0$. Hence the problem is to describe the structure of the defect measure and to find conditions under which it is canceled.

For the compressible Navier-Stokes equations, the concentration problem is

poorly investigated. Recently, the Hausdorff and parabolic dimensions of the support of the defect measure were estimated in [21] and [22]. The absence of concentrations in the critical case $\gamma = 3/2$ have been proved in [17]. The structure of the concentrations set may be very complicated. Therefore, we restrict our considerations by the case of rotationally symmetric flows.

Denote by $O(\alpha)$ the orthogonal matrix defined by the equalities

$$O(\alpha) = \begin{pmatrix} O_{12}(\alpha) & 0 \\ 0 & 1 \end{pmatrix}, \quad O_{12}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{1.10}$$

Definition 1.2. *We say that a solution to equations (1.3) is rotationally symmetric if the equalities*

$$\Omega = O(\alpha)\Omega, \quad \mathbf{u}\big(O(\alpha)x, t\big) = O(\alpha)\mathbf{u}(x,t), \quad \varrho(x,t) = \varrho\big(O(\alpha)x, t\big) \tag{1.11}$$

holds for every $\alpha \in \mathbb{R}$.

We are now in a position to formulate the main results of this paper. Introduce the regularized internal energy

$$\begin{aligned} e &= \varrho \ln \varrho - \varrho + e_1(\varrho) + e_2(\varrho), \\ e_1 &= \varepsilon \varrho^{1+\varepsilon} \ln(1+\varepsilon\varrho), \quad e_2 = \frac{1}{2}\delta^3\varrho^3, \end{aligned} \tag{1.12}$$

depending on the small parameters $\varepsilon, \delta \in (0, 1/2)$. The corresponding pressure function $p = e'\varrho - e$ is defined by the equalities

$$\begin{aligned} p &= \varrho + p_1(\varrho) + p_2(\varrho), \\ p_1 &= \varepsilon \varrho^{1+\varepsilon}\big(\varepsilon \ln(1+\varepsilon\varrho) + \frac{\varepsilon\varrho}{1+\varepsilon\varrho}\big), \quad p_2 = \delta^3\varrho^3. \end{aligned} \tag{1.13}$$

For positive ε, δ, the function p meet all requirements of existence theorem proved in [6], which leads to the following

Proposition 1.3. *Let $\varrho_0 > 0$ and*

$$\|\varrho_0\|_{L^\infty(\Omega)} + \|\mathbf{u}_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(Q_T)} \le c_E. \tag{1.14}$$

Then the boundary value problem

$$\begin{gathered} \partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \operatorname{div}\mathbb{S}(\mathbf{u}) + \varrho\mathbf{f} \quad \text{in } Q_T, \\ \partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0, \quad \varrho \ge 0 \quad \text{in } Q_T, \\ \mathbf{u} = 0 \quad \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \varrho(x,0) = \varrho_0(x) > 0 \quad \text{in } \Omega \end{gathered} \tag{1.15}$$

has a weak solution, which admits the estimate

$$\|\varrho|\mathbf{u}|^2\|_{L^\infty(0,T;L^1(\Omega))} + \|e(\varrho)\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega))} \le E. \tag{1.16}$$

Here the constant E depends only on Ω, T, and c_E.

Note that
$$\varrho\ln(2+\varrho)\leq 2(\varrho\ln\varrho-\rho)+c,$$
where c is some absolute constant. It follows from this and the expressions for e and p that
$$\begin{aligned}\|\varrho\ln(2+\varrho)\|_{L^\infty(0,T;L^1(\Omega))} &+ \|e_1(\varrho)\|_{L^\infty(0,T;L^1(\Omega))} \\ +\|e_2(\varrho)\|_{L^\infty(0,T;L^1(\Omega))} &\leq cE+c.\end{aligned}\tag{1.17}$$
and
$$\|p\|_{L^\infty(0,T;L^1(\Omega))}+\|p_1(\varrho)\|_{L^\infty(0,T;L^1(\Omega))}+\|p_2(\varrho)\|_{L^\infty(0,T;L^1(\Omega))}\leq cE+c.\tag{1.18}$$

The following remark is important.

Remark 1.4. *Recall definition (1.10) of the matrix $O(\alpha)$. If the domain Ω, initial data $(\varrho_0,\mathbf{u}_0)$ and mass force $\mathbf{f}$ satisfy the symmetry conditions*
$$\begin{gathered}\Omega=O(\alpha)\Omega,\quad \mathbf{u}_0\big(O(\alpha)x,t\big)=O(\alpha)\mathbf{u}_0(x,t),\\ \varrho_0(x,t)=\varrho_0\big(O(\alpha)x,t\big),\quad \mathbf{f}\big(O(\alpha)x,t\big)=O(\alpha)\mathbf{f}(x,t),\end{gathered}\tag{1.19}$$
for all $\alpha\in\mathbb{R}$, then problem (1.15) has a rotationally -symmetric solution which meets all requirements of Definition 1.2 and Proposition 1.3. We do not claim that all weak solutions to problem (1.15) given by Proposition 1.3 are symmetric. We mean that at least one of such solutions has symmetry properties.

The particular cases of rotationally symmetric solutions are the axially symmetric solutions with zero azimuth velocity component, and radially symmetric solutions with ϱ and $\mathbf{u}$ depending only on $|x|$. In the radially symmetric case, the existence of global weak solutions for all adiabatic exponents $\gamma>1$ was proved in [10], see also [5]. The similar results for axially symmetric and helically symmetric solutions were proved in [11] and [19]. However, the obtained solutions are defined only in the flow domain with punctured axis of symmetry. In paper [20] the existence of rotationally symmetric solutions was proved for all $\gamma>(7+\sqrt{73})/12$. The end-point case of isothermal flows with $\gamma=1$ is still poorly studied. Our goal is to fill this gap.

The following theorem is the main result of this paper. Assume that the flow domain Ω with the boundary of the class C^∞ satisfies the symmetry condition
$$O(\alpha)\Omega=\Omega\ \text{ for all }\quad\alpha\in\mathbb{R},$$
and the intersection of Ω with the symmetry axis Ox_3 is an interval
$$\mathcal{I}=\{x:\ x_1=x_2=0,\ x_3\in(-a,a)\}.$$
Furthermore assume that the initial data $(\varrho_0,\mathbf{u}_0)$ and mass force f satisfy condition (1.14) and symmetry condition (1.19). Denote by $(\varrho_{\varepsilon,\delta},\mathbf{u}_{\varepsilon,\delta})$ the rotationally-symmetric solutions to problem (1.15) satisfying conditions of Proposition 1.3.

Theorem 1.5. *Under the above assumptions, there is a subsequence of the sequence*

$$\varrho_\varepsilon = \varrho_{\varepsilon,\kappa(\varepsilon)}, \quad \mathbf{u}_\varepsilon = \mathbf{u}_{\varepsilon,\kappa(\varepsilon)}, \quad \kappa(\varepsilon) = e^{-\frac{1}{\varepsilon^9}},$$

still denoted by ϱ_ε, $\mathbf{u}_\varepsilon$, *with the following properties.*

$$\begin{aligned} \varrho_\varepsilon \to \varrho, \ \ \varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \quad & \textit{weakly in} \quad L^1(Q_T), \\ \mathbf{u}_\varepsilon \to \mathbf{u} \quad & \textit{weakly in} \quad L^2(0,T; W_0^{1,2}(\Omega)). \end{aligned} \tag{1.20}$$

The functions ϱ, $\mathbf{u}$ *satisfy symmetry condition* (1.11) *and admit energy estimate* (1.16), (1.17), (1.18).

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \otimes \mathbf{u} + \boldsymbol{\mu}, \quad p(\varrho_\varepsilon) \to \varrho \tag{1.21}$$

star weakly in the space of parametrized measures $L^r_w(0,T; C_0(\Omega)')$ *for every* $r \in (1,\infty)$. *(See Section* 6 *and Lemma* 6.2 *for details.)*

The parametrized Radon measure

$$\boldsymbol{\mu}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu_{33}(t) \end{pmatrix}$$

is supported on the segment $\mathcal{I}$ *of the symmetry axis for a.e.* $t \in (0,T)$.

The functions ϱ, $\mathbf{u}$ *serve as a rotationally -symmetric weak solution to the boundary value problem*

$$\begin{aligned} \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \varrho + \operatorname{div} \boldsymbol{\mu} = \ & \operatorname{div} \mathbb{S}(\mathbf{u}) + \varrho \mathbf{f} \quad \textit{in} \ \ Q_T, \\ \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \quad & \varrho \geq 0 \ \ \textit{in} \ \ Q_T, \\ \mathbf{u} = 0 \quad & \textit{on} \ \ \partial\Omega \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad & \varrho(x,0) = \varrho_0(x) > 0 \quad \textit{in} \ \Omega. \end{aligned} \tag{1.22}$$

The rest of the paper is devoted to the proof of this theorem. In order to obtain a priori estimates of rotationally symmetric weak solutions to the regularized problem, we rewrite equations (1.15) in the cylinder coordinate. The next section is devoted to this procedure.

2 Cylindrical coordinates. Integral identities

Denote by r, ϑ, x_3 the cylindrical coordinates in the space $\mathbb{R}^3$ of points $x = (x_1, x_2, x_3)$,

$$x_1 = r\cos\vartheta, \quad x_2 = r\sin\vartheta, \quad x_3 = x_3. \tag{2.1}$$

It is convenient to use the notation

$$y = (y_1, y_3) = (r, x_3), \quad z = (z_1, z_2, z_3) = (r, \theta, x_3) = (y_1, \theta, y_3). \tag{2.2}$$

Let $G \subset \mathbb{R}^2$ be an intersection of the rotationally symmetric flow domain Ω with the half-plane

$$\Pi^+ = \{x_1 > 0, x_2 = 0\},$$

i.e.,

$$G = \{x \in \Omega : x_1 > 0, x_2 = 0\}.$$

It can be regarded as the domain in the plane of variable y. Assume that Ω and G are simply-connected. We also assume that the intersection Ω and the symmetry axis Ox_3 is the interval

$$\mathcal{I} = \big\{\, x :\ x_3 \in (-a, a), \quad x_1 = x_2 = 0 \,\big\}.$$

We will consider $\mathcal{I}$ as the interval $-a < y_3 < a$, $y_1 = 0$ in the plane of variable y. It follows that the boundary G consists of the interval $\mathcal{I}$, the arc $\partial\Omega \cap \Pi^+$, and the end-points of $\mathcal{I}$. In the cylindrical coordinates, we have

$$\Omega = (G \times \mathbb{S}^1) \cup \mathcal{I}.$$

Denote by v_1, v_2, and v_3 the radial, azimuth, and axial components of the velocity $\mathbf{u}$, i.e.,

$$\mathbf{u} = \mathbf{U}\,\mathbf{v}, \quad \mathbf{v} = (v_1, v_2, v_3), \quad \mathbf{U} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

The flow is rotationally symmetric if and only if ϱ и $\mathbf{v}$ are independent of θ,

$$\varrho = \varrho(y, t), \quad \mathbf{v} = \mathbf{v}(y, t).$$

The main energy dissipation estimate (1.16) has analogy in the cylindrical coordinates. This results from the following technical lemma.

Lemma 2.1. *Let $\mathbf{u} : \Omega \to \mathbb{R}^3$ be a rotationally symmetric vector field such that $\mathbf{u} = \mathbf{U}(\theta)\,\mathbf{v}(y)$, $y = (r, x_3) = (y_1, y_3)$. Then*

$$\int_\Omega |\nabla_x \mathbf{u}|^2\, dx = 2\pi \int_G |\nabla_y \mathbf{v}|^2\, y_1 dy + 2\pi \int_G (v_1^2 + v_2^2)\frac{1}{y_1} dy. \tag{2.4}$$

Proof. The proof given in Section 7, Lemma 7.3. □

It follows from this lemma and (1.16) that the rotationally symmetric solution $(\rho, \mathbf{u})$ defined by Proposition 1.8 in the cylindrical coordinates admits the estimate

$$\begin{gathered} \int_0^T \int_G \Big(|\nabla \mathbf{v}|^2 + \frac{1}{y_1^2}(v_1^2 + v_2^2) \Big)\, y_1 dy dt \le c, \\ \operatorname*{ess\ sup}_{t \in (0,T)} \int_G \Big(\varrho |\mathbf{v}|^2 + e(\varrho) \Big)\, y_1 dy \le c, \end{gathered} \tag{2.5}$$

Here the constant E depends only on Ω, T, and c_E. Further we will use the special integral identity for functions v and ρ.

Proposition 2.2. *Let $\varrho(x, t)$, $\mathbf{u}(x, t)$ be a rotationally symmetric solution to problem (1.15) given by Proposition 1.3. Furthermore, assume that the vector fields $\mathbf{v}(y, t)$ and $\overline{\mathbf{v}}(y, t)$ are defined by the equalities*

$$\mathbf{u} = \mathbf{U}(\theta)\mathbf{v}(y, t), \quad \overline{\mathbf{v}}(y, t) = (v_1(y, t),\, v_3(y, t)).$$

Then the integral identities

$$\int_{G\times(0,T)}\Big(\varrho\overline{\mathbf{v}}\cdot\partial_t\overline{\boldsymbol{\varsigma}}+\varrho\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}:\nabla\overline{\boldsymbol{\varsigma}}+p(\varrho)\ \mathrm{div}\ \overline{\boldsymbol{\varsigma}}-\mathbf{B}(\overline{\mathbf{v}}):\nabla\overline{\boldsymbol{\varsigma}}$$
$$+\mathbf{b}(\overline{\mathbf{v}},\varrho)\varsigma_1+\varrho\overline{\mathbf{f}}\cdot\overline{\boldsymbol{\varsigma}}\Big)\,y_1dy\,dt=0, \quad (2.6)$$

$$\int_{G\times(0,T)}\Big(\ \varrho\partial_t\psi+\varrho\overline{\mathbf{v}}\cdot\nabla\psi\ \Big)\,y_1\,dy\,dt=0. \quad (2.7)$$

hold for every continuously differentiable, compactly supported in $G\times(0,T)$ *vector field*

$$\overline{\boldsymbol{\varsigma}}=(\varsigma_1,\varsigma_3):G\times(0,T)\to\mathbb{R}^2.$$

and continuously differentiable, compactly supported in $G\times(0,T)$ *function* $\psi:G\times(0,T)\to\mathbb{R}$.

Here $\nabla\mathbf{a}=\nabla_y\mathbf{a}$ *for each vector field* $\mathbf{a}(y,t)$*. The matrix* $\mathbf{B}$ *and vector field* $\mathbf{b}$ *are defined by the equalities*

$$\begin{aligned}\mathbf{B}&=(\nabla\overline{\mathbf{v}}+\nabla\overline{\mathbf{v}}^\top)-\frac{2}{3}(\mathrm{div}\ \overline{\mathbf{v}}+\frac{1}{r}v_1)\,\mathbf{I},\\ \mathbf{b}&=\frac{1}{y_1}\varrho v_2^2+\frac{1}{y_1}p-\frac{4}{3y_1^2}v_1+\frac{2}{3y_1}\mathrm{div}\ \overline{\mathbf{v}}.\end{aligned} \quad (2.8)$$

For every compact set $K\Subset G$*, the matrix* $\mathbf{B}$ *and vector field* $\mathbf{b}$ *admit the estimate*

$$\|\mathbf{B}\|_{L^2(K\times(0,T))}+\|\mathbf{b}\|_{L^1(K\times(0,T))}\le c, \quad (2.9)$$

where the constant c *depends only on* K *and the constant in estimate* (2.5)*.*

Proof. The proof given in Section 7, Proposition 7.6. □

3 Localization. Potential estimate

The main difficulty of our problem is the proof that the kinetic energy tensors $\varrho\mathbf{u}\times\mathbf{u}$ of rotationally symmetric solutions to problem 1.15 given by Proposition 1.3 are equiintegrable on ever compact set $K\subset\Omega\setminus\mathcal{I}$. In order to prove this result we developed the approach based on estimates of the density potentials. The main ingredient of the method is the L^2 estimate of the function $(-\Delta)^{1/4}\varrho$. In the cylindrical coordinates, the vector field v demonstrates a complicated behavior near the boundary of G. Since we are interesting in interior estimate, it is convenient to localize the problem and work with compactly supported functions defined in the whole space.

3.1. Localization. Fix a compact set $K\Subset G\times(0,T)$, and choose two functions $\zeta,\eta\in C_0^\infty(G_T)$ such that

$$\zeta=\eta=1 \text{ on } K,\quad \eta=1 \text{ on supp } \zeta,\quad 0\le\zeta,\eta\le 1, \quad (3.1)$$

Assume that they are extended by zero to the whole space $\mathbb{R}^2\times\mathbb{R}$. Set

$$\rho(y,t)=\zeta(y,t)\varrho(y,t),\quad \mathbf{w}(y,t)=\eta(y,t)\mathbf{v}(y,t),\quad \overline{\mathbf{w}}=(w_1,w_3)=\eta\overline{\mathbf{v}}. \quad (3.2)$$

Extend these functions by zero to $\mathbb{R}^3$. The following Lemma characterizes the properties of the extended functions.

Proposition 3.1. *The functions ρ and $\mathbf{w}$ are compactly supported in $G\times(0,T)$. There is $R>0$, depending only on K, such that*

$$0<R^{-1}\le y_1\le R \quad \textit{on the supports of} \quad \rho \quad \textit{and} \quad \zeta.$$

There is a constant c, depending only on K and the constants in estimates (1.16)-(1.18), such that

$$\begin{aligned}\|\zeta\varrho\ln(2+\varrho)\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}^2))}+\|\zeta\, e_1\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}^2))}+\|\zeta\, e_2\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}^2))}\le c,\\ \|\rho|\mathbf{w}|^2\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}^2))}+\|\mathbf{w}\|_{L^2(\mathbb{R};W^{1,2}(\mathbb{R}^2))}\le c.\end{aligned} \tag{3.3}$$

Moreover, the integral identities

$$\begin{aligned}\int_{\mathbb{R}^3}\Big(\rho\overline{\mathbf{w}}\cdot\partial_t\overline{\boldsymbol{\varsigma}}+\rho\overline{\mathbf{w}}\otimes\overline{\mathbf{w}}:\nabla\overline{\boldsymbol{\varsigma}}+\zeta p(\varrho)\ \mathrm{div}\ \overline{\boldsymbol{\varsigma}}\\ -\zeta\mathbf{B}:\nabla\overline{\boldsymbol{\varsigma}}+\mathbf{F}\cdot\overline{\boldsymbol{\varsigma}}\Big)\,y_1dydt=0,\end{aligned} \tag{3.4}$$

$$\int_{\mathbb{R}^3}\big(\rho\partial_t\psi+\rho\overline{\mathbf{w}}\cdot\nabla\psi+\varrho\,\mathbf{h}\,\psi\big)\,y_1dydt=0, \tag{3.5}$$

hold true for all $\overline{\boldsymbol{\varsigma}},\psi\in C^1_{\mathrm{loc}}(\mathbb{R}^3\setminus\{y_1=0\})$. The vector field $\mathbf{F}$ and function $\mathbf{h}$ are compactly supported in $G\times(0,T)$ and admit the estimates

$$\|\mathbf{F}\|_{L^1(\mathbb{R}^3)}+\|\mathbf{h}\|_{L^2(\mathbb{R}^3)}+\|\varrho\mathbf{h}\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}^2))}\le c, \tag{3.6}$$

where c depends on K and does not depend on the regularization parameters ε, δ.

Proof. Compactness of the support of $(\rho,\mathbf{w})$ obviously follows from (3.2).

For the proof of integral identities (3.4) and (3.5) choose arbitrary functions $\overline{\boldsymbol{\varsigma}},\psi\in C^1_{\mathrm{loc}}(\mathbb{R}^3)$. Next, replace $\overline{\boldsymbol{\varsigma}}$ and ψ in integral identities (2.6) and (2.7) by $\zeta\overline{\boldsymbol{\varsigma}}$ and $\zeta\psi$. By doing so we arrive at integral identities (3.4), (3.5) with

$$\begin{aligned}\mathbf{F}\ &=\ \partial_t\zeta\varrho\overline{\mathbf{w}}+\varrho\overline{\mathbf{w}}\otimes\overline{\mathbf{w}}\nabla\zeta-\mathbf{B}\nabla\zeta+p\nabla\zeta+\\ &\quad\ \zeta\mathbf{b}\,\mathbf{e}_1+\rho\overline{\mathbf{f}},\\ \mathbf{h}\ &=\ \overline{\mathbf{w}}\cdot\nabla\zeta.\end{aligned}$$

It remains to note that estimate (3.6) follows from estimates (3.3) and (2.8). $\square$

3.2. Potential estimate. In this section, we prove the estimate of the Riesz potential for the function ρ. Recall that for every integrable, compactly supported in $\mathbb{R}^2$ function ρ the fractional Laplacian $(-\Delta)^{-1/2}\rho$ is defined by the equality, [1],

$$(-\Delta)^{-1/2}\rho(y)=c_{1/2}\int_{\mathbb{R}^2}\frac{\rho(z)}{|y-z|}\,dz. \tag{3.7}$$

We have $(-\Delta)^{-1/2} = (-\Delta)^{-1/4}(-\Delta)^{-1/4}$, where

$$(-\Delta)^{-1/4}\rho(y) = c_{1/4}\int_{\mathbb{R}^2}\frac{\rho(z)}{|y-z|^{3/2}}\,dz. \tag{3.8}$$

Here $c_{1/2}$, $c_{1/4}$ are some unessential absolute constants. The following proposition is the main result of this section.

Proposition 3.2. *Under the assumptions of Proposition 3.1, the truncated density ρ admits the estimate*

$$\|(-\Delta)^{-1/4}\rho\|_{L^2(\mathbb{R}^3)} \le c. \tag{3.9}$$

Here the constant c depends on the compact set $K \subset G \times (0,T)$ and does not depend on the regularization parameters ε, δ in the expressions for the internal energy.

The proof falls into a sequence of lemmas. Introduce the kernels

$$\begin{aligned} K_0(y) &= \frac{y}{|y|},\ K_{0i} = \frac{y_i}{|y|} \\ K_{ij}(y) &= \partial_j K_{0i}(y) = \frac{|y|^2\delta_{ij} - y_i y_j}{|y|^3} \quad \text{for} \quad y \in \mathbb{R}^2 \setminus \{0\}, \quad i,j = 1,3. \end{aligned} \tag{3.10}$$

We adapt the convention that these kernels equal zero at $y = 0$. Set

$$\overline{\varsigma}_i(y,t) = K_{0i} * \rho(y,t) \equiv \int_{\mathbb{R}^2} K_{0i}(y-z)\rho(t,z)\,dz, \quad i = 1,3. \tag{3.11}$$

Lemma 3.3. *Under the above assumptions, the vector field $\overline{\boldsymbol{\varsigma}}$ belongs to the class $L^\infty(\mathbb{R}; C^1(\mathbb{R}^2))$. Its time derivative $\partial_t\overline{\boldsymbol{\varsigma}}$ belongs to the class $L^2(\mathbb{R}; C(\mathbb{R}^2))$ and is defined by the equality*

$$\partial_t\overline{\varsigma}_i = -\sum_{j=1,3} K_{ij} * (\rho w_j) + \sigma_i. \tag{3.12}$$

The vector field $\boldsymbol{\sigma} = (\sigma_1, \sigma_3)$ admits the estimate

$$\|\sigma_i\|_{L^\infty(\mathbb{R}^3)} \le c(K). \tag{3.13}$$

Proof. We begin with the observation that estimate (3.3) and formula (1.12) imply

$$\delta^3\|\rho^3\|_{L^\infty(\mathbb{R}; L^1(\mathbb{R}^2))} \le c \tag{3.14}$$

From this and the Hölder inequality we obtain the estimate

$$\begin{aligned} \|\varrho\mathbf{w}\|^2_{L^2(\mathbb{R}; L^s(\mathbb{R}^2))} = \int_{\mathbb{R}} \|\rho\mathbf{w}\|^2_{L^s(\mathbb{R}^2)}dt &\le \int_{\mathbb{R}} \|\rho\|^2_{L^3(\mathbb{R}^2)}\,\|\mathbf{w}\|^2_{L^{3s/(3-s)}(\mathbb{R}^2)}\,dt \\ &\le c(s)\delta^{-2}\int_{\mathbb{R}} \|\nabla\mathbf{w}\|^2_{L^2(\mathbb{R}^2)}dt \le c(\delta, s), \end{aligned} \tag{3.15}$$

which holds true for all $1 \le s < 3$. From this and (3.5) we conclude that the integral identity

$$\int_{\mathbb{R}^3} \big(\rho(z,t)\partial_t\psi(z,t)\big)\, z_1 dydt + \sum_{j=1,3}\int_{\mathbb{R}^3} \big(\rho(z,t)w_j(z,t)\partial_j\psi(z,t)\big) z_1 dydt \\ + \int_{\mathbb{R}^3} \varrho(z,t)\,\mathbf{h}(z,t)\,\psi(z,t)\, z_1 ydt = 0 \tag{3.16}$$

holds for all $\psi(z,t)$ satisfying the conditions

$$\psi \in C^1_{loc}(\mathbb{R}; L^\infty_{loc}(\mathbb{R}^2 \setminus \{z_1 = 0\})) \cap C_{loc}(\mathbb{R}; W^{1,\sigma}_{loc}(\mathbb{R}^2 \setminus \{z_1 = 0\})), \quad \sigma > 3/2. \tag{3.17}$$

Now choose an arbitrary function $\chi \in C_0^\infty(\mathbb{R})$. It is easily seen that for an arbitrary fixed y, the function $\psi = z_1^{-1}\chi(t)K_{0i}(y-z)$ satisfies condition (3.17). Substituting ψ into (3.16) leads to the integral identity

$$\int_{\mathbb{R}} \chi'(t)\Big\{\int_{\mathbb{R}^2} K_{0i}(y-z)\rho(z,t)\, dz\Big\}dt - \sum_{j=1,3}\int_{\mathbb{R}} \chi(t)\Big\{\int_{\mathbb{R}^2} K_{ij}(y-z)(\rho w_j(z,t))\, dz\Big\}dt \\ + \int_{\mathbb{R}} \chi(t)\Big\{\int_{\mathbb{R}^2} K_{0i}(y-z)\big(\varrho\mathbf{h}(z,t) - \frac{1}{z_1}\rho w_1\big)\, dz\Big\}\, dt = 0.$$

Since χ is an arbitrary function of the class C_0^∞, it follows that the vector field

$$\overline{\boldsymbol{\varsigma}} = (\overline{\varsigma}_1, \overline{\varsigma}_3), \quad \overline{\varsigma}_i = K_{0i} * \varrho$$

satisfies the desired equality

$$\partial_t\overline{\varsigma}_i = -\sum_{j=1,3} K_{ij} * (\rho w_j(z,t)) + \sigma_i, \tag{3.18}$$

where

$$\sigma_i = K_{0i} * \big(\varrho\mathbf{h}(z,t) - \frac{1}{z_1}\rho w_1\big). \tag{3.19}$$

Note that $1/|z_1| \le R$ on the support of ρ and $\mathbf{h}$. On the other hand, estimates (3.3) and (3.6) imply the inequality

$$\|\varrho\mathbf{h}\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}^2))} + \|\varrho|\mathbf{w}|\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}^2))} \le c(K).$$

Since the the kernel $K_0(y)$ is bounded it follows that

$$\|\sigma_i\|_{L^\infty(\mathbb{R};C(\mathbb{R}^2))} \le c(K). \tag{3.20}$$

Next, the boundedness and continuity of $\overline{\boldsymbol{\varsigma}}$ obviously follows from equality (3.3) and estimate (3.14). We have

$$\partial_j\overline{\varsigma}_i = K_{ij} * \rho = \int_{\mathbb{R}^2} \frac{|y-z|^2\delta_{ij} - (y_i - z_i)(y_j - z_j)}{|y-z|^3}\rho(z,t)dz.$$

The kernel in the right hand side is smooth for $y \ne z$ and has the weak singularity of order $\sim |y-z|^{-1}$ at $y = z$. Since the function $\rho \in L^\infty(\mathbb{R}; L^3(\mathbb{R}^2))$ is compactly supported and $3 > 2$, it follows from the standard estimates of potential operators, see [1], that $\partial_j \overline{\varsigma}_i$ belongs to the class $L^\infty(\mathbb{R}; C(\mathbb{R}^2))$. Hence $\overline{\boldsymbol{\varsigma}}$ belongs to the class $L^\infty(\mathbb{R}; C(\mathbb{R}^2))$

Repeating these arguments and using estimate (3.15) we conclude that $K_{ij} * (\rho w_j)$ belongs to the class $L^2(\mathbb{R}; C(\mathbb{R}^2))$. This result along with equality (3.18) and estimate (3.20) imply the inclusion $\partial_t \overline{\boldsymbol{\varsigma}} \in L^2(\mathbb{R}; C(\mathbb{R}^2))$. These completes the proof of the lemma. □

Lemma 3.3 implies the following important identity.

Lemma 3.4. *Under the above assumptions, we have*

$$\frac{1}{2}\int_{\mathbb{R}^5} \rho(y,t)\rho(z,t)\frac{|(y-z)^\perp \cdot (\overline{\mathbf{w}}(y,t)-\overline{\mathbf{w}}(z,t))|^2}{|y-z|^3}\, y_1 dydzdt + \int_{\mathbb{R}^5} \zeta(y,t)p(y,t)\rho(z,t)\,\frac{y_1 dydzdt}{|z-y|} + \Xi = 0, \tag{3.21}$$

where

$$(y-z)^\perp = (y_2 - z_2, z_1 - y_1), \quad \overline{\mathbf{w}} = (w_1, w_3), \quad |\Xi| \le c,$$

the constant c does not depend on the regularization parameters.

Proof. Substituting the vector field $\overline{\boldsymbol{\varsigma}}$ given by formula (3.11) into integral identity (3.4) and using formula (3.12) for $\partial_t \overline{\boldsymbol{\varsigma}}$ we arrive at the equality

$$\int_{\mathbb{R}^3} \Big((\rho w_i w_j)(K_{ij} * \rho) - (\rho w_i)(K_{ij} * (\rho w_j))\Big)\, y_1 dydt \;+ \int_{\mathbb{R}^3} \zeta p (K_{ii} * \rho)\, y_1 dydt + \Xi \;=\; 0, \tag{3.22}$$

where

$$\Xi = \int_{\mathbb{R}^3} \Big(\rho w_i \sigma_i - \zeta B_{ij} K_{ij} * \rho + F_i K_{0i} * \rho\Big)\, y_1 dydt. \tag{3.23}$$

Let us prove that $|\Xi|$ is bounded by the constant independent of the regularization parameters. Since $|K_0|$ does not exceed 1, and ϱ is compactly supported in $G \times (0,T)$, we have

$$|K_0 * \rho| \le c\|\rho\|_{L^\infty(\mathbb{R}, L^1(\mathbb{R}^2))} \le c.$$

From this and estimates (3.3) and (3.13) we conclude that

$$\Big|\int_{\mathbb{R}^3} \Big(\rho w_i \sigma_i + F_i K_{0i} * \rho\Big) y_1 dydt\Big| \le c. \tag{3.24}$$

It remains to estimate the product of the matrix $\mathbf{B}$ and the convolution $K_{ij} * \rho$. Formula (3.10) for K_{ij} implies the inequality

$$\Big|\int_{\mathbb{R}^3} \zeta B_{ij} K_{ij} * \rho y_1 dydt\Big| \le c \int_{(0,T)\times G} \Big(\int_G \frac{\rho(z,t)dz}{|y-z|}\Big\}|\zeta \mathbf{B}(y,t)|\, y_1 dydt. \tag{3.25}$$

Recall that there is a constant R, depending only on the compact set K such that the inequalities

$$0 < R^{-1} \le y_1 \le R \;\text{ on the support of }\; \zeta. \tag{3.26}$$

Since the supports of $\zeta\mathbf{B}$ and ρ are contained in the support of ζ, it follows from this and (3.25) that

$$\Big|\int_{\mathbb{R}^3}\zeta B_{ij}K_{ij}*\rho y_1 dydt\Big| \le c\int_{(0,T)\times G}\Big(\int_G \frac{\rho(z,t)dz}{|y-z|}\Big\}|\zeta\mathbf{B}(y,t)|dydt = c\int_{(0,T)\times G}\Big(\int_G \frac{|\zeta\mathbf{B}(y,t)|dy}{|y-z|}\Big\}\rho(z,t)dzdt. \tag{3.27}$$

Introduce the functions

$$M(t)=\|\zeta\mathbf{B}(\cdot,t)\|_{L^2(\mathbb{R}^2)}$$

$$\mathbf{S}(y,t)=\frac{1}{M(t)}\zeta\mathbf{B}(y,t) \quad\text{for}\quad M(t)>0 \quad\text{and}\quad \mathbf{S}(y,t)=0 \quad\text{otherwise} \quad ,$$

$$\mathbf{V}(y,t)=\int_G\frac{|\mathbf{S}(t,z)|dz}{|y-z|}$$

It is clear that

$$\|\mathbf{V}(\cdot,t)\|_{W^{1,2}(G)}\le c. \tag{3.28}$$

With this notation inequality (3.27) becomes

$$\Big|\int_{\mathbb{R}^3}\zeta B_{ij}K_{ij}*\rho y_1 dydt\Big| \le c\int_0^T M(t)\Big\{\int_G\Big(\int_G\frac{|S(z,t)|dz}{|y-z|}\Big)\rho(y,t)\,dy\Big\}\,dt \le c\int_0^T M(t)\Big\{\int_G \mathbf{V}(y,t)\rho(y,t)\,dy\Big\}\,dt. \tag{3.29}$$

Now our task is to estimate the integral in the right hand side of this inequality. Applying the Young inequality we obtain

$$\mathbf{V}\,\rho\le e^{\mathbf{V}}-\mathbf{V}-1+(1+\rho)\ln(1+\rho)-\rho. \tag{3.30}$$

We have

$$\ln(1+\rho)\le\rho\ln(2+\rho)\quad\text{for}\quad\rho\ge1\quad\text{and}\quad\ln(1+\rho)\le\ln2\quad\text{for}\quad\rho\le1,$$

which yields

$$(1+\rho)\ln(1+\rho)-\rho\le2\rho\ln(2+\rho)+\ln2.$$

It follows from this and estimate (3.3)in Proposition 3.1, that

$$\int_G\big((1+\rho(t))\ln(1+\rho(t))-\rho(t)\big)\,dy \le \int_G\big(2\rho(t)\ln(2+\rho(t))+\ln2\big)\,dy = \int_G\big(2\zeta\varrho(t)\ln(2+\zeta\varrho(t))+\ln2\big)\,dy \le 2\int_G\zeta\varrho(t)\ln(2+\varrho(t))\,dy+c\le c \tag{3.31}$$

for a.e. $t \in \mathbb{R}$. On the other hand the Pokhozhaev- Trudinger inequality along with estimate (3.28) implies

$$\int_G \big(e^{\mathbf{V}(y,t)} - \mathbf{V}(y,t) - 1\big)\, dy \le c(\|\mathbf{V}(\cdot,t)\|_{W^{1,2}(G)}) \le c \text{ for a.e. } \quad t \in \mathbb{R}.$$

It follows from this estimate, estimate (3.31) and the Young inequality (3.30) that

$$\int_G V(t)\rho(t)\, dy \le c \text{ for a.e. } \quad t \in \mathbb{R}. \tag{3.32}$$

Combining this result with inequality (3.29) we arrive at the estimate

$$\Big|\int_{\mathbb{R}^3} \zeta B_{ij}\, K_{ij} * \rho dt dy\Big| \le c \int_0^T M(t) dt.$$

Since $\zeta\mathbf{B}$ is compactly supported in $G \times (0,T)$, there is a compact $K' \subset G$ such that spt $\zeta \subset K' \times (0,T)$. In fact, K' is the orthogonal projection of the support of ζ onto G. Next, estimate (2.9) in Proposition 2.2 implies the inequality

$$\int_0^T M(t)^2 dt \le \int_{K' \times (0,T)} |\mathbf{B}(y,t)|^2 dy dt \le c.$$

Thus we get

$$\Big|\int_{\mathbb{R}^3} \zeta B_{ij}\, K_{ij} * \rho y_1 dy dt\Big| \le c. \tag{3.33}$$

It follows from the expression (3.23) for the quantity Ξ that

$$|\Xi| \le \int_{\mathbb{R}^3} \big|\, \rho w_i \sigma_i + F_i K_{0i} * \rho\big|\, y_1 dy dt + \Big|\int_{\mathbb{R}^3} \zeta B_{ij} K_{ij} * \rho\, y_1 dy dt\Big|.$$

It remains to note that desired estimate $|\Xi| \le c$ is an obviously follows from this and estimates (3.24) (3.33). Finally notice that the main identity (3.21)is a straightforward consequence of relation (3.22) and the identities

$$\int_{\mathbb{R}^3} \Big((\rho w_i w_j)(K_{ij} * \rho) - (\rho w_i)(K_{ij} * (\rho w_j))\Big) y_1\, dy dt =$$

$$\frac{1}{2}\int_{\mathbb{R}^5} \rho(t,y)\rho(t,z) \frac{|(y-z)^{\perp} \cdot (\overline{\mathbf{w}}(t,y) - \overline{\mathbf{w}}(z,t))|^2}{|y-z|^3} y_1\, dy dz dt,$$

$$\int_{\mathbb{R}^3} \zeta p (K_{ii} * \rho) y_1\, dy dt = \int_{\mathbb{R}^5} \zeta(y,t) p(y,t) \rho(z,t)\, \frac{y_1 dy dz t}{|z-y|}.$$

This completes the proof of Lemma 3.4

□

We are now in a position to complete the proof of Proposition 3.2. To this end, note that the first integral in the left hand side of identity (3.21) is nonnegative. From this and estimate for Ξ in Lemma 3.4 we get the inequality

$$\int_{\mathbb{R}^3} \rho(y,t) \Big\{ \int_{\mathbb{R}^2} \zeta(z,t) p(z,t)\, \frac{dz}{|z-y|} \Big\} y_1 dy dt \le c.$$

Since ρ and ζp are compactly supported in $G\times(0,T)$, and

$$0<R^{-1}<y_1<R$$

on the supports ζ and ρ, we have

$$\begin{gathered}\int_{\mathbb{R}^3}\rho(y,t)\Big\{\int_{\mathbb{R}^2}\zeta(z,t)p(z,t)\,\frac{dz}{|z-y|}\Big\}dydt=\\ \int_{(0,T)\times G}\rho(y,t)\Big\{\int_G\zeta(z,t)p(z,t)\,\frac{dz}{|z-y|}\Big\}dydt\le c.\end{gathered}\tag{3.34}$$

Formula (1.13) for p and relation $\rho=\zeta\varrho$ imply the inequality

$$\zeta p=\zeta\varrho+\zeta\varepsilon\varrho^{1+\varepsilon}\big(\varepsilon\ln(1+\varepsilon\varrho)+\frac{\varepsilon\varrho}{1+\varepsilon\varrho}\big)+\zeta\delta^3\varrho^3\ge\rho,$$

which yields the estimate

$$\int_G\zeta(z,t)p(z,t)\,\frac{dz}{|z-y|}\ge\int_G\frac{\rho(t,z)dz}{|z-y|}.$$

From this and (3.34) we obtain

$$\int_{G\times(0,T)}\rho(y,t)\Big\{\int_G\frac{\rho(z,t)dz}{|z-y|}\Big\}dydt\le c.$$

Noting that ρ is compactly supported in $G\times(0,T)$ we arrive at the inequality

$$\int_{\mathbb{R}^3}\rho\,(-\Delta)^{-1/2}\rho\,dydt=c_{1/2}\int_{\mathbb{R}^3}\rho(y,t)\Big\{\int_{\mathbb{R}^2}\frac{\rho(z,t)dz}{|z-y|}\Big\}\,dydt\le c.$$

It remains to note

$$\int_{\mathbb{R}^3}(-\Delta)^{-1/4}\rho\,(-\Delta)^{-1/4}\rho\,dydt=\int_{\mathbb{R}^3}\rho\,(-\Delta)^{-1/2}\rho\,dydt,$$

and the proposition follows □

4 Estimates of rotationally symmetric functions in Orlicz spaces

In this section we prove the Orlicz type estimates for the density ϱ and the velocity field $\mathbf{u}$. The result is of general character and does not related with the theory of Navier-Stokes equations. Suppose that the bounded flow domain $\Omega\subset\mathbb{R}^3$ is rotationally symmetric and has a smooth boundary $\partial\Omega$. The intersection of Ω and the symmetry axis Ox_3 consists of the only interval $\mathcal{I}$.

Recall that the cylindrical coordinates (y,ϑ), $y=(y_1,y_3)$ are defined by the equalities.

$$x_1=y_1\cos\vartheta,\quad x_2=y_1\sin\vartheta,\quad x_3=y_3.\tag{4.1}$$

In particular, Ω is defined by the relation

$$\Omega\setminus\mathcal{I}=\big\{\,y,\vartheta:\ y\in G,\ \theta\in[0,2\pi]\,\big\}.\tag{4.2}$$

Here the domain $G \subset \mathbb{R}^2$ is the intersection of the rotationally symmetric flow domain Ω and the half-plane Π^+:$x_1 > 0$, $x_2 = 0$. Since $y_1 = x_1$ and $y_3 = x_3$, G can be regarded as a domain in the plane of the variable y.

Let us consider the function $\varrho : \Omega \times (0,T) \to \mathbb{R}^+$ and the vector field $\mathbf{u} : \Omega \times (0,T) \to \mathbb{R}^3$ satisfying the following conditions.

C.1 The functions $\varrho \in L^\infty(0,T;L^1(\Omega))$ and $\mathbf{u} \in L^2(0,T;W^{1,2}_0(\Omega))$ admit the estimate

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega \big(\,\varrho \ln(2+\varrho) + \varrho|\mathbf{u}|^2\,\big)\, dx + \|\nabla \mathbf{u}\|_{L^2(\Omega\times(0,T))} \le c_0. \tag{4.3}$$

C.2 The functions ϱ and $\mathbf{u}$ are rotationally symmetric, i.e.,

$$\varrho = \varrho(y,t), \quad \mathbf{u}(y,\theta,t) = \mathbf{U}(\theta)\,\mathbf{v}(y,t), \tag{4.4}$$

where the orthogonal matrix $\mathbf{U}$ is defined by (2.3).

C.3 For every compact set $K \Subset G\times(0,T)$, the function $\varrho(y,t)\mathbf{1}_K$ admits the estimate

$$\int_K \big((-\Delta)^{-1/4}(\varrho\mathbf{1}_K)\big)^2 dydt \le c(K). \tag{4.5}$$

Here

$$(-\Delta)^{-1/4}\varrho(y,t) = c_{1/4}\int_G \frac{1}{|y-z|^{3/2}}\,\varrho(z,t)\,dz.$$

The main result of this section is the following theorem.

Theorem 4.1. *Let $\varrho : \Omega\times(0,T) \to \mathbb{R}^+$ and $\mathbf{u} : \Omega\times(0,T) \to \mathbb{R}^3$ satisfy conditions (**C.1**)-(**C.1**). Then the inequality*

$$\int_{K'} \varrho|\mathbf{u}|^2\, \ln(2+\varrho|\mathbf{u}|^2)^\alpha\, dxdt \le c \tag{4.6}$$

*holds for all compact sets $K' \subset (\Omega\setminus\mathcal{I})\times(0,T)$ and for all $0<\alpha<1/4$. Here the constant c depends only on K', α, and the constant c_0 in condition (**C.1**).*

The proof falls into three steps. First, we reformulate the theorem in the cylindrical coordinates.

4.1. Step 1. Cylindrical coordinates. Localization. Without loss of generality we may assume that K' is a toroidal compact set in the form

$$K' = \big\{(y,\theta,t) : (y,t)\in K, \quad \theta\in[0,2\pi]\big\},$$

where K is a compact subset $G\times(0,T)$. In other words, K' is obtained by rotation of K around the symmetry axis Ox_3. By virtue of Condition (**C.2**), it suffices to prove that the inequality

$$\int_K \varrho|\mathbf{v}|^2\, \ln(2+\varrho|\mathbf{v}|^2)^\alpha\, y_1 dydt \le c \tag{4.7}$$

holds for all compact sets $K \subset G\times(0,T)$ and for all $0<\alpha<1/4$. In order to simplify the proof, we use the localization procedure defined in Section 3.

Fix a compact set $K \Subset G\times(0,T)$, and choose two functions $\zeta, \eta \in C_0^\infty(G_T)$ such that

$$\zeta = \eta = 1 \text{ на } K, \quad \eta = 1 \text{ на supp } \zeta, \quad 0 \le \zeta, \eta \le 1. \tag{4.8}$$

Assume that they are extended by zero to the whole space $\mathbb{R}^2 \times \mathbb{R}$. Set

$$\rho(y,t) = \zeta(y,t)\varrho(y,t), \quad \mathbf{w}(y,t) = \eta(y,t)\mathbf{v}(y,t), \quad \overline{\mathbf{w}} = (w_1, w_3) = \eta\overline{\mathbf{v}}. \tag{4.9}$$

Extend these functions by zero to $\mathbb{R}^3$. Note that the function ρ and vector field $\mathbf{w}$ meet all requirements of Proposition 3.1. Recall that the is $R > 0$, depending on K, such that $R^{-1} \le y_1 \le R$ on the support of ζ. Therefore, in order to prove Theorem (4.1), it suffices to show that the estimate

$$\int_{\mathbb{R}^3} \rho|\mathbf{w}|^2 \, \ln(2 + \rho|\mathbf{w}|^2)^\alpha \, dydt \le c \tag{4.10}$$

holds for all $0 < \alpha < 1/4$.

4.2. Step 2. Weak estimates of the kinetic energy. Now we prove the weak type estimates for the localized kinetic energy $\rho|\mathbf{w}|^2$. Our considerations are based on the following proposition.

Proposition 4.2. *Let the vector field* $\mathbf{W} : \mathbb{R}^3 \to \mathbb{R}^3$ *compactly supported in* $G \times (0,T)$ *and*

$$\|\mathbf{W}(t)\|_{W^{1,2}(\mathbb{R}^2)} \le 1 \quad \textit{for a.e.} \quad t \in \mathbb{R}. \tag{4.11}$$

Furthermore assume that ϱ *satisfies conditions* ***C.1-C.3*** *of Theorem 4.1. Then, there is a constant* c*, depending on* K *and the constant* c_0 *in condition* ***C.1****, such that*

$$\int_{\mathbb{R}^3} \rho|\mathbf{W}|^2 \, \ln(2 + \rho|\mathbf{W}|^2)^{1/2} \, dydt \le c, \tag{4.12}$$

where c *depends on* K.

Proof. The proof is given in Appendix 9. □

The main goal of this section is the weak-type estimate for the kinetic energy tensor. The result is given by the following proposition. Fix an arbitrary $N \ge 0$ and denote by $\mathcal{A}_N \subset G \times (0,T)$ the set

$$\mathcal{A}_N = \big\{ \, (y,t) \in G \times (0,T) : \; \rho(y,t)|\mathbf{w}(y,t)|^2 \ge N \, \big\}. \tag{4.13}$$

Proposition 4.3. *Under the above assumptions, we have*

$$\int_{\mathcal{A}_N} \rho|\mathbf{w}|^2 \, dydt \le c \, \ln(2 + N)^{-1/4}. \tag{4.14}$$

Here the constant c *depends on the compact set* K*, and does not depend on the regularization parameters* ε*,* δ *in the expressions for the internal energy* e *and the pressure function* p*.*

Proof. Since the function $\rho|\mathbf{w}|^2$ is integrable, it suffices to prove the lemma for $N \geq 3$. In this case, we may replace $\ln(2+N)$ by $\ln N$. Therefore, it suffices to prove the inequality

$$\int_{\mathcal{A}_N} \rho|\mathbf{w}|^2 \, dydt \leq c \, \ln N^{-1/4} \quad \text{for all} \quad N \geq 3. \tag{4.15}$$

Our considerations are based on the following construction. Denote by $M(t)$ the function

$$M(t) = \|\mathbf{w}(\cdot, t)\|_{W^{1,2}(\mathbb{R}^2)}.$$

Note that $M(t)$ is compactly supported in $(0, T)$. Choose an arbitrary $\sigma \geq 1$ and set

$$\begin{aligned} \mathcal{B} &= \big\{(y,t) \in G \times (0,T) : \ (y,t) \in \mathcal{A}_N, \, M(t) \leq \sigma \big\}, \\ \mathcal{C} &= \big\{(y,t) \in G \times (0,T) : \ (y,t) \in \mathcal{A}_N, \, M(t) > \sigma \big\}. \end{aligned}$$

It is clear that

$$\int_{\mathcal{A}_N} \rho|\mathbf{w}|^2 \, dydt = \int_{\mathcal{B}} \rho|\mathbf{w}|^2 \, dydt + \int_{\mathcal{C}} \rho|\mathbf{w}|^2 \, dydt. \tag{4.16}$$

Let us estimate the second integral in the right hand side. It follows from the energy estimate (3.3) that

$$\begin{aligned} \int_{\mathcal{C}} \rho|\mathbf{w}|^2 \, dydt &\leq \int_{t: M(t) > \sigma} \Big(\int_G \rho|\mathbf{w}|^2 \, dy \Big) \, dt \\ &\leq c \ \text{ meas } \ \{M(t) > \sigma\} \end{aligned}$$

Next, energy estimate (3.3) implies

$$\int_{\mathbb{R}} M(t)^2 \, dt = \int_{\mathbb{R}^2} \|\mathbf{w}(t)\|^2_{W^{1,2}(\mathbb{R}^2)} \, dt \leq c,$$

which gives

$$\text{meas } \ \{M(t) > \sigma\} \leq c/\sigma^2. \tag{4.17}$$

Combining the obtained inequalities we arrive at the estimate

$$\int_{\mathcal{C}} \rho|\mathbf{w}|^2 \, dydt \leq c/\sigma^2. \tag{4.18}$$

Now our task is to estimate the first integral in the right hand side of (4.16). Introduce the function

$$\mathbf{W}(y,t) = M(t)^{-1}\mathbf{w}(y,t) \text{ when } M(t) > 0, \text{ and } \mathbf{W}(y,t) = 0 \ \text{ otherwise}.$$

It is clear that $(y,t) \in \mathcal{B}$ if and only if

$$M(t) \leq \sigma, \quad \rho(y,t)|\mathbf{W}(y,t)|^2 \geq \frac{N}{M(t)^2}$$

It follows from this that

$$\mathcal{B} \subset \mathcal{B}^* = \big\{ \, (y,t) \in G \times (0,T) : M(t) \leq \sigma, \ \rho(y,t)|\mathbf{W}(y,t)|^2 \geq N/\sigma^2 \, \big\}. \tag{4.19}$$

Thus we get

$$\begin{aligned}\int_{\mathcal{B}} \rho|\mathbf{w}|^2\,dydt &\le \int_{\mathcal{B}^*} \rho|\mathbf{w}|^2\,dydt \\ &= \int_{\mathcal{B}^*} M(t)^2\rho|\mathbf{W}|^2\,dydt \\ &\le \sigma^2\int_{\mathcal{B}^*} \rho|\mathbf{W}|^2\,dydt.\end{aligned} \tag{4.20}$$

On the other hand, definition (4.19) of the set $\mathcal{B}^*$ implies

$$\begin{aligned}\ln(2+N/\sigma^2)^{1/2}\int_{\mathcal{B}^*} \rho|\mathbf{W}|^2\,dydt &\le \int_{\mathcal{B}^*} \rho|\mathbf{W}|^2\ln(2+\rho|\mathbf{W}|^2)^{1/2}dydt \\ &\le \int_{G\times(0,T)} \rho|\mathbf{W}|^2\ln(2+\rho|\mathbf{W}|^2)^{1/2}dydt.\end{aligned}$$

From this and estimate (4.12) in Proposition 4.2 we obtain

$$\begin{aligned}\int_{\mathcal{B}^*} \rho|\mathbf{W}|^2\,dydt &\le \\ &\qquad \ln(2+N/\sigma^2)^{-1/2}\int_{G\times(0,T)} \rho|\mathbf{W}|^2\ln(2+\rho|\mathbf{W}|^2)^{1/2}dydt \\ &\le c\ln(2+N/\sigma^2)^{-1/2}\le c\ln(N/\sigma^2)^{-1/2}.\end{aligned}$$

Substituting this inequality into (4.20) we obtain the estimate

$$\int_{\mathcal{B}} \rho|\mathbf{w}|^2\,dydt \le c\sigma^2\ln(N/\sigma^2)^{-1/2}. \tag{4.21}$$

Next, substituting inequalities (4.18) and (4.21) into identity (4.16) we arrive at the estimate

$$\int_{\mathcal{A}_N} \rho|\mathbf{w}|^2\,dydt \le c/\sigma^{-2} + c\sigma^2\ln(N/\sigma^2)^{-1/2}. \tag{4.22}$$

Now set $\sigma = \ln N^{1/8}$. We have

$$\ln\left(N/(\ln N)^{1/4}\right) \ge \ln\left(N/N^{1/4}\right) = \frac{3}{4}\ln N.$$

Substituting these relations into (4.22) we finally obtain

$$\int_{\mathcal{A}_N} \rho|\mathbf{w}|^2\,dydt \le c\ln N^{-1/4} + c\ln N^{1/4}/(\ln N)^{1/2} \le c\ln N^{-1/4},$$

and the proposition follows. □

4.3. Step 3. The proof of Theorem 4.1. We are now in a position to complete the proof of Theorem 4.1. To this end, we use the weak-type estimate (4.14) in Proposition 4.3. Denote by $E(\lambda)$, $\lambda \ge 0$ the monotone function defined by the relation

$$E(\lambda) = \text{meas}\ \left\{ (y,t)\in G\times(0,T):\ \rho(y,t)|\mathbf{w}(y,t)|^2 \ge \lambda \right\}.$$

It is the standard fact of calculus, that the equality

$$\int_{G\times(0,T)} f(\rho|\mathbf{w}|^2)\,dydt = -\int_{[0,\infty)} f(\lambda)\;dE(\lambda)$$

holds for every piecewise continuous function f such that $f(\rho|\mathbf{w}|^2) \in L^1(G\times(0,T))$. Here $-dE(\lambda)$ is the Borel-Stieltjes measure associated with the function $-E(\lambda)$. In particular, we have

$$\int_{\mathcal{A}_N} \rho|\mathbf{w}|^2\,dydt = -\int_{[N,\infty)} \lambda\,dE(\lambda). \tag{4.23}$$

With this notation the desired inequality (4.10) reads

$$\int_{G\times(0,T)} \rho|\mathbf{w}|^2 \ln(2+\rho|\mathbf{w}|^2)^\alpha\,dydt \equiv -\int_{[0,\infty)} \lambda\ln(2+\lambda)^\alpha\,dE(\lambda) \le c. \tag{4.24}$$

Introduce the monotone function F and the corresponding Borel-Stieltjes measure $-dF$ defined by the equalities

$$F(s) = -\int_{[s,\infty)} \lambda\,dE(\lambda), \quad dF(\lambda) = \lambda\,dE(\lambda).$$

Note that identity (4.23) and estimate (4.14) in Proposition 4.3 imply

$$F(s) \le c\ln(2+s)^{-1/4} \;\text{ for }\; s\ge 0. \tag{4.25}$$

It is clear that

$$-\int_{[0,\infty)} \lambda\ln(2+\lambda)^\alpha\,dE(\lambda) = -\int_{[0,\infty)} \ln(2+\lambda)^\alpha\,dF(\lambda)$$

Now our task is to prove that the integral in the right hand side exists and admits the effective estimate. Let us consider the regularized integral

$$I(\epsilon,N) = -\int_{[0,N)} g(\lambda)dF(\lambda), \quad g(\lambda) = \frac{\ln(2+\lambda)^\alpha}{1+\epsilon^2\lambda^2}$$

Note that g is bounded and continuously differentiable. On the other hand, the total variation of F is finite. Hence, we may integrate by parts to obtain

$$I(\epsilon,N) = \int_{[0,N)} g'(\lambda)\,F(\lambda)\,d\lambda + g(0)F(0) - g(N)\lim_{\lambda\to N-0} F(\lambda). \tag{4.26}$$

We have

$$\begin{aligned} g'(\lambda) &= g_0(\lambda) - \mathbf{R}(\lambda),\\ g_0(\lambda) &= \alpha\ln(2+\lambda)^{\alpha-1}(2+\lambda)^{-1}(1+\epsilon^2\lambda^2)^{-1}\\ \mathbf{R}(\lambda) &= 2\epsilon^2\lambda\ln(2+\lambda)^\alpha(1+\epsilon^2\lambda^2)^{-2} \end{aligned}$$

Since $\alpha\in(0,1/4)$, estimate (4.25) implies

$$|g'(\lambda)F(\lambda)| \le c(1+\epsilon^2\lambda^2)^{-1}$$

which yields $g'F \in L^1(\mathbb{R}^+)$. On the other hand, we have

$$0 \le g(N)F(N) \le c\ln(2+N)^{\alpha-1/4}(1+\epsilon^2N^2)^{-1} \to 0 \;\text{ as }\; N\to\infty.$$

Letting $N \to \infty$ in equality (4.26) we conclude that there exists the integral

$$\begin{aligned} -\int_{[0,\infty)} g(\lambda)\, dF(\lambda) &= \int_0^\infty g_0(\lambda)\, F(\lambda)\, d\lambda \\ &- \int_0^\infty \mathbf{R}(\lambda)\, F(\lambda)\, d\lambda + g(0)F(0). \end{aligned}$$

Since $\mathbf{R}$ and F are nonnegative, we have

$$-\int_{[0,\infty)} g(\lambda)\, dF(\lambda) \le \int_0^\infty g_0(\lambda)\, F(\lambda)\, d\lambda + g(0)F(0). \tag{4.27}$$

It follows from (4.13) and (4.23) that

$$\begin{aligned} F(0) &= -\int_{[0,\infty)} dF(\lambda) = -\int_{[0,\infty)} \lambda dE(\lambda) \\ &= \int_{\mathcal{A}_0} \rho|\mathbf{w}|^2\, dydt = \int_{G\times(0,T)} \rho|\mathbf{w}|^2\, dydt \le c. \end{aligned} \tag{4.28}$$

Next, estimate (4.25) yields

$$\begin{aligned} \int_0^\infty g_0 F\, d\lambda &\le \alpha \int_0^\infty \ln(2+\lambda)^{\alpha-1}(2+\lambda)^{-1}\, F(\lambda) d\lambda \\ &\le \int_0^\infty \ln(2+\lambda)^{\alpha-1-1/4}(2+\lambda)^{-1}\, d\lambda \le c(\alpha). \end{aligned}$$

It follows from this and (4.28) that

$$-\int_{[0,\infty)} g_0(\lambda)\, dF(\lambda) \le c(\alpha) + \ln 2^\alpha c \le c. \tag{4.29}$$

Recall that $-dF$ is a nonnegative Borel measure. Notice that g_0 is an increasing function of ϵ and

$$g_0(\lambda) \nearrow \ln(2+\lambda)^\alpha \quad \text{as} \quad \epsilon \searrow 0.$$

Letting $\epsilon \to 0$ in (4.29) and applying the Fatou theorem we conclude that there exists the integral

$$-\int_{[0,\infty)} \ln(2+\lambda)^\alpha\, dF(\lambda) \equiv -\int_{[0,\infty)} \lambda \ln(2+\lambda) dE(\lambda) \le c.$$

Combining this inequality with relation (4.24) we obtain desired estimate (4.10). This completes the proof of Theorem 4.1.

5 Singular limits

Until now, we have been dealing with solutions of regularized equations with internal energy e and pressure functions p given by the equalities

$$\begin{aligned} e &= \varrho \ln \varrho - \varrho + e_1(\varrho) + e_2(\varrho), \\ e_1 &= \varepsilon \varrho^{1+\varepsilon} \ln(1+\varepsilon\varrho), \quad e_2 = \frac{1}{2}\delta^3 \varrho^3, \end{aligned} \tag{5.1}$$

$$\begin{aligned} p &= \varrho + p_1(\varrho) + p_2(\varrho), \\ p_1 &= \varepsilon\varrho^{1+\varepsilon}\big(\varepsilon \ln(1+\varepsilon\varrho) + \frac{\varepsilon\varrho}{1+\varepsilon\varrho}\big), \quad p_2 = \delta^3\varrho^3. \end{aligned} \tag{5.2}$$

Here $\varepsilon, \delta \in (0, 1/2)$ are the regularization parameters.

Proposition 1.3 guarantees the existence of weak rotationally symmetric solutions of regularized equations. These solutions depend on the regularization parameters ε and δ. In this section, we will show this dependence and use the following notation for solutions of regularized equations (1.15)

$$\varrho = \varrho_{\varepsilon,\delta}, \quad \mathbf{u} = \mathbf{u}_{\varepsilon,\delta}$$

Recall that the main goal of the article is to prove the existence od weak rotationally symmetric solutions to the isothermal Navier-Stokes equations (1.22). The main Theorem 1.5 gives a precise formulation of this result. We obtain the solution of equations (1.22) as the limit of the solutions to the regularized equations (1.15). The first step on this way is to prove that the pressure functions p_1, p_2 converge to zero on compact subsets of $\Omega\setminus\mathcal{I}$ as the regularization parameters ε, δ tend to zero.

The main idea is the following. We assume that the regularization parameters ε and δ are dependent,

$$\delta = \kappa(\varepsilon) \ll \varepsilon, \quad \kappa(\varepsilon) = e^{-\frac{1}{\varepsilon^9}}, \quad \varepsilon \in (0, 1/2). \tag{5.3}$$

In this case, we use the denotations

$$\varrho_\varepsilon = \varrho_{\varepsilon,\kappa(\varepsilon)}, \quad \mathbf{u}_\varepsilon = \mathbf{u}_{\varepsilon,\kappa(\varepsilon)}. \tag{5.4}$$

The following theorem is the main result of this section.

Theorem 5.1. *Under the above assumptions,*

$$\int_{K'} (p_1(\varrho_\varepsilon) + p_2(\varrho_\varepsilon))\, dxdt \to 0 \quad as \quad \varepsilon \to 0 \tag{5.5}$$

for every compact set $K' \subset (\Omega\setminus\mathcal{I}) \times (0, T)$

We reformulate this theorem in the cylindrical coordinates arguing as in the beginning of Section 4. Recall that the cylindrical coordinates (y, ϑ), $y = (y_1, y_3)$ are defined by the equalities.

$$x_1 = y_1 \cos\vartheta, \quad x_2 = y_1 \sin\vartheta, \quad x_3 = y_3.$$

In particular, we have

$$\Omega\setminus\mathcal{I} = \big\{\, y, \vartheta, t : \; y \in G, \; \theta \in [0, 2\pi] \,\big\}.$$

Here the domain $G \subset \mathbb{R}^2$ is the intersection of the rotationally symmetric flow domain Ω and the half-plane $\{x_1 > 0, x_2 = 0\}$. It can be regarded as the domain in the plane of variable y. Without loss of generality we may assume that K' is a toroidal compact set

$$K' = \big\{(y, \theta, t) : (y, t) \in K, \quad \theta \in [0, 2\pi]\big\}.$$

Here K is a compact subset of $G \times (0,T)$. In other words, K' is obtained by rotation of K around the symmetry axis Ox_3. A rotationally symmetric weak solution to problem (1.15) admits the representation, see Section 2,

$$\mathbf{u}_{\varepsilon,\delta}(x,t) = \mathbf{U}(\theta)\,\mathbf{v}_{\varepsilon,\delta}(y,t), \quad \varrho_{\varepsilon,\delta} = \varrho_{\varepsilon,\delta}(y,t), \quad (y,t) \in G \times (0,T),$$

where the orthogonal matrix $\mathbf{U}$ is given by (2.3). Hence Theorem 5.1 can be reformulated as follows

Theorem 5.2. *Under the above assumptions,*

$$\int_K p_1(\varrho_\varepsilon)\,dydt + \int_K p_2(\varrho_\varepsilon)\,dydt \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{5.6}$$

for every compact set $K \subset G \times (0,T)$

The rest of the section is devoted to the proof of Theorems 5.1 and 5.2. We start with the analysis of the behavior of the function p_2 for small $\delta \ll \varepsilon$.

5.1. Estimates of p_2. Auxiliary theorem. The proofs of Theorems 5.1 and 5.2 are based on the intermediate technical result. In order to formulate it, recall the localization procedure, see Section 3 for details.

Fix a compact set $K \Subset G\times(0,T)$, and choose two functions $\zeta, \eta \in C_0^\infty(G_T)$ such that

$$\zeta = \eta = 1 \text{ on } K, \quad \eta = 1 \text{ on supp } \zeta, \quad 0 \le \zeta, \eta \le 1.$$

Assume that they are extended by zero to the whole space $\mathbb{R}^3$. Set

$$\rho_{\varepsilon,\delta}(y,t) = \zeta(y,t)\varrho_{\varepsilon,\delta}(y,t), \quad \mathbf{w}_{\varepsilon,\delta}(y,t) = \eta(y,t)\mathbf{v}_{\varepsilon,\delta}(y,t).$$

Extend these functions by zero to $\mathbb{R}^3$. Note that the function $\rho = \rho_{\varepsilon,\delta}$ and and the vector field $\mathbf{w} = \mathbf{w}_{\varepsilon,\delta}$ meet all requirements of Proposition 3.1. We also notice that $\varrho_{\varepsilon,\delta} = \rho_{\varepsilon,\delta}$ in K. The following auxiliary theorem is the main result of this subsection.

Theorem 5.3. *For every compact set $K \in G \times (0,T)$, there is a constant c such that the inequality*

$$\int_{\mathbb{R}^3} p_2(\rho_{\varepsilon,\delta})\,dydt = \int_{\mathbb{R}^3} \delta^3\,\rho_{\varepsilon,\delta}^3\,dydt \le c\varepsilon \tag{5.7}$$

holds for all (ε, δ) satisfying the inequalities

$$0 < \varepsilon \le 1/2, \quad 0 < \delta \le \kappa(\varepsilon) = e^{-\frac{1}{\varepsilon^9}}. \tag{5.8}$$

The proof falls into three steps. By abuse of notation, we will write simply ρ and $\mathbf{w}$ instead of $\rho_{\varepsilon,\delta}$ and $\mathbf{w}_{\varepsilon,\delta}$.

Step 1. The basic inequality. Introduce the function

$$\varphi(\rho) = \ln(1+\delta\rho) \ \text{ such that } \ p_2\varphi = \delta^3\rho^3\,\ln(1+\delta\rho) \tag{5.9}$$

Recall that the potential $\nabla\Delta^{-1}$ admits the representation

$$\nabla\Delta^{-1}u = c\int_{\mathbb{R}^2} \frac{y-z}{|y-z|^2}\,u(z)\,dz.$$

Our first goal is the derivation of the special integral inequality for $p_2\varphi$. The result is given by the following lemma.

Lemma 5.4. *Let* $\mathbf{B}$ *and* $\mathbf{F}$ *be defined by Propositions 2.2 and 3.1. Then we have*

$$\int_{\mathbb{R}^3} p_2(\rho)\,\varphi(\rho)\,y_1 dydt \le \sum_{i=1}^{5} I_i, \tag{5.10}$$

where

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^3} \big\{\nabla\Delta^{-1}\text{ div }(\varphi\overline{\mathbf{w}})\big\}\cdot(\rho\overline{\mathbf{w}})\,y_1dydt\\ I_2 &= -\int_{\mathbb{R}^3}(\rho\overline{\mathbf{w}}\otimes\overline{\mathbf{w}}):(\nabla^2\Delta^{-1}\varphi)\,y_1dydt,\\ I_3 &= \int_{\mathbb{R}^3}\zeta\mathbf{B}:(\nabla^2\Delta^{-1}\varphi)\,y_1dydt,\\ I_4 &= -\int_{\mathbb{R}^3}\mathbf{F}\cdot(\nabla\Delta^{-1}\varphi)\,y_1dydt,\\ I_5 &= -\int_{\mathbb{R}^3}(\nabla\Delta^{-1}\boldsymbol{\pi})\cdot(\rho\overline{\mathbf{w}})\,y_1dydt, \end{aligned} \tag{5.11}$$

Here $\overline{\mathbf{w}} = (w_1, w_3)$, ζ *is a smooth cutoff function such that* $\rho = \zeta\varrho$. *The function* $\boldsymbol{\pi}$ *is defined by the equality*

$$\boldsymbol{\pi} = \frac{\delta\rho}{1+\delta\rho}\big(\nabla\zeta\cdot\overline{\mathbf{w}} - \frac{1}{y_1}w_1 - \text{div }\overline{\mathbf{w}}\big) + \varphi\text{ div }\overline{\mathbf{w}}. \tag{5.12}$$

Proof. The proof is given in Appendix 10. □

Step 2. Estimates of I_i Now our task is to estimate the quantities I_i in the right hand side of inequality (5.10). The derivation of these estimate is based on the following auxiliary proposition, which is not related directly to Navier-Stokes equations. Let us consider the function $\rho : \mathbb{R}^2 \to \mathbb{R}^+$ and vector field $\mathbf{W} = (W_1, W_3) : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following condition.

Condition 5.5. *The functions* ρ *and* $\mathbf{W}$ *are compactly supported in* G. *They admit the estimate*

$$\begin{gathered}\int_{\mathbb{R}^2}\rho\ln(2+\rho)\,dy + \int_{\mathbb{R}^2}p_1(\rho)\,dy \le c_1 < \infty, \quad \|\mathbf{W}\|_{W^{1,2}(\mathbb{R}^3)} \le 1,\\ 0 < c_1^{-1} \le y_1 \le c_1 \quad \textit{on the supports of} \;\; \rho \;\; \textit{and} \;\; \mathbf{W}.\end{gathered} \tag{5.13}$$

Proposition 5.6. *Let* ρ *and* $\mathbf{W}$ *satisfy Condition 5.5. Then there is a constant* c, *depending on* c_1 *and* G *such that*

$$\|\rho\mathbf{W}\|_{L^{1+\frac{\varepsilon}{2}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^2}, \qquad \|\rho|\mathbf{W}|^2\|_{L^{1+\frac{\varepsilon}{2}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^3}, \tag{5.14}$$

$$\|\nabla\Delta^{-1}\text{ div }(\varphi\mathbf{W})\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^6}, \qquad \|\nabla^2\Delta^{-1}\varphi\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^5}, \tag{5.15}$$

$$\|\nabla\Delta^{-1}\boldsymbol{\pi}^*\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^3}, \qquad \|\nabla\Delta^{-1}\varphi\|_{L^\infty(\mathbb{R}^2)} \le c. \tag{5.16}$$

Here

$$\varphi = \ln(1+\delta\rho), \quad \boldsymbol{\pi}^* = \frac{\delta\rho}{1+\delta\rho}\big(\nabla\zeta\cdot\mathbf{W} - \frac{1}{y_1}W_1 - \text{ div } \mathbf{W}\big) + \varphi \text{ div } \mathbf{W}. \quad (5.17)$$

Proof. The proof is given in Appendix 11. □

We are now in a position to estimate the quantities I_i. The result is given by the following lemma.

Lemma 5.7. *Under the assumptions of Theorem 5.2, the quantities I_i admit the estimates*

$$|I_i| \le c\varepsilon^{-8}, \quad \varepsilon, \delta \in (0, 1/2). \quad (5.18)$$

Here the constant c does not depend on ε and δ

Proof. We estimate I_i step by step. Introduce the functions

$$\begin{aligned} M(t) &= \|\overline{\mathbf{w}}(\cdot,t)\|_{W^{1,2}(\mathbb{R}^2)}, \quad t \in \mathbb{R}, \\ \mathbf{W}(y,t) &= M(t)^{-1}\overline{\mathbf{w}}(y,t) \text{ for } M(t) \neq 0, \\ \mathbf{W}(y,t) &= 0 \text{ otherwise.} \end{aligned} \quad (5.19)$$

Recall that the cutoff function $\zeta : \mathbb{R}^3 \to [0,1]$ is compactly supported in $G \times (0,T)$. In particular, we have

$$\rho\ln(2+\rho) = \zeta\varrho\ln(2+\zeta\varrho) \le \zeta\varrho\ln(2+\varrho).$$

From this and estimate (3.3) in Proposition 3.1 we conclude that the inequality

$$\int_{\mathbb{R}^2} \rho(t)\ln(2+\rho(t))\, dydt \le c. \quad (5.20)$$

holds for a.e. $t \in \mathbb{R}$. Here c does not depend on t. Therefore, the functions ρ and $\mathbf{W}$ meet al requirements of Proposition 5.6 and satisfy inequalities (5.14)-(5.16). Now we may estimate I_i.

Estimate of I_1. We have

$$\big\{\nabla\Delta^{-1} \text{ div } (\varphi\overline{\mathbf{w}})\big\} \cdot (\rho\overline{\mathbf{w}}) = M(t)^2\rho\mathbf{W}\cdot\big\{\nabla\Delta^{-1} \text{ div } (\varphi\mathbf{W})\big\}.$$

Recall that

$$0 < R^{-1} < y_1 < R \text{ on the support of } \rho.$$

From this and expression (5.11) for I_1 we conclude that

$$|I_1| \le c\int_{\mathbf{R}} M(t)^2\Big(\int_{\mathbf{R}^2} |\rho(t)\mathbf{W}(t)|\,|\nabla\Delta^{-1} \text{ div } (\varphi(t)\mathbf{W}(t))|\, dy\Big)\, dt.$$

Note that $\rho(t)$ and $\mathbf{W}(t)$ meet all requirements of Proposition 5.6 and admit estimates (5.14), (5.15) with the constant c independent of t. It follows from these estimate and the Hölder inequality that

$$\int_{\mathbf{R}^2} |\rho(t)\mathbf{W}(t)|\,|\nabla\Delta^{-1} \text{ div } (\varphi(t)\mathbf{W}(t))|\, dy \le$$

$$\|\rho(t)\mathbf{W}(t)\|_{L^{1+\frac{\varepsilon}{2}}(\mathbb{R}^2)}\,\|\nabla\Delta^{-1} \text{ div } (\varphi(t)\mathbf{W}(t)\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^8}.$$

This gives the desired estimate

$$|I_1| \le \frac{c}{\varepsilon^6}\int_{\mathbf{R}} M(t)^2\,dt \le \frac{c}{\varepsilon^6}\|\overline{\mathbf{w}}\|^2_{L^2(\mathbb{R};W^{1,2}(\mathbb{R}^2))} \le \frac{c}{\varepsilon^8}.$$

Estimate of I_2. Arguing as before we obtain

$$|I_2| \le c\int_{\mathbf{R}} M(t)^2\Big(\int_{\mathbf{R}^2}\rho(t)|\mathbf{W}(t)|^2\,|\nabla^2\Delta^{-1}\varphi(t)|\,dy\Big)\,dt.$$

Estimates (5.14) and (5.15) in Proposition 5.6 imply the inequality

$$\int_{\mathbf{R}^2}\rho(t)|\mathbf{W}(t)|^2\,|\nabla^2\Delta^{-1}\varphi(t)|\,dy \le$$
$$\|\rho(t)|\mathbf{W}(t)^2\|_{L^{1+\frac{\varepsilon}{2}}(\mathbb{R}^2)}\,\|\nabla^2\Delta^{-1}\varphi(t)\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^8}.$$

Thus we get

$$|I_2| \le \frac{c}{\varepsilon^6}\int_{\mathbf{R}} M(t)^2\,dt \le \frac{c}{\varepsilon^6}\|\overline{\mathbf{w}}\|^2_{L^2(\mathbb{R};W^{1,2}(\mathbb{R}^2))} \le \frac{c}{\varepsilon^8}.$$

Estimate of I_3.Note that the function ζ is compactly supported in a compact subset of $G\times(0,T)$. Moreover, this subset depends only on the compact set K in Proposition 5.3. It follows from this and estimate (2.9) in this proposition that

$$\|\zeta\mathbf{B}\|_{L^2(\mathbb{R}^3)} \le c. \tag{5.21}$$

On the other hand the inequality

$$\varphi(\rho(t))^2 = \ln(1+\delta\rho(t))^2 \le c\rho(t)$$

along with estimate (5.20) implies

$$\|\varphi(t)\|_{L^2(\mathbb{R}^2)} \le c.$$

From this and estimate (8.5) for the Riesz singular integral in Appendix 8 we obtain

$$\|\nabla^2\Delta^{-1}\varphi(t)\|_{L^2(\mathbb{R}^2)} \le c,$$

where c is independent on t. Since φ is compactly supported in $G\times(0,T)$ it follows that

$$\|\nabla^2\Delta^{-1}\varphi\|_{L^2(\mathbb{R}^3)} \le c. \tag{5.22}$$

Combining estimates (5.21) and (5.22) we arrive at desired estimate for I_3,

$$|I_3| \le \|\zeta\mathbf{B}\|_{L^2(\mathbb{R}^3)}\,\|\nabla^2\Delta^{-1}\varphi\|_{L^2(\mathbb{R}^3)} \le c.$$

Estimate of I_4. Since $\varphi(t)$ meet all requirements of Proposition 5.6 and is compactly supported in $G\times(0,T)$, it follows from estimate (5.16) in Proposition 5.6 that

$$\|\nabla\Delta^{-1}\varphi\|_{L^\infty(\mathbb{R}^3)} \le c \tag{5.23}$$

From this, estimate (3.6) for $\mathbf{F}$, and expression (5.11) for I_4 we obtain the needed estimate

$$|I_4| \le \|\mathbf{F}\|_{L^1(\mathbb{R}^3)}\,\|\nabla\Delta^{-1}\varphi\|_{L^\infty(\mathbb{R}^3)} \le c.$$

Estimate of I_5. Rewrite formula (5.11) for I_5 in the equivalent form

$$I_5 = -\int_{\mathbb{R}} M(t)^2\Big(\int_{\mathbb{R}^2} (\nabla\Delta^{-1}\boldsymbol{\pi}^*)\cdot(\rho\mathbf{W})y_1 dy\Big)\,dt.$$

It follows that

$$|I_5| \le \int_{\mathbb{R}} M(t)^2\Big(\int_{\mathbb{R}^2} |\nabla\Delta^{-1}\boldsymbol{\pi}^*|\,|\rho\mathbf{W}|dy\Big)\,dt.$$

It follows from estimates (5.14) and (5.16) that

$$\int_{\mathbb{R}^2} |\nabla\Delta^{-1}\boldsymbol{\pi}^*(t)|\,|\rho(t)\mathbf{W}(t)|dy \le$$
$$\|\rho(t)\mathbf{W}(t)\|_{L^{1+\frac{\varepsilon}{2}}(\mathbb{R}^2)}\,\|\nabla\Delta^{-1}\boldsymbol{\pi}^*\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^6}.$$

This gives the desired estimate

$$|I_5| \le \frac{c}{\varepsilon^6}\int_{\mathbf{R}} M(t)^2\,dt \le \frac{c}{\varepsilon^6}\|\overline{\mathbf{w}}\|^2_{L^2(\mathbb{R};W^{1,2}(\mathbb{R}^2))} \le \frac{c}{\varepsilon^6}.$$

□

Step 3. Proof of Theorem 5.3 We are now in a position to complete the proof of the auxiliary Theorem 5.3. We begin with the observation that

$$0 < R^{-1} \le y_1 \le R \ \text{ in the support of } \ \rho.$$

Here the constant R depends only on the compact set K in Theorem 5.3. It follows from this, inequality (5.10) in Proposition 5.4, and estimates (5.18) in Lemma 5.7 that

$$\int_{\mathbb{R}^3} \delta^3\rho^3\ln(1+\delta\varrho)\,dydt = \int_{\mathbb{R}^3} p_2\varphi\,dydt \le c\sum_{i=1}^{5} I_i \le \frac{c}{\varepsilon^8}, \tag{5.24}$$

where c depends only on K. Let us prove that there is a constant c, independent of ε and δ, such that

$$\int_{\mathbb{R}^3} \delta^3\rho^3\,dydt \le c\Big(\delta^{3/8} + \frac{1}{\ln(1+\delta^{-1/6})}\,\frac{1}{\varepsilon^6}\Big). \tag{5.25}$$

To this end, choose two arbitrary numbers m and N satisfying the following conditions

$$m, N \ge 1, \quad \frac{N}{\delta} \ge m.$$

Introduce two sets $A_N \subset A_m \subset G\times(0,T)$ defined by the equalities

$$\begin{aligned} A_m &= \{\,(y,t):\ \rho(y,t)\ge m\,\},\\ A_N &= \{\,(y,t):\ \delta\rho(y,t)\ge N\,\}.\end{aligned}$$

We have

$$\begin{aligned}\int_{\mathbb{R}^3}(\delta\rho)^3\,dydt &= \int_{(G\times(0,T))\setminus A_m}(\delta\rho)^3\,dydt \\ &+ \int_{A_m\setminus A_N}(\delta\rho)^3\,dydt + \int_{A_N}(\delta\rho)^3\,dydt.\end{aligned} \tag{5.26}$$

It is easy to see that

$$\int_{(G\times(0,T))\setminus A_m} (\delta\rho)^3\, dydt \le \delta^3 m^3\, T \,\text{meas}\; G \le c\delta^3 m^3. \tag{5.27}$$

Obviously we have

$$(\delta\rho)^3 \le \frac{1}{\ln(1+N)}\,(\delta\rho)^3\,\ln(1+\delta\rho) \;\text{ in }\; A_N.$$

From this and estimate (5.24) we conclude that

$$\int_{A_N} (\delta\rho)^3\, dydt \le c\,\frac{1}{\ln(1+N)}\,\frac{1}{\varepsilon^8}. \tag{5.28}$$

Next, the Chebyshev inequality and the energy estimate imply

$$\text{meas}\; A_m \le \frac{1}{m}\int_{\mathbb{R}^3} \rho(y,t)\, dydt \le \frac{c}{m}.$$

Noting that $\delta\rho \le N$ in $A_m \setminus A_N$ we obtain

$$\int_{A_m\setminus A_N} (\delta\rho)^3\, dydt \le c\,\frac{N^3}{m}. \tag{5.29}$$

Estimates (5.27)-(5.29) along with equality (5.26) imply

$$\int_{\mathbb{R}^3} (\delta\rho)^3\, dydt \le c\Big(\,\delta^3 m^3 + \frac{N^3}{m} + \frac{1}{\ln(1+N)}\frac{1}{\varepsilon^8}\,\Big).$$

Substituting $m = \delta^{-5/6}$, $N = \delta^{-1/9}$ in this inequality we obtain desired estimate (5.25). It remains to note that for $\delta \le \exp(-1/\varepsilon^9)$, inequality (5.25) implies the estimate

$$\begin{aligned}\int_{\mathbb{R}^3} \delta^3\rho^3\, dydt \;&\le\; c\Big(\,\delta^{1/2} + \frac{1}{\ln(1+\delta^{-1/9})}\,\frac{1}{\varepsilon^8}\,\Big)\\ &\le\; ce^{-\frac{1}{2\varepsilon^9}} + \frac{1}{\ln(\,e^{\frac{1}{9\varepsilon^9}}\,)}\,\frac{c}{\varepsilon^6} \le c\varepsilon.\end{aligned}$$

This completes the proof of Theorem 5.3.

5.2. Proof of Theorems 5.1 and 5.2..

Proof of Theorem 5.2. Introduce the sequence of compact sets $K_n \subset G\times(0,T)$, $n \ge 1$ such that

$$K_n \subset K_{n+1}, \quad \bigcup_n K_n = G\times(0,T).$$

For every n, choose a function $\zeta_n \in C_0^\infty(\mathbb{R}^3)$ with the properties

$$\text{the support of }\; \zeta_n \Subset G\times(0,T), \quad 0 \le \zeta_n \le 1, \quad \zeta_n = 1 \;\text{ on }\; K_n.$$

Set

$$\rho_{n,\varepsilon} := \zeta_n\varrho_\varepsilon, \;\text{ where }\; \varrho_\varepsilon = \varrho_{\varepsilon,\kappa(\varepsilon)}, \quad \kappa(\varepsilon) = e^{-\frac{1}{\varepsilon^9}}.$$

It is easy to see that the function

$$\zeta_n(y,t)\varrho_{\varepsilon,\delta}(y,t)$$

meets all requirements of Theorem 5.3 with ζ and K replaced by ζ_n and K_n. It follows from this and estimate (5.7) in Theorem 5.3 that

$$\int_{\mathbb{R}^3} p_2(\rho_{n,\varepsilon}(y,t))\,dydt \le c_n\varepsilon \quad \text{for all} \quad \varepsilon\in(0,1/2]. \tag{5.30}$$

for all

$$\varepsilon\in(0,1/2],\quad 0<\delta\le\kappa(\varepsilon).$$

Here the constant c_n is independent of ε and δ.

Now choose an arbitrary compact set $K\subset G\times(0,T)$. It is clear that $K\subset K_n$ for some $n\ge 1$. Hence

$$\varrho_\varepsilon=\rho_{n,\varepsilon}\quad\text{on}\quad K.$$

Since the functions p_1 and p_2 are nonnegative, we have

$$\int_K p_1(\varrho_\varepsilon)\,dydt+\int_K p_2(\varrho_\varepsilon)\,dydt\le\int_{\mathbb{R}^3} p_1(\rho_{n,\varepsilon})\,dydt+\int_{\mathbb{R}^3} p_2(\rho_{n,\varepsilon})\,dydt.$$

Therefore, it suffices to prove that

$$\int_{\mathbb{R}^3} p_1(\rho_{n,\varepsilon})\,dydt+\int_{\mathbb{R}^3} p_2(\rho_{n,\varepsilon})\,dydt\to 0\quad\text{as}\quad\varepsilon\to 0. \tag{5.31}$$

By virtue of (5.30), we have

$$\int_{\mathbb{R}^3} p_1(\rho_{n,\varepsilon})\,dydt+\int_{\mathbb{R}^3} p_2(\rho_{n,\varepsilon})\,dydt\le\int_{\mathbb{R}^3} p_1(\rho_{n,\varepsilon})\,dydt+c_n\varepsilon. \tag{5.32}$$

Consider the integral in the right hand side of this inequality in more details. *By abuse of notation, we will write ϱ, ζ, and ρ instead of ϱ_ε, ζ_n, and $\rho_{n,\varepsilon}$.*

Note that the function $\varrho(y,t)$ meets all requirements of Proposition 3.1. In particular, it admits the estimate

$$\operatorname*{ess\,sup}_{t\in\mathbb{R}}\int_{\mathbb{R}^2}\zeta(y,t)\;e_1(\varrho)\,dydt\le C(K).$$

Recall that ζ is compactly supported in the bounded domain $G\times(0,T)$. Since

$$\rho=\zeta\varrho,\quad 0\le\zeta\le 1,$$

we have

$$\zeta e_1(\varrho)=\zeta\varepsilon\varrho^{1+\varepsilon}\ln(1+\varepsilon\varrho)\ge\varepsilon\rho^{1+\varepsilon}\ln(1+\varepsilon\rho)=e_1(\rho),$$

which yields the estimate

$$\text{ess}\quad\sup_{t\in\mathbb{R}}\int_{\mathbb{R}^2} e_1(\rho)\,dydt\le C(K). \tag{5.33}$$

Introduce the function $\lambda(y,t)$ defined by the equalities

$$\lambda=\varepsilon\,\rho,\quad \rho=\lambda/\varepsilon.$$

Formulae (5.1) and (5.2) implies

$$p_1(\rho)=\varepsilon\,e_1(\rho)+\varepsilon\left(\frac{\lambda}{\varepsilon}\right)^{1+\varepsilon}\frac{\lambda}{1+\lambda}=\varepsilon\,e_1(\rho)+\varepsilon^{-\varepsilon}\,\frac{\lambda^{2+\varepsilon}}{1+\lambda}.$$

From this, estimate (5.33), and the inequality $\varepsilon^{-\varepsilon} \le c$ we conclude that the estimate

$$\int_{\mathbb{R}^2} p_1(\rho(t))\,dy \le c\varepsilon + c\int_G \frac{\lambda(t)^{2+\varepsilon}}{1+\lambda(t)}\,dy \tag{5.34}$$

holds for a.e. $t \in \mathbb{R}$. Recall that ρ and λ are compactly supported in $G\times(0,T)$. Let us evaluate the integral in the right hand side. We split it into three parts

$$\int_{\mathbb{G}} \frac{\lambda(y,t)^{2+\varepsilon}}{1+\lambda(y,t)}\,dy = J_1(t) + J_2(t) + J_3(t), \tag{5.35}$$

where

$$\begin{aligned}
J_1(t) &= \int_{G^-(t)} \frac{\lambda(y,t)^{2+\varepsilon}}{1+\lambda(y,t)} dy,\\
J_2(t) &= \int_{G^+(t)} \frac{\lambda(y,t)^{2+\varepsilon}}{1+\lambda(y,t)} dy\\
J_3(t) &= \int_{G\setminus(G^-(t)\cup G^+(t))} \frac{\lambda(y,t)^{2+\varepsilon}}{1+\lambda(y,t)} dy,
\end{aligned}$$

$$G^-(t) = \{\, y\in G:\ \lambda(y,t) \le m^{-1}\,\}, \quad G^+(t) = \{\, y\in G:\ \lambda(y,t)\ge m\,\},$$

$m > 2$ is an arbitrary number. It is easily seen that

$$J_1(t) \le cm^{-2}\int_G dy = c\,m^{-2}. \tag{5.36}$$

For every $\lambda \ge m$, we have

$$\frac{\lambda^{2+\varepsilon}}{1+\lambda} \le \lambda^{1+\varepsilon}\ln(1+\lambda)\frac{1}{\ln(1+m)}.$$

Next, we have

$$\lambda^{1+\varepsilon}\ln(1+\lambda) = \varepsilon^{\varepsilon}\varepsilon\rho^{1+\varepsilon}\ln(1+\varepsilon\rho) = \varepsilon^{\varepsilon} e_1(\rho) \le c\,e_1(\rho).$$

From this we conclude that

$$J_2(t) = \int_{G^+(t)} \frac{\lambda(y,t)^{2+\varepsilon}}{1+\lambda(y,t)} dy \;\le\; \frac{c}{\ln(1+m)}\int_G e_1(\rho(y,t))\,dy \;\le\; \frac{c}{\ln(1+m)}. \tag{5.37}$$

It remains to estimate $J_3(t)$. To this end, note that the estimate

$$\int_G \rho(y,t)\,dy \;\le\; c$$

and the Chebyshev inequality imply

$$\begin{aligned}
J_3(t) &\le \int_{m^{-1}\le\lambda\le m} \frac{\lambda(y,t)^{2+\varepsilon}}{1+\lambda(y,t)} dy \le m^{1+\varepsilon}\int_{m^{-1}\le\lambda} dy \qquad (5.38)\\
&= m^{1+\varepsilon}\int_{(\varepsilon m)^{-1}\le\rho} dy \le c\,m^{1+\varepsilon}(\varepsilon m) \;\le\; c\,\varepsilon\, m^{2+\varepsilon}.
\end{aligned}$$

Substituting (5.36)-(5.37) in (5.35) leads to the inequality

$$\int_G \frac{\lambda(y,t)^{2+\varepsilon}}{1+\lambda(y,t)}\,dy \le c\,m^{-2} + \frac{c}{\ln(1+m)} + c\,\varepsilon\,m^{2+\varepsilon}.$$

Note that the constant c does not depend on t, m, δ, and ε. Since $\lambda(y,t)$ is compactly supported in $G\times(0,T)$, we conclude that

$$\int_{\mathbb{R}^2} \frac{\lambda(y,t)^{2+\varepsilon}}{1+\lambda(y,t)}\,dy \le c\,m^{-2} + \frac{c}{\ln(1+m)} + c\,\varepsilon\,m^{2+\varepsilon}.$$

Since $p_1(\rho)$ is compactly supported in $G\times(0,T)$, itt follows from this and (5.34) that

$$\int_{\mathbb{R}^2} p_1(\rho(t))\,dy \le c\varepsilon + c\,m^{-2} + \frac{c}{\ln(1+m)} + c\,\varepsilon\,m^{2+\varepsilon}.$$

Combining this inequality with (5.32) and recalling that $\rho=\rho_{n,\varepsilon}$ we arrive at the estimate

$$\int_{\mathbb{R}^3} p_1(\rho_{n,\varepsilon})\,dydt + \int_{\mathbb{R}^3} p_2(\rho_{n,\varepsilon})\,dydt \le c\varepsilon + c\,m^{-2} + \frac{c}{\ln(1+m)} + c\,\varepsilon\,m^{2+\varepsilon}.$$

Letting $\varepsilon\to 0$ we obtain

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^3} (p_1(\rho_{n,\varepsilon}) + p_2(\rho_{n,\varepsilon})\,dydt \le c\,m^{-2} + \frac{c}{\ln(1+m)}.$$

Letting $m\to\infty$ we arrive at desired relation (5.31). This completes the proof of Theorem 5.2.

Proof of Theorem 5.1. Theorem 5.1 is a straightforward consequence of Theorem 5.2.

6 Proof of Theorem 1.5

6.1. Part 1. Weak limits. We obtain a solution to isothermal compressible Navier- Stokes equations (1.22) as a weak limit of solutions $\varrho_{\varepsilon,\delta}=\varrho$, $\mathbf{u}_{\varepsilon,\delta}=\mathbf{u}$ to regularized equations (1.15). Note that these solutions are defined by Proposition 1.3. Set

$$\varrho_\varepsilon = \varrho_{\varepsilon,\kappa(\varepsilon)}, \quad \mathbf{u}_\varepsilon = \mathbf{u}_{\varepsilon,\kappa(\varepsilon)}, \quad \kappa(\varepsilon) = e^{-\frac{1}{\varepsilon^9}}. \tag{6.1}$$

In this subsection, we study the weak compactness properties of sequences ϱ_ε and $\mathbf{u}_\varepsilon$. It follows from estimates (1.16)-(1.18) in Proposition 1.3 that

$$\|\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} \le c. \tag{6.2}$$

$$\begin{aligned}\|\varrho_\varepsilon \ln(2+\varrho_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \quad &+ \quad \|e_1(\varrho_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \\ +\|e_2(\varrho_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \quad &\le \quad c.\end{aligned} \tag{6.3}$$

$$\|p(\varrho_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} + \|p_1(\varrho_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} + \|p_2(\varrho_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le c. \tag{6.4}$$

Here the constant c depends only on Ω, T, and the constant c_E in condition (1.14) of Proposition 1.3. Our further considerations are based on the following lemma.

Lemma 6.1. *The estimate*

$$\|\varrho_\varepsilon|\mathbf{u}_\varepsilon|\ln(2+\varrho_\varepsilon|\mathbf{u}_\varepsilon|)^\alpha\|_{L^\infty(0,T;L^1(\Omega))}\leq c \tag{6.5}$$

holds for every $\alpha<1/2$. Here the constant c depends only on α and the constants in estimates (6.2)-(6.4).

Proof. The proof is given in Appendix 12.

□

Estimates (6.3) and (6.5) imply that the sequences ϱ_ε and $\varrho_\varepsilon\mathbf{u}_\varepsilon$ are equi-integrable. It follows from this and the Dunford-Pettis Theorem (see [8], thm. 2.54) that there are subsequences, still denoted by ϱ_ε, $\varrho_\varepsilon\mathbf{u}_\varepsilon$, and functions ϱ, $\overline{\varrho\mathbf{u}}$ such that

$$\varrho_\varepsilon\to\varrho,\quad \varrho_\varepsilon\mathbf{u}_\varepsilon\to\overline{\varrho\mathbf{u}}\ \text{ weakly in }\ L^1(Q_T)\ \text{ as }\ \varepsilon\to 0. \tag{6.6}$$

In particular, for every measurable set $\mathcal{T}\subset(0,T)$, we have

$$\begin{aligned}\int_{\Omega\times\mathcal{T}}\varrho\,dxdt &= \lim_{\varepsilon\to 0}\int_{\Omega\times\mathcal{T}}\varrho_\varepsilon\,dxdt\\ \leq \text{ meas }\mathcal{T}\limsup_{\varepsilon\to 0}\|\varrho_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} &\leq c\text{ meas }\mathcal{T},\end{aligned}$$

$$\begin{aligned}\Big|\int_{\Omega\times\mathcal{T}}\overline{\varrho\mathbf{u}}\,dxdt\Big| &= \lim_{\varepsilon\to 0}\Big|\int_{\Omega\times\mathcal{T}}\varrho_\varepsilon\mathbf{u}_\varepsilon\,dxdt\Big|\\ \leq \text{ meas }\mathcal{T}\limsup_{\varepsilon\to 0}\|\varrho_\varepsilon\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} &\leq c\text{ meas }\mathcal{T}.\end{aligned}$$

It follows from this that $\varrho,\overline{\varrho\mathbf{u}}\in L^\infty(0,T;L^1(\Omega))$. By virtue of estimate (6.2), there is a subsequence, still denoted by $\mathbf{u}_\varepsilon$, and a vector field $\mathbf{u}\in L^2(0,T;W^{1,2}_0(\Omega))$ such that

$$\mathbf{u}_\varepsilon\to\mathbf{u}\ \text{ weakly in }\ L^2(0,T;W^{1,2}(\Omega))\ \text{ as }\ \varepsilon\to 0. \tag{6.7}$$

Let us consider the sequence K_n, $n\geq 1$, of compact subsets of $(\Omega\setminus\mathcal{I})\times(0,T)$ such that

$$K_1\subset\cdots\subset K_n\subset K_{n+1}\subset\cdots\subset(\Omega\setminus\mathcal{I})\times(0,T),\quad \bigcup_{n\geq 1}K_n=(\Omega\setminus\mathcal{I})\times(0,T).$$

By virtue of Proposition 3.2, the sequences ϱ_ε and $\mathbf{u}_\varepsilon$ satisfy conditions (**C.1**)-(**C.3**) of Theorem 4.1. It follows from this theorem that for every $n\geq 1$ and $\alpha\in(0,1/4)$,

$$\int_{K_n}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\,\ln(2+\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2)\,dxdt\leq c_n,$$

where c_n is independent of ε. It follows that the sequence $\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2$ is equi-integrable on every compact set $K \subset (\Omega \setminus \mathcal{I}) \times (0,T)$. Using the Dunford-Pettis Theorem and a diagonal procedure we may extract a subsequuence of $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ (not relabeled) such that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u} \otimes \mathbf{u}} \quad \text{weakly in} \quad L^1(K) \quad \text{as} \quad \varepsilon \to 0 \tag{6.8}$$

for every compact set $K \subset (\Omega \setminus \mathcal{I}) \times (0,T)$. Here $\overline{\varrho \mathbf{u} \otimes \mathbf{u}} \in L^1(Q_T)$ is a matrix valued function.

Let us consider the weak compactness properties of the sequences of the pressure functions $p(\varrho_\varepsilon)$ and kinetic energy tensors $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$. To this end, recall briefly the basic facts concerning L^p spaces on Banach spaces.

Let Y be a Banach space and let Y' be a Banach space dual to Y. A function $\nu : (0,T) \to Y'$ is said to be weakly star measurable if for any $L \in Y$, the scalar function

$$t \in (0,T) \mapsto \langle\, \nu(t)\,,\, L \,\rangle \quad \text{is measurable}\ .$$

Further the notation $L^r_w(0,T;Y')$, $1 \le r < \infty$, stands for the Banach space of weakly star measurable functions with the finite norm

$$\|\nu\|_{L^r_w(0,T;Y')} = \Big(\int_0^T \|\nu(t)\|^r_{Y'}\, dt \Big)^{1/r}.$$

If a Banach space Y is separable, then every weakly measurable mapping $\varphi : (0,T) \to Y$ is strongly measurable. In this case, $L^r(0,T;Y)$ is the standard space of functions integrable in the sense of Bochner. This leads to the following lemma (see [8], thm. 2.112)

Lemma 6.2. *Assume that Y is separable. If $L \in \big(L^r(0,T;Y)\big)'$, $1 < r < \infty$, then there exists a unique $\nu \in L_w^{\frac{r}{r-1}}(0,T;Y')$ such that*

$$\langle\, L\,,\, \varphi \,\rangle = \int_0^T \langle\, \nu(t),\, \varphi(t) \,\rangle\, dt \tag{6.9}$$

for every $\varphi \in L^r(0,T;Y)$. Moreover,

$$\|L\|_{\big(L^r(0,T;Y)\big)'} = \|\nu\|_{L_w^{\frac{r}{r-1}}(0,T;Y')}.$$

In particular, we have

$$L^r_w(0,T;Y') \,=\, \big(\, L^{\frac{r}{r-1}}(0,T;C_0(\Omega))\,\big)' \quad \textit{for} \quad 1 \le r < \infty.$$

Denote by $\mathbb{M}(\Omega)$ the Banach space of all Radon measures in Ω. Note that $\mathbb{M}(\Omega) = C_0(\Omega)'$ is dual to the space $C_0(\Omega)$ of continuous functions $u : \overline{\Omega} \to \mathbb{R}$ vanishing at $\partial\Omega$. By the Riesz representation theorem (see [8], thm. 1.195), for every $\nu \in \mathbb{M}(\Omega)$ there is a unique signed Borel measure ν^* defined on the algebra of Borel subsets of Ω such that

$$\langle\, \nu\,,\, \varphi \,\rangle \,=\, \int_\Omega \varphi\, d\nu^*.$$

for all $\varphi \in C_0$. The norm of ν admits the representation

$$\|\nu\|_{\mathbb{M}(\Omega)} = \|\nu^*\|(\Omega),$$

where $\|\nu^*\|(\Omega)$ is the total variation of a signed measure ν^* over Ω. Further we will identify ν and ν^*.

Every function $g \in L^1(\Omega)$ determines a unique Radon measure

$$d\nu = g\,dx, \quad \langle \nu\,,\,\varphi \rangle \;=\; \int_\Omega \varphi\, g dx \;\text{ for all }\; \varphi \in C_0(\Omega). \tag{6.10}$$

It is clear that $\|\nu\|_{\mathbb{M}(\Omega)} = \|g\|_{L^1(\Omega)}$. Relation (6.10) defines the isometry

$$L^r(0,T;L^1(\Omega)) \hookrightarrow L^r_w(0,T;\mathbb{M}(\Omega)).$$

In particular, L^1-functions $p(\varrho_\varepsilon(t))$ and $\varrho_\varepsilon(t)u_{\varepsilon,i}(t)u_{\varepsilon,j}(t)$ determine the Radon measures $\sigma_\varepsilon(t)$ and $\nu_{\varepsilon,ij}$ given by the equalities

$$\int_\Omega \varphi\, d\sigma_\varepsilon(t) = \int_\Omega \varphi\, p(\varrho_\varepsilon)(t)\, dx, \quad \int_\Omega \varphi\, d\nu_{\varepsilon,ij} = \int_\Omega \varphi\, \varrho_\varepsilon u_{\varepsilon,i} u_{\varepsilon,j}\, dx$$

for all $\varphi \in C_0(\Omega)$. It follows from estimates (6.2) and (6.3) that the sequences $\sigma_\varepsilon(t)$ and $\nu_{\varepsilon,ij}$ are bounded in every space $L^r_w(0,T;\mathbb{M}(\Omega))$, $1 \le r \le \infty$.

Now fix an arbitrary $r \in (1,\infty)$ and note that the space $L^{\frac{r}{r-1}}(0,T;C_0(\Omega))$ is separable. From this and Lemma 6.2 we conclude that

$$L^r_w(0,T;\mathbb{M}(\Omega)) = \big(\, L^{\frac{r}{r-1}}(0,T;C_0(\Omega))\,\big)'$$

By virtue of the Alaoglu theorem, there are subsequences of the sequences $\sigma_\varepsilon(t)$, $\nu_{\varepsilon,ij}$, still denoted by $\sigma_\varepsilon(t)$, $\nu_{\varepsilon,ij}$, and the parametrized measures σ, $\nu_{ij} \in L^r_w(0,T;\mathbb{M}(\Omega))$ such that

$$\sigma_\varepsilon \to \sigma, \quad \nu_{\varepsilon,ij} \to \nu_{ij} \;\text{ weakly star in }\; L^r_w(0,T;\mathbb{M}(\Omega)) \;\text{ as }\; \varepsilon \to 0.$$

This means that

$$\begin{aligned} \int_{Q_T} p(\varrho_\varepsilon)\varphi\, dxdt \;&\to\; \int_0^T \Big(\int_\Omega \varphi\, d\sigma(t) \Big) dt, \\ \int_{Q_T} \varrho_\varepsilon u_{\varepsilon,i} u_{\varepsilon,j}\, \varphi\, dxdt \;&\to\; \int_0^T \Big(\int_\Omega \varphi\, d\nu_{ij}(t) \Big) dt \end{aligned} \tag{6.11}$$

for all $\varphi \in L^{\frac{r}{r-1}}(0,T;C_0(\Omega))$.

Recall that ϱ_ε and $\mathbf{u}_\varepsilon$ are weak solutions to boundary value problem (1.3). They satisfy integral identities (1.8) and (1.9) with the pressure function

$$\begin{aligned} p(\varrho) &= \varrho + p_1(\varrho) + p_2(\varrho), \\ p_1(\varrho) &= \varepsilon \varrho^{1+\varepsilon}\big(\varepsilon \ln(1+\varepsilon\varrho) + \frac{\varepsilon\varrho}{1+\varepsilon\varrho}\big), \quad p_2(\varrho) = \kappa(\varepsilon)^3 \varrho^3. \end{aligned}$$

Substituting ϱ_ε, $\mathbf{u}_\varepsilon$ into (1.8) and (1.9), letting $\varepsilon \to 0$, and using relations (6.6), (6.7), and (6.11) we arrive at the integral identities

$$\begin{aligned}\int_{Q_T} \overline{\varrho\mathbf{u}} \cdot \partial_t \boldsymbol{\xi}\, dxdt + \int_0^T \Big(\int_\Omega \nabla\boldsymbol{\xi} : d\boldsymbol{\nu}(t) \Big)\, dt \\ + \int_0^T \Big(\int_\Omega \operatorname{div} \boldsymbol{\xi}\, d\sigma(t) \Big)\, dt - \int_{Q_T} \mathbb{S}(\mathbf{u}) : \nabla\boldsymbol{\xi}\, dxdt \\ + \int_{Q_T} \varrho\mathbf{f} \cdot \boldsymbol{\xi}\, dxdt + \int_\Omega \varrho_0 \mathbf{u}_0(x) \cdot \boldsymbol{\xi}(x,0)\, dx = 0,\end{aligned} \tag{6.12}$$

$$\int_{Q_T} \big(\varrho\partial_t\psi + \overline{\varrho\mathbf{u}} \cdot \nabla\psi\big)\, dxdt + \int_\Omega \varrho_0(x)\psi(x,0)\, dx = 0. \tag{6.13}$$

Here $\boldsymbol{\nu}(t) = \big(\nu_{ij}(t)\big)_{1\le i,j\le 3}$ is a symmetric matrix-valued Radon measure in Ω.

Identities (6.12) and (6.13) hold for all vector fields $\boldsymbol{\xi} \in C^\infty(Q_T)$ vanishing in a neighborhood of $\partial\Omega \times [0,T]$ and $\Omega \times \{t = T\}$, and for all functions $\psi \in C^\infty(Q_T)$ vanishing in a neighborhood of $\Omega \times \{t = T\}$.

6.2. Part 2. Weak limits of the momentum and kinetic energy tensor. In this subsection we study in details the vector field $\overline{\varrho\mathbf{u}}$ and the measures σ, $\boldsymbol{\nu}$. The following lemma is the first result in this direction.

Proposition 6.3. *Under the assumptions of Theorem 1.5, we have*

$$\overline{\varrho\mathbf{u}} = \varrho\mathbf{u} \quad in \quad Q_T, \tag{6.14}$$

$$\overline{\varrho\mathbf{u} \otimes \mathbf{u}} = \varrho\mathbf{u} \otimes \mathbf{u} \quad in \quad Q_T. \tag{6.15}$$

In order to avoid repetition, we prove the relation (6.14). The proof of (6.15) is similar. The proof of (6.14) falls into three steps.

Step1. By virtue of the mass balance equation (1.15), the density function ϱ_ε satisfies the integral identity

$$\int_{Q_T} \varrho_\varepsilon\, \partial_t\psi\, dxdt = -\int_{Q_T} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\psi\, dxdt$$

for every $\psi \in C_0^\infty(Q_T)$. It follows from the energy estimates (6.2), (6.3), and the embedding $W_0^{3,2}(\Omega) \hookrightarrow C^1(\Omega)$ that

$$\Big| \int_{Q_T} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\psi\, dxdt \Big| \le c \int_0^T \|\psi(t)\|_{W_0^{3,2}(\Omega)}\, dt.$$

This means that

$$\|\partial_t\varrho_\varepsilon\|_{L^\infty(0,T;W^{-3,2}(\Omega))} \le c \quad\text{and}\quad \|\varrho_\varepsilon\|_{Lip(0,T;W^{-3,2}(\Omega))} \le c.$$

On the other hand, we have

$$\|\varrho_\varepsilon(t)\|_{W^{-2,2}(\Omega)} \le c.$$

Note that the embedding $W^{-2,2}(\Omega) \hookrightarrow W^{-3,2}(\Omega)$ is compact. Hence the sequence $\varrho_\varepsilon : [0,T] \to W^{-3,2}(\Omega)$ meets all requirements of the Arzelà-Ascoli

Theorem (see [4], thm. 7.5.7). Therefore, after passing to a subsequence we may assume that

$$\varrho_\varepsilon \to \varrho \quad \text{in} \quad C(0,T;W^{-3,2}(\Omega)). \tag{6.16}$$

Step 2. Choose an arbitrary integer $N \geq 3$. Introduce the truncated vector field

$$\begin{aligned}
\mathbf{u}_{N,\varepsilon}(x,t) &= \mathbf{u}_\varepsilon(x,t) \quad \text{if} \quad |\mathbf{u}_\varepsilon(x,t)| \leq N,\\
\mathbf{u}_{N,\varepsilon}(x,t) &= \frac{N}{|\mathbf{u}_\varepsilon(x,t)|}\,\mathbf{u}_\varepsilon(x,t) \quad \text{if} \quad |\mathbf{u}_\varepsilon(x,t)| \geq N.
\end{aligned}$$

It is clear that $\mathbf{u}_{N,\varepsilon} \in L^2(0,T;W^{1,2}_0(\Omega))$ and

$$\|\mathbf{u}_{N,\varepsilon}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c, \quad |\mathbf{u}_{N,\varepsilon}| \leq N. \tag{6.17}$$

Using the diagonal procedure we may extract a subsequence (nor relabeled) such that

$$\mathbf{u}_{N,\varepsilon} \to \overline{\mathbf{u}}_N \quad \text{weakly in} \quad L^2(0,T;W^{1,2}(\Omega)) \quad \text{and star weakly in} \quad L^\infty(Q_T). \tag{6.18}$$

The limiting vector field $\overline{\mathbf{u}}_N$ admits estimates (6.18). The following lemma is the main result of this paragraph

Lemma 6.4.

$$\int_{Q_T} \varphi\,\varrho_\varepsilon\,\mathbf{u}_{N,\varepsilon}\,dxdt \to \int_{Q_T} \varphi\,\varrho\,\overline{\mathbf{u}}_N\,dxdt \quad \textit{as} \quad \varepsilon \to 0 \quad \textit{for all} \quad \varphi \in C^\infty_0(Q_T). \tag{6.19}$$

Proof. Introduce the mollifier

$$\mathbf{T}_\omega u(x) = \frac{1}{\omega^3}\int_{\mathbb{R}^3} \vartheta\big(\,\frac{x-y}{\omega}\,\big)\,u(y)\,dy \tag{6.20}$$

where the kernel

$$\theta \in C^\infty_0(\mathbb{R}^3), \quad \text{support } \theta \subset \{|x| \leq 1\}, \quad \int_{\mathbb{R}^3} \theta(x)\,dx = 1.$$

Note that

$$\int_{Q_T} \varphi\,\big(\,\varrho_\varepsilon\,\mathbf{u}_{N,\varepsilon} - \varrho\,\overline{\mathbf{u}}_N\,\big)\,dxdt = \Pi_1 + \Pi_2 + \Pi_3, \tag{6.21}$$

where

$$\begin{aligned}
\Pi_1 &= \int_{Q_T} \varphi\varrho_\varepsilon\,\big(\mathbf{u}_{N,\varepsilon} - \mathbf{T}_\omega\mathbf{u}_{N,\varepsilon}\,\big)\,dxdt,\\
\Pi_2 &= \int_{Q_T} \varphi\varrho\,\big(\mathbf{T}_\omega\overline{\mathbf{u}}_N - \overline{\mathbf{u}}_N\,\big)\,dxdt,\\
\Pi_3 &= \int_{Q_T} \varphi\big(\varrho_\varepsilon\mathbf{T}_\omega\mathbf{u}_{N,\varepsilon} - \varrho\mathbf{T}_\omega\overline{\mathbf{u}}_N\,\big)\,dxdt.
\end{aligned}$$

Note that the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. It follows that

$$\begin{aligned}
\|\mathbf{u}_{N,\varepsilon}(t) - \mathbf{T}_\omega\mathbf{u}_{N,\varepsilon}(t)\|_{L^2(\Omega)} &\leq \delta(\omega)\,\|\mathbf{u}_{N,\varepsilon}(t)\|_{W^{1,2}(\Omega)}, \qquad (6.22)\\
\|\overline{\mathbf{u}}_N(t) - \mathbf{T}_\omega\overline{\mathbf{u}}_N(t)\|_{L^2(\Omega)} &\leq \delta(\omega)\,\|\overline{\mathbf{u}}_N(t)\|_{W^{1,2}(\Omega)},
\end{aligned}$$

where $\delta(\omega) \sim c\omega^{1/2} \to 0$ as $\omega \to 0$. We also have

$$\begin{aligned} \|\mathbf{u}_{N,\varepsilon}(t) - \mathbf{T}_\omega \mathbf{u}_{N,\varepsilon}(t)\|_{L^\infty(\Omega)} &\leq 2N, \\ \|\overline{\mathbf{u}}_N(t) - \mathbf{T}_\omega \overline{\mathbf{u}}_N(t)\|_{L^\infty(\Omega)} &\leq 2N. \end{aligned} \tag{6.23}$$

Fix an arbitrary $m > 3$ and set

$$A_m = \{(x,t) \in Q_T : \ \rho_\varepsilon > m\} \subset Q_T.$$

We have

$$\begin{aligned} |\Pi_1| &\leq \Big| \int_{Q_T \setminus A_m} \varphi \varrho_\varepsilon \big(\mathbf{u}_{N,\varepsilon} - \mathbf{T}_\omega \mathbf{u}_{N,\varepsilon}\big)\, dxdt \Big| \\ &+ \Big| \int_{A_m} \varphi \varrho_\varepsilon \big(\mathbf{u}_{N,\varepsilon} - \mathbf{T}_\omega \mathbf{u}_{N,\varepsilon}\big)\, dxdt \Big|. \end{aligned} \tag{6.24}$$

By virtue of (6.22), the first term in the right hand side admits the estimate

$$\begin{aligned} \Big| \int_{Q_T \setminus A_m} \varphi \varrho_\varepsilon \big(\mathbf{u}_{N,\varepsilon} - \mathbf{T}_\omega \mathbf{u}_{N,\varepsilon}\big)\, dxdt \Big| &\leq cm \int_{Q_T} |\mathbf{u}_{N,\varepsilon} - \mathbf{T}_\omega \mathbf{u}_{N,\varepsilon}|\, dxdt \\ &\leq cm \int_0^T \|\mathbf{u}_{N,\varepsilon}(t) - \mathbf{T}_\omega \mathbf{u}_{N,\varepsilon}(t)\|_{L^2(\Omega)}\, dt \\ &\leq cm\delta(\omega) \int_0^T \|\mathbf{u}_{N,\varepsilon}(t)\|_{W^{1,2}(\Omega)}\, dt \leq cm\delta(\omega). \end{aligned}$$

On the other hand, energy estimate (6.3) and estimate (6.23) imply the inequality

$$\Big| \int_{A_m} \varphi \varrho_\varepsilon \big(\mathbf{u}_{N,\varepsilon} - \mathbf{T}_\omega \mathbf{u}_{N,\varepsilon}\big)\, dxdt \Big| \leq \frac{cN}{\ln m}.$$

Substituting the obtained results into (6.24) we arrive at the estimate

$$|\Pi_1| \leq c\,\big(m\delta(\omega) + \frac{N}{\ln m}\big).$$

Now set $m = 1/(\delta|\ln\delta|)$ and choose ω so small that $\delta(\omega) < 1$. We thus get

$$m\delta(\omega) + \frac{N}{\ln m} = \frac{1}{|\ln\delta(\omega)|} + \frac{N}{|\ln(\delta(\omega)|\ln\delta(\omega)|)|} \leq \frac{cN}{|\ln(\delta(\omega)|\ln\delta(\omega)|)|},$$

which gives

$$|\Pi_1| \leq \frac{cN}{|\ln(\delta(\omega)|\ln\delta(\omega)|)|}. \tag{6.25}$$

Repeating these arguments and using estimates (6.22) and (6.23) for $\overline{\mathbf{u}}_N$ we obtain the similar estimate for Π_2,

$$|\Pi_2| \leq \frac{cN}{|\ln(\delta(\omega)|\ln\delta(\omega)|)|}. \tag{6.26}$$

Note that c is independent on ω. Let us consider the quantity Π_3. We have

$$\Pi_3 = \int_{Q_T} (\varrho_\varepsilon - \varrho)\big(\varphi \mathbf{T}_\omega \mathbf{u}_{N,\varepsilon}\big)\, dxdt + \int_{Q_T} \varrho\varphi \mathbf{T}_\omega \big(\mathbf{u}_{N,\varepsilon} - \overline{\mathbf{u}}_N\big)\, dxdt. \tag{6.27}$$

Recall that the function $\mathbf{u}_{N,\varepsilon}(t) \in W_0^{1,2}(\Omega)$ equals zero outside of Ω. It follows from the properties of the mollifiers that

$$\|\mathbf{T}_\omega \mathbf{u}_{N,\varepsilon}(t)\|_{W^{3,2}(\Omega)} \le c(\omega)\|\mathbf{u}_{N,\varepsilon}\|_{L^2(\Omega)} \le c(\omega)\|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)}.$$

Since $\varphi \in C^\infty(Q_T)$ is compactly supported in the cylinder $Q_T = \Omega \times (0,T)$, the vector field $\varphi(t)\mathbf{u}_{N,\varepsilon}(t) \in W_0^{3,2}(\Omega)$ admits the estimate

$$\|\varphi(t)\mathbf{T}_\omega \mathbf{u}_{N,\varepsilon}(t)\|_{W^{3,2}(\Omega)} \le c(\omega)\|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)}.$$

Here c depends on ω, φ and is independent on t. We thus get

$$\begin{aligned}\Big|\int_{Q_T}(\varrho_\varepsilon-\varrho)\big(\varphi\mathbf{T}_\omega\mathbf{u}_{N,\varepsilon}\big)\,dxdt\Big| &\le \|\varrho_\varepsilon-\varrho\|_{C(0,T;W^{-3,2}(\Omega))}\int_0^T \|\varphi\mathbf{T}_\omega\mathbf{u}_{N,\varepsilon}(t)\|_{W^{3,2}(\Omega)}\,dt\\ &\le c(\omega)\|\varrho_\varepsilon-\varrho\|_{C(0,T;W^{-3,2}(\Omega))}\int_0^T \|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)}\,dt\\ &\le c(\omega)\|\varrho_\varepsilon-\varrho\|_{C(0,T;W^{-3,2}(\Omega))}.\end{aligned}$$

From this and (6.16) we conclude that

$$\int_{Q_T}(\varrho_\varepsilon-\varrho)\big(\varphi\mathbf{T}_\omega\mathbf{u}_{N,\varepsilon}\big)\,dxdt \to 0 \quad \text{as} \quad \varepsilon\to 0. \tag{6.28}$$

Next we have

$$\int_{Q_T}\varrho\varphi\mathbf{T}_\omega\big(\mathbf{u}_{N,\varepsilon}-\overline{\mathbf{u}}_N\big)\,dxdt = \int_{Q_T}\mathbf{T}_\omega(\varrho\varphi)\big(\mathbf{u}_{N,\varepsilon}-\overline{\mathbf{u}}_N\big)\,dxdt.$$

Since the mapping $\mathbf{T}_\omega : L^1(\Omega) \to L^\infty(\Omega)$ is bounded, we have

$$\|\mathbf{T}_\omega(\varrho\varphi)\|_{L^\infty(Q_T)} \le c(\omega).$$

On the other hand,

$$\mathbf{u}_{N,\varepsilon} \to \overline{\mathbf{u}}_N \quad \text{weakly in} \quad L^2(Q_T) \quad \text{as} \quad \varepsilon \to 0. \tag{6.29}$$

It follows that

$$\int_{Q_T}\varrho\varphi\mathbf{T}_\omega\big(\mathbf{u}_{N,\varepsilon}-\overline{\mathbf{u}}_N\big)\,dxdt \to 0 \quad \text{as} \quad \varepsilon\to 0.$$

Combining this relation with (6.27) and (6.28) we conclude that

$$\Pi_3 \to 0 \quad \text{as} \quad \varepsilon \to 0$$

for fixed ω and N. From this, estimates (6.25)-(6.26), and identity (6.21) we obtain

$$\limsup_{\varepsilon\to 0}\Big|\int_{Q_T}\varphi\,\big(\,\varrho_\varepsilon\,\mathbf{u}_{N,\varepsilon}-\varrho\,\overline{\mathbf{u}}_N\,\big)\,dxdt\Big| \le \frac{cN}{|\ln(\delta(\omega)|\ln\delta(\omega)|)|}.$$

Letting $\omega \to 0$ we arrive at the desired relation (6.19). □

Step 3. We are now in a position to complete the proof of Proposition 6.3. The main idea is to consider the truncated sequence $\varrho_\varepsilon \mathbf{u}_{N,\varepsilon}$ as an approximation of the sequence $\varrho_\varepsilon \mathbf{u}_\varepsilon$. Let us estimate the discrepancy

$$\mathbf{D}_1(N,\varepsilon) = \int_{Q_T} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{u}_{N,\varepsilon}|\, dxdt. \tag{6.30}$$

It follows from estimate (6.5) in Lemma 6.1 and De la Vallèe Poussin criterion that the sequence $\varrho_\varepsilon \mathbf{u}_\varepsilon$ is equi-integrable. This means that there is a function $\delta_1 : \mathbb{R}^+ \to \mathbb{R}^+$, independent of ε, such that

$$\lim_{\omega\to 0} \delta_1(\omega) = 0, \quad \int_C \varrho_\varepsilon |\mathbf{u}_\varepsilon|\, dxdt \le \delta_1(\text{ meas } C) \ \text{ for every } \ C \subset Q_T. \tag{6.31}$$

Note that

$$\varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{u}_{N,\varepsilon}| \le 2\varrho_\varepsilon |\mathbf{u}_\varepsilon| \ \text{ in } \ Q_T, \quad \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{u}_{N,\varepsilon}| = 0 \ \text{ in } \ Q_T \setminus C_{N,\varepsilon},$$

where

$$C_{N,\varepsilon} = \big\{\, (x,t) \in Q_T : |\mathbf{u}_\varepsilon(x,t)| > N \,\big\}.$$

We thus get

$$\mathbf{D}_1(N,\varepsilon) \le c \int_{C_{N,\varepsilon}} \varrho_\varepsilon |\mathbf{u}_\varepsilon| dxdt \le c\delta_1(\text{ meas } C_{N,\varepsilon}). \tag{6.32}$$

The Chebyshev inequality and estimate (6.2) imply

$$\text{meas } C_{N,\varepsilon} \le \frac{1}{N^2} \int_{Q_T} |\mathbf{u}_\varepsilon|^2\, dxdt \le \frac{c}{N^2}. \tag{6.33}$$

From this and (6.32) we conclude that

$$\mathbf{D}_1(N,\varepsilon) \le c\delta_1(cN^{-2}). \tag{6.34}$$

Note that we have no information about the limiting vector field $\varrho\mathbf{u}$. Now our task is to prove that it is integrable and

$$\mathbf{D}_2(N) := \int_{Q_T} |\varrho\mathbf{u} - \varrho\overline{\mathbf{u}}_N|\, dxdt \to 0 \ \text{ as } \ N \to \infty. \tag{6.35}$$

By virtue of the estimate (6.5) in Lemma 6.1, we have

$$\int_{Q_T} \varrho_\varepsilon |\mathbf{u}_{N,\varepsilon}| \ln(2 + \varrho_\varepsilon |\mathbf{u}_{N,\varepsilon}|)^\alpha\, dxdt \le \int_{Q_T} \varrho_\varepsilon |\mathbf{u}_\varepsilon| \ln(2 + \varrho_\varepsilon |\mathbf{u}_\varepsilon|)^\alpha\, dxdt \le c$$

for all $0 < \alpha < 1/2$. Therefore, the sequence $\varrho_\varepsilon \mathbf{u}_{N,\varepsilon}$ is equi-integrable. From this and (6.19) we conclude that $\varrho_\varepsilon \mathbf{u}_{N,\varepsilon}$ converge weakly in $L^1(Q_T)$ to $\varrho\overline{\mathbf{u}}_N$. Choose an arbitrary measurable $C \subset Q_T$ and set

$$\boldsymbol{\varsigma} = \frac{\overline{\mathbf{u}}_N(x,t)}{|\overline{\mathbf{u}}_N(x,t)|}\, \mathbf{1}_{\boldsymbol{C}} \ \text{ for } \ |\overline{\mathbf{u}}_N(x,t)| > 0 \ \text{ and } \ \boldsymbol{\varsigma} = 0 \ \text{ otherwise.}$$

It is clear that the measurable function $\boldsymbol{\varsigma}$ satisfies the inequality $|\boldsymbol{\varsigma}| \leq 1$ and is supported in C. It follows from this and inequality (6.31) that

$$\begin{aligned} \delta_1(\text{meas C}) &\geq \int_C \varrho_\varepsilon |\mathbf{u}_\varepsilon| dxdt \geq \int_C \varrho_\varepsilon |\mathbf{u}_{N,\varepsilon}| dxdt \\ &\geq \int_{Q_T} \varrho_\varepsilon \mathbf{u}_{N,\varepsilon} \cdot \boldsymbol{\varsigma} dxdt \to \int_{Q_T} \varrho \overline{\mathbf{u}}_N \cdot \boldsymbol{\varsigma} dxdt \\ &= \int_C \varrho |\overline{\mathbf{u}}_N| dxdt. \end{aligned}$$

It follows that

$$\int_C \varrho |\overline{\mathbf{u}}_N| dxdt \leq \delta_1(\text{ meas } C) \text{ for all measurable } C \subset Q_T.$$

Hence the sequence $\varrho\overline{\mathbf{u}}_N$, $N \geq 3$, is equi-interable. Let us prove that $\overline{\mathbf{u}}_N$ converges to $\mathbf{u}$ in $L^1(Q_T)$. Recalling estimate (6.33) and applying the Cauchy inequality we obtain

$$\begin{aligned} \int_{Q_T} |\mathbf{u}_{N,\varepsilon} - \mathbf{u}_\varepsilon|\, dxdt &= \int_{C_{N,\varepsilon}} |\mathbf{u}_{N,\varepsilon} - \mathbf{u}_\varepsilon|\, dxdt \\ \leq c(\text{ meas } C_{N,\varepsilon})^{1/2} &\leq cN^{-1}. \end{aligned}$$

Since

$$\mathbf{u}_\varepsilon \to \mathbf{u}, \quad \mathbf{u}_{N,\varepsilon} \to \overline{\mathbf{u}}_N \text{ weakly in } L^2(Q_T) \text{ as } \varepsilon \to 0,$$

we have

$$\int_{Q_T} |\mathbf{u} - \overline{\mathbf{u}}_N| dxdt \leq \liminf_{\varepsilon \to 0} \int_{Q_T} |\mathbf{u}_\varepsilon - \mathbf{u}_{N,\varepsilon}| dxdt \leq \frac{c}{N}.$$

It follows that

$$\|\mathbf{u} - \overline{\mathbf{u}}_N\|_{L^1(Q_T)} \to 0 \text{ as } N \to \infty.$$

In particular, the the vector fields $\overline{\mathbf{u}}_N$ converges to $\mathbf{u}$ in measure. Therefore, the sequence $\varrho\overline{\mathbf{u}}_N$ is equi-integrable in Q_T and converges to $\varrho\mathbf{u}$ in measure. It follows that

$$\varrho\overline{\mathbf{u}}_N \to \varrho\mathbf{u} \text{ in } L^1(Q_T) \text{ as } N \to \infty.$$

This gives the desired relation (6.35) In particular, we prove that the vector field $\varrho\mathbf{u}$ is integrable. For every fixed $\varphi \in L^\infty(Q_T)$, we have

$$\begin{aligned} \int_{Q_T} \varphi(\varrho_\varepsilon \mathbf{u}_\varepsilon - \varrho\mathbf{u})\, dxdt &= \int_{Q_T} \varphi\varrho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{u}_{N,\varepsilon})\, dxdt \\ + \int_{Q_T} \varphi\varrho(\overline{\mathbf{u}}_N - \mathbf{u})\, dxdt &+ \int_{Q_T} \varphi(\varrho_\varepsilon \mathbf{u}_{N,\varepsilon} - \varrho\overline{\mathbf{u}}_N)\, dxdt. \end{aligned}$$

It follows that

$$\Big|\int_{Q_T} \varphi(\varrho_\varepsilon \mathbf{u}_\varepsilon - \varrho\mathbf{u})\, dxdt\Big| \leq c\mathbf{D}_1(N,\varepsilon) + c\mathbf{D}_2(N) + \mathbf{D}_3(N,\varepsilon),$$

where

$$\mathbf{D}_3(N,\varepsilon)=\Big|\int_{Q_T}\varphi(\varrho_\varepsilon\mathbf{u}_{N,\varepsilon}-\varrho\overline{\mathbf{u}}_N)\,dxdt\Big|\to 0\quad\text{as}\quad\varepsilon\to 0$$

The latter follows from relation (6.19) in Lemma 6.4. From this and estimate (6.34) we obtain

$$\Big|\int_{Q_T}\varphi(\varrho_\varepsilon\mathbf{u}_\varepsilon-\varrho\mathbf{u})\,dxdt\Big|\le c\delta_1(cN)+c\mathbf{D}_2(N)+\mathbf{D}_3(N,\varepsilon).$$

Letting $\varepsilon\to 0$ we arrive at the inequality

$$\limsup_{\varepsilon\to 0}\Big|\int_{Q_T}\varphi(\varrho_\varepsilon\mathbf{u}_\varepsilon-\varrho\mathbf{u})\,dxdt\Big|\le c\delta_1(cN)+c\mathbf{D}_2(N).$$

Note that the left hand side of this inequality is independent on N. Letting $N\to\infty$ and using (6.35) we arrive at the desired relation

$$\lim_{\varepsilon\to 0}\int_{Q_T}\varphi(\varrho_\varepsilon\mathbf{u}_\varepsilon-\varrho\mathbf{u})\,dxdt=0.$$

It remains to note that $\varrho_\varepsilon\mathbf{u}_\varepsilon$ converge weakly to $\overline{\varrho\mathbf{u}}$ as $\varepsilon\to 0$. This completes the proof of Proposition 6.3.

6.3. Part 3. Specification of the measures $\boldsymbol{\nu}$ and σ. In this subsection we study in details the structure of the measures $\boldsymbol{\nu}(t)$ and $\sigma(t)$ introduced in Subsection 6.1. Note that $\boldsymbol{\nu}(t)$ and $\sigma(t)$ are Borel measures defined on the σ- algebra of Borel sets $B\subset\Omega$. By the Lebesgue dominant convergence theorem,

$$\int_\Omega\varphi_n\,d\boldsymbol{\nu}(t)\to\int_\Omega\varphi\;\;d\boldsymbol{\nu}(t),\quad\int_\Omega\varphi_n\,d\sigma(t)\to\int_\Omega\varphi\;\;d\sigma(t)\quad\text{as}\quad n\to\infty$$

for every sequence of Borel functions φ_n, $n\ge 1$, satisfying the conditions

$$\varphi_n:\Omega\to\mathbb{R},\quad|\varphi_n|\le c,\quad\varphi_n(x)\to\varphi(x)\;\;\text{for every}\quad x\in\Omega.$$

Denote by $\boldsymbol{\mu}(t)$ and $\varpi(t)$ the restrictions of the measures $\boldsymbol{\nu}(t)$ and $\sigma(t)$ on $\mathcal{I}$,

$$\boldsymbol{\mu}(t)=\lfloor_{\mathcal{I}}\,\boldsymbol{\nu}(t),\quad\varpi(t)=\lfloor_{\mathcal{I}}\,\sigma(t).$$

Recall that the interval $\mathcal{I}$ is the intersection of Ω and the symmetry axis. The following proposition is the main result of this subsection

Proposition 6.5. *Assume that the function $\varphi:Q_T\to\mathbb{R}$ satisfies the conditions*

$$\varphi\in C^\infty(Q_T),\quad\varphi(\cdot,t)\in C_0^\infty(\Omega)\;\;\textit{for every}\;\;t\in[0,T].$$

Then

$$\int_0^T\Big(\int_\Omega\varphi\,d\boldsymbol{\nu}(t)\Big)\,dt=\int_{Q_T}\varphi\varrho\mathbf{u}\otimes\mathbf{u}\,dxdt+\int_0^T\Big(\int_{\mathcal{I}}\varphi\,d\boldsymbol{\mu}(t)\Big)\,dt,\tag{6.36}$$

$$\int_0^T\Big(\int_\Omega\varphi\,d\sigma(t)\Big)\,dt=\int_{Q_T}\varphi\varrho\,dxdt+\int_0^T\Big(\int_{\mathcal{I}}\varphi\,d\varpi(t)\Big)\,dt.\tag{6.37}$$

Proof. The proof is based on the following construction. Fix an arbitrary C^∞ function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the conditions

$$0 \le \Theta \le 1, \quad \Theta(s) = 1 \text{ for } 0 \le s \le 1, \quad \Theta(s) = 0 \text{ for } s \ge 2.$$

Set

$$\theta_r(x) = \Theta\big(\frac{1}{r}\sqrt{x_1^2 + x_2^2}\big), \quad r > 0. \tag{6.38}$$

Let us consider the integral

$$\int_{Q_T} \varphi\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \, dxdt = \int_{Q_T} (1-\theta_r)\varphi\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \, dxdt + \int_{Q_T} \theta_r\varphi\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \, dxdt.$$

Note that $(1-\theta_r)\varphi$ is compactly supported in $\Omega \setminus \mathcal{I}$. It follows from relation (6.8) and equality (6.15) that

$$\lim_{\varepsilon\to 0} \int_{Q_T} (1-\theta_r)\varphi\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \, dxdt = \int_{Q_T} (1-\theta_r)\varphi\varrho \mathbf{u} \otimes \mathbf{u} \, dxdt.$$

On the other hand, relation (6.11) implies

$$\lim_{\varepsilon\to 0} \int_{Q_T} \theta_r\varphi\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \, dxdt = \int_0^T \Big(\int_\Omega \theta_r\varphi d\boldsymbol{\nu}(t) \Big) \, dt.$$

We thus get

$$\lim_{\varepsilon\to 0} \int_{Q_T} \varphi\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \, dxdt = \int_{Q_T} (1-\theta_r)\varphi\varrho \mathbf{u} \otimes \mathbf{u} \, dxdt + \int_0^T \Big(\int_\Omega \theta_r\varphi d\boldsymbol{\nu}(t) \Big) dt. \tag{6.39}$$

Note that the integral in the left hand side is independent of r. We have for fixed $t \in [0,T]$,

$$\theta_r(x)\varphi(x,t) \to 0 \text{ as } r \to 0 \text{ for } x \in \Omega\setminus\mathcal{I}, \quad \theta_r(x)\varphi(x,t) = \varphi(x,t) \text{ for } x \in \mathcal{I}.$$

For every $t \in (0,T)$, the limiting function equals 0 outside of $\mathcal{I} \times (0,T)$ and equals φ on $\mathcal{I}$. It is a Borel function as a pointwise limit of Borel functions. Obviously, the sequence $\theta_r\varphi$ is uniformly bounded. Applying the Lebesgue dominating convergence Theorem we conclude that

$$J_r(t) := \int_\Omega \theta_r(x)\varphi(x,t) d\boldsymbol{\nu}(t) \to \int_{\mathcal{I}} \varphi(x,t) \, d\boldsymbol{\mu}(t) =: J(t).$$

It is clear that

$$|J_r(t)| \le c \|\boldsymbol{\nu}(t)\|_{\mathbb{M}(\Omega)}.$$

Recall that $\|\boldsymbol{\nu}(t)\|_{\mathbb{M}(\Omega)} \in L^r(0,T)$ for every $r \in [1,\infty)$. Hence the sequence $J_r(t)$ meets all requirements of the Lebesgue dominated convergence theorem on the interval $(0,T)$. Applying this theorem we obtain

$$\int_0^T J_r(t) \, dt \to \int_0^T J(t) \, dt,$$

which gives

$$\lim_{r\to 0} \int_0^T \Big(\int_\Omega \theta_r(x,t)\varphi(x,t) \, d\boldsymbol{\nu}(t) \Big) dt = \int_0^T \Big(\int_{\mathcal{I}} \varphi(x,t) \, d\boldsymbol{\mu}(t) \Big) \, dt. \tag{6.40}$$

Since $\varrho\mathbf{u}\otimes\mathbf{u}\in L^1(Q_T)$, we have

$$\lim_{r\to 0}\int_{Q_T}(1-\theta_r)\varphi\varrho\mathbf{u}\otimes\mathbf{u}\,dxdt=\int_{Q_T}\varphi\varrho\mathbf{u}\otimes\mathbf{u}\,dxdt. \tag{6.41}$$

Letting $r\to 0$ in (6.39) and using (6.40), (6.41) we obtain

$$\lim_{\varepsilon\to 0}\int_{Q_T}\varphi\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\,dxdt=\int_{Q_T}\varphi\varrho\mathbf{u}\otimes\mathbf{u}\,dxdt+\int_0^T\Big(\int_{\mathcal{I}}\varphi(x,t)\,d\boldsymbol{\mu}(t)\Big)\,dt.$$

It remains to note that by virtue of (6.11),

$$\lim_{\varepsilon\to 0}\int_{Q_T}\varphi\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\,dxdt=\int_0^T\Big(\int_{\Omega}\varphi d\boldsymbol{\nu}(t)\Big)\,dt.$$

This completes the proof of (6.36). Let us prove (6.37). We have

$$\int_{Q_T}\varphi p(\varrho_\varepsilon)\,dxdt=\int_{Q_T}(1-\theta_r)\varphi p(\varrho_\varepsilon)\,dxdt+\int_{Q_T}\theta_r\varphi p(\varrho_\varepsilon)\,dxdt.$$

Recall the equality

$$p(\varrho_\varepsilon)=\varrho_\varepsilon+p_1(\varrho_\varepsilon)+p_2(\varrho_\varepsilon).$$

Note that $(1-\theta_r)\varphi$ is compactly supported in $\Omega\setminus\mathcal{I}$. From this and relation (5.5) in Theorem 5.1 we conclude that

$$\lim_{\varepsilon\to 0}\int_{Q_T}(1-\theta_r)\varphi(p_1(\varrho_\varepsilon)+p_2(\varrho_\varepsilon))\,dxdt=0.$$

Since $\varrho_\varepsilon\to\varrho$ weakly in $L^1(Q_T)$, we have

$$\lim_{\varepsilon\to 0}\int_{Q_T}(1-\theta_r)\varphi\,p(\varrho_\varepsilon)\,dxdt=\int_{Q_T}(1-\theta_r)\varphi\,\varrho\,dxdt. \tag{6.42}$$

It follows from relation (6.11) that

$$\lim_{\varepsilon\to 0}\int_{Q_T}\theta_r\varphi p(\varrho_\varepsilon)\,dxdt=\int_0^T\Big(\int_\Omega\theta_r\varphi\,d\sigma(t)\Big)dt.$$

Combining this relation with (6.42) we arrive at the equality

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{Q_T}\varphi\,p(\varrho_\varepsilon)\,dxdt=&\int_{Q_T}(1-\theta_r)\varphi\,\varrho\,dxdt\\&+\int_0^T\Big(\int_\Omega\theta_r\varphi\,d\sigma(t)\Big)dt.\end{aligned}$$

Letting $r\to 0$ and arguing as before we finally obtain

$$\lim_{\varepsilon\to 0}\int_{Q_T}\varphi\,p(\varrho_\varepsilon)\,dxdt=\int_{Q_T}\varphi\,\varrho\,dxdt+\int_0^T\Big(\int_{\mathcal{I}}\varphi\,d\varpi(t)\Big)dt.$$

It remains to note that

$$\lim_{\varepsilon\to 0}\int_{Q_T}\varphi\,p(\varrho_\varepsilon)\,dxdt=\int_0^T\Big(\int_\Omega\varphi\,d\sigma(t)(t)\Big)dt.$$

This completes the proof of (6.37).

□

Corollary 6.6. *Under the above assumptions, the integral identities*

$$\begin{gathered}\int_{Q_T}\big(\,\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\xi}+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla\boldsymbol{\xi}+\varrho\ \mathrm{div}\ \boldsymbol{\xi}\,\big)\,dxdt+\\ \int_0^T\Big(\int_{\mathcal{I}}\nabla\boldsymbol{\xi}:d\boldsymbol{\mu}(t)\Big)\,dt+\int_0^T\Big(\int_{\mathcal{I}}\mathrm{div}\ \boldsymbol{\xi}\,d\varpi(t)\Big)\,dt\\ -\int_{Q_T}\mathbb{S}(\mathbf{u}):\nabla\boldsymbol{\xi}\,dxdt+\int_{Q_T}\varrho\mathbf{f}\cdot\boldsymbol{\xi}\,dxdt\\ +\int_\Omega\varrho_0\mathbf{u}_0(x)\cdot\boldsymbol{\xi}(x,0)\,dx=0,\end{gathered}\tag{6.43}$$

$$\int_{Q_T}\big(\varrho\partial_t\psi+\varrho\mathbf{u}\cdot\nabla\psi\big)\,dxdt+\int_\Omega\varrho_0(x)\psi(x,0)\,dx=0\tag{6.44}$$

hold for all vector fields $\boldsymbol{\xi}\in C^\infty(Q_T)$ *vanishing in a neighborhood of* $\partial\Omega\times[0,T]$ *and* $\Omega\times\{t=T\}$*, and for all functions* $\psi\in C^\infty(Q_T)$ *vanishing in a neighborhood of* $\Omega\times\{t=T\}$*.*

Proof. The proof obviously follows from integral identities (6.12)- (6.13) and equalities (6.14)-(6.15), (6.36)-(6.37). □

6.4. Part 4. Proof of Theorem 1.5 . We are now in a position to complete the proof of Theorem 1.5. It remains to show that matrix value parametrized measure $\boldsymbol{\mu}=(\mu_{ij})_{i,j=1,2,3}$ has the only nonzero entry μ_{33}, and the parametrized measure $\varpi=0$. To this and choose an arbitrary functions

$$\eta\in C_0^\infty(0,T),\quad \chi\in C_0^\infty(\mathcal{I}).$$

Introduce the vector field $\boldsymbol{\xi}:Q_T\to\mathbb{R}^3$ defined by the equalities

$$\boldsymbol{\xi}=(\xi_1,0,0),\quad \xi_1=x_1\theta_r(x)\chi(x_3)\eta(t),$$

where θ_r is defined by (6.38). For sufficiently small r, the smooth vector field $\boldsymbol{\xi}$ is compactly supported in Q_T. Denote by $Q_{T,r}$ the cylinder

$$Q_{T,r}=\{\,(x,t)\in Q_T:\ \sqrt{x_1^2+x_2^2}\le 2r\,\}.$$

Obviously, $\boldsymbol{\xi}$ is supported in $Q_{T,r}$. Substituting $\boldsymbol{\xi}$ in integral identity (6.43) we obtain

$$\begin{gathered}\int_{Q_{T,r}}\big(\,\varrho u_1\partial_t\xi_1+\varrho u_1^2\partial_1\xi_1+\varrho u_1u_2\partial_2\xi_1+\varrho u_1u_3\partial_3\xi_1\,\big)\,dxdt+\\ \int_{Q_{T,r}}\big(\,\varrho\partial_1\xi_1-S_{11}\partial_1\xi_1-S_{12}\partial_2\xi_1-S_{13}\partial_3\xi_1+\varrho\mathbf{f}_1\xi_1\big)\,dxdt\\ +\int_0^T\eta(t)\Big(\int_{\mathcal{I}}\chi(x_3)d\mu_{11}(t)\Big)\,dt+\int_0^T\eta(t)\Big(\int_{\mathcal{I}}\chi(x_3)\,d\varpi(t)\Big)\,dt.\end{gathered}\tag{6.45}$$

Note that

$$|\nabla\xi_1|+|\partial_t\xi_1|\le c\ \text{ in }\ Q_T,\quad \varrho,\varrho|\mathbf{u}|^2,S_{ij}\in L^1(Q_T).\tag{6.46}$$

Moreover, we have

$$\operatorname{meas} Q_{T,r} \to 0 \quad \text{as} \quad r \to 0. \tag{6.47}$$

It follows that

$$\int_{Q_{T,r}} \big(\,\varrho u_1 \partial_t \xi_1 + \varrho u_1^2 \partial_1 \xi_1 + \varrho u_1 u_2 \partial_2 \xi_1 + \varrho u_1 u_3 \partial_3 \xi_1\,\big)\, dxdt +$$
$$\int_{Q_{T,r}} \big(\,\varrho \partial_1 \xi_1 - S_{11} \partial_1 \xi_1 - S_{12} \partial_2 \xi_1 - S_{13} \partial_3 \xi_1 + \varrho \mathbf{f}_1 \xi_1\big)\, dxdt \to 0$$

as $r \to 0$. Letting $r \to 0$ in (6.45) we obtain

$$\int_0^T \eta(t)\Big(\int_{\mathcal{I}} \chi(x_3) d\mu_{11}(t)\Big)\, dt + \int_0^T \eta(t)\Big(\int_{\mathcal{I}} \chi(x_3)\, d\varpi(t)\Big)\, dt = 0$$

for all $\chi \in C_0^\infty(\mathcal{I})$ and $\eta \in C_0^\infty(0,T)$. Recalling that $\mu_{11} \geq 0$ and $\varpi \geq 0$ we conclude that

$$\mu_{11} = \varpi = 0. \tag{6.48}$$

In order to prove the equality $\mu_{12} = 0$, introduce the vector field

$$\boldsymbol{\xi} = (\xi_1, 0, 0), \quad \xi_1 = x_2 \theta_r(x) \chi(x_3) \eta(t).$$

Substituting $\boldsymbol{\xi}$ in (6.43) and noting that $\varpi = 0$ we arrive at the equality

$$\begin{aligned}\int_{Q_{T,r}} \big(\,\varrho u_1 \partial_t \xi_1 + \varrho u_1^2 \partial_1 \xi_1 + \varrho u_1 u_2 \partial_2 \xi_1 + \varrho u_1 u_3 \partial_3 \xi_1\,\big)\, dxdt +\\
\int_{Q_{T,r}} \big(\,\varrho \partial_1 \xi_1 - S_{11} \partial_1 \xi_1 - S_{12} \partial_2 \xi_1 - S_{13} \partial_3 \xi_1 + \varrho \mathbf{f}_1 \xi_1\big)\, dxdt\\
+ \int_0^T \eta(t)\Big(\int_{\mathcal{I}} \chi(x_3) d\mu_{12}(t)\Big)\, dt = 0.\end{aligned} \tag{6.49}$$

Recalling relations (6.46) and (6.47) we obtain

$$\int_{Q_{T,r}} \big(\,\varrho u_1 \partial_t \xi_1 + \varrho u_1^2 \partial_1 \xi_1 + \varrho u_1 u_2 \partial_2 \xi_1 + \varrho u_1 u_3 \partial_3 \xi_1\,\big)\, dxdt +$$
$$\int_{Q_{T,r}} \big(\,\varrho \partial_1 \xi_1 - S_{11} \partial_1 \xi_1 - S_{12} \partial_2 \xi_1 - S_{13} \partial_3 \xi_1 + \varrho \mathbf{f}_1 \xi_1\big)\, dxdt \to 0$$

as $r \to 0$. Letting $r \to 0$ in (6.49) we get

$$\int_0^T \eta(t)\Big(\int_{\mathcal{I}} \chi(x_3) d\mu_{12}(t)\Big)\, dt = 0$$

for all $\chi \in C_0^\infty(\mathcal{I})$ and $\eta \in C_0^\infty(0,T)$. We thus get

$$\mu_{12} \equiv \mu_{21} = 0. \tag{6.50}$$

Let us prove that $\mu_{22} = 0$. To this end, take the vector field $\boldsymbol{\xi}$ in the form

$$\boldsymbol{\xi} = (0, \xi_2, 0), \quad \xi_2 = x_2 \theta_r(x) \chi(x_3) \eta(t).$$

Substituting $\boldsymbol{\xi}$ in (6.43) and noting that $\varpi = 0$ we obtain

$$\int_{Q_{T,r}} \big(\, \varrho u_2 \partial_t \xi_2 + \varrho u_1 u_2 \partial_1 \xi_1 + \varrho u_2^2 \partial_2 \xi_2 + \varrho u_2 u_3 \partial_3 \xi_1 \,\big)\, dxdt +$$

$$\int_{Q_{T,r}} \big(\, \varrho \partial_2 \xi_2 - S_{12} \partial_1 \xi_2 - S_{22} \partial_2 \xi_2 - S_{23} \partial_3 \xi_2 + \varrho \mathbf{f}_2 \xi_2 \big)\, dxdt$$

$$+ \int_0^T \eta(t) \Big(\int_{\mathcal{I}} \chi(x_3) d\mu_{22}(t) \Big)\, dt = 0.$$

Arguing as before we conclude that $\mu_{22} = 0$. Combining this result with (6.48) and (6.50) we arrive at the equalities

$$\mu_{ij} = 0 \ \text{ for } \ 1 \le i, j \le 2, \quad \varpi = 0. \tag{6.51}$$

It remains to prove that

$$\mu_{13} \equiv \mu_{31} = \mu_{23} \equiv \mu_{32} = 0. \tag{6.52}$$

We begin with the observation that

$$\varrho_\varepsilon u_{\varepsilon,i} u_{\varepsilon,j} \varphi_i \varphi_j = \varrho_\varepsilon (\mathbf{u}_\varepsilon \cdot \boldsymbol{\varphi})^2 \ge 0 \ \text{ for all } \ \boldsymbol{\varphi} \in \mathbb{R}^3.$$

Choose an arbitrary functions

$$\boldsymbol{\varphi} : \mathcal{I} \to \mathbb{R}^3, \quad \varphi_i \in C_0(\mathcal{I}), \quad \eta \in C_0^+(0, T).$$

and set

$$\boldsymbol{\xi} = \theta_r(x) \eta(t)\, \boldsymbol{\varphi}(x_3).$$

It follows from this and (6.11), (6.16) that

$$0 \le \int_{Q_T} \theta_r \eta \varrho_\varepsilon u_{\varepsilon,i} u_{\varepsilon_j} \varphi_i \varphi_j \, dxdt \ \ \to \ \ \int_{Q_T} \theta_r \eta \varrho u_i u_j \varphi_i \varphi_j \, dxdt$$

$$+ \ \ \int_0^T \eta(t) \Big(\int_{\mathcal{I}} \varphi_i(x_3) \varphi_j(x_3)\, d\mu_{i,j}(t) \Big)\, dt.$$

Note that the functions $\theta_r \eta \boldsymbol{\varphi}$ are uniformly bounded and are compactly supported in the small cylinder $Q_{T,r}$, which measure tends to 0 as $r \to 0$. Since the function $\varrho |\mathbf{u}|^2$ is integrable, we conclude that

$$\lim_{r \to 0} \int_{Q_T} \theta_r \eta \varrho u_i u_j \varphi_i \varphi_j \, dxdt = 0.$$

We thus get

$$\int_0^T \eta(t) \Big(\int_{\mathcal{I}} \varphi_i(x_3) \varphi_j(x_3)\, d\mu_{i,j}(t) \Big)\, dt \ge 0 \ \text{ for all } \ \eta \in C_0^+(0, T).$$

It follows that

$$\int_{\mathcal{I}} \varphi_i(x_3) \varphi_j(x_3)\, d\mu_{i,j}(t) \, \ge 0 \ \text{ for all } \ \boldsymbol{\varphi} \in C_0(\mathcal{I}) \ \text{ and for a.e. } t \in (0, T).$$

Recalling (6.51) we conclude that the inequality

$$0 \le \int_{\mathcal{I}} \big(2\varphi_1 \varphi_3 d\mu_{13}(t) + 2\varphi_2 \varphi_3 d\mu_{23}(t) + \varphi_3^2 d\mu_{33}(t) \,\big)$$

holds for all $\varphi_j \in C_0(\mathcal{I})$ and for a.e. $t \in (0,T)$. It is possible if and only if $\mu_{13} = \mu_{23} = 0$. Therefore, the matrix μ_{ij} may have the only nonzero entry μ_{33}. This completes the proof of Theorem 1.5.

7 Appendix. Navier-Stokes equations in the cylindrical coordinates

In this section we give the proof of the lemma and proposition formulated in Section 2. To this end we recall the denotations for the cylindrical coordinates Denote by r, ϑ, x_3 the cylindrical coordinates in the space $\mathbb{R}^3$ of points $x = (x_1, x_2, x_3)$,

$$x_1 = r\cos\vartheta, \quad x_2 = r\sin\vartheta, \quad x_3 = x_3, \tag{7.1}$$

$$y = (y_1, y_3) = (r, x_3), \quad z = (z_1, z_2, z_3) = (r, \theta, x_3) = (y_1, \theta, y_3). \tag{7.2}$$

and the domain

$$G = \{x \in \Omega : x_1 > 0, x_2 = 0\}.$$

We will consider G as a domain in the plane of variable y. The radial, azimuth and axial components v_i of the velocity field $\mathbf{u}$ form the vector field $\mathbf{v}$ such that

$$\mathbf{u} = \mathbf{U}\,\mathbf{v}, \quad \mathbf{v} = (v_1, v_2, v_3), \quad \mathbf{U} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.3}$$

We start with the proof of the following technical lemma. Throughout of this section the denotation

$$\mathbf{A} \;=\; \big[\, A_1, A_2, A_3 \,\big]$$

stands for a matrix $\mathbf{A}$ with the columns A_i. Next, we denote by $\mathbf{A} : \mathbf{B}$ the scalar product

$$\mathbf{A} : \mathbf{B} \;=\; \mathrm{tr}\;(\mathbf{A}^\top\,\mathbf{B}).$$

Note that $|\mathbf{A}|^2 = \mathbf{A} : \mathbf{A}$ is the squared Hilbert-Schmidt norm of $\mathbf{A}$.

Lemma 7.1. *Let Ω be rotationally symmetric and $\mathbf{u} \in W^{1,2}(\Omega)$. Then*

$$\nabla_x \mathbf{u}(x) = \mathbf{U}(\theta)\,\big(\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v})\,\big)\,\mathbf{U}^\top. \tag{7.4}$$

Here the matrices $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ are defined by the equalities

$$\boldsymbol{\lambda}(\mathbf{v}) = \begin{pmatrix} \partial_r v_1 & \dfrac{1}{r}\partial_\theta v_1 & \partial_{x_3} v_1 \\ \partial_r v_2 & \dfrac{1}{r}\partial_\theta v_2 & \partial_{x_3} v_2 \\ \partial_r v_3 & \dfrac{1}{r}\partial_\theta v_3 & \partial_{x_3} v_3 \end{pmatrix}, \quad \boldsymbol{\nu}(\mathbf{v}) = \begin{pmatrix} 0 & -\dfrac{1}{r} v_2 & 0 \\ 0 & \dfrac{1}{r} v_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.5}$$

Proof. By virtue of the chain rule, the equality

$$\nabla_x \mathbf{u} = \nabla_z \mathbf{u}\,\nabla_x z$$

holds true for almost all $z \in G \times \mathbb{S}^1$. We have

$$z = (\sqrt{x_1^2 + x_2^2},\ \arctan \frac{x_2}{x_1},\ x_3)$$

Which gives

$$\nabla_x z = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{R}\mathbf{U}^\top, \quad \mathbf{R} = \text{ diag } \{1, \frac{1}{r}, 1\}.$$

Thus we get

$$\nabla_x \mathbf{u} = \nabla_z \mathbf{u}\, \mathbf{R}\, \mathbf{U}^\top \tag{7.6}$$

On the other hand, we have

$$\nabla_z \mathbf{u} = \big[\, \partial_r \mathbf{u}, \partial_\theta \mathbf{u}, \partial_{x_3} \mathbf{u} \,\big] = \big[\, \mathbf{U}\partial_r \mathbf{v}, \mathbf{U}\partial_\theta \mathbf{v}, \mathbf{U}\partial_{x_3} \mathbf{v} \,\big] + \big[\, 0, \partial_\theta \mathbf{U}\mathbf{v}, 0 \,\big] \tag{7.7}$$

Notice that

$$\mathbf{U}^\top \partial_\theta \mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{S}.$$

Since $\mathbf{U}\mathbf{U}^\top = \mathbf{I}$, we can rewrite identity (7.7) in the equivalent form

$$\nabla_z \mathbf{u} = \big[\, \mathbf{U}\partial_r \mathbf{v}, \mathbf{U}\partial_\theta \mathbf{v}, \mathbf{U}\partial_{x_3} \mathbf{v} \,\big] + \big[\, 0, \mathbf{U}(\mathbf{S}\mathbf{v}), 0 \,\big].$$

Note that for every matrix $\mathbf{A}$ and columns a, b, c, we have

$$\big[\, \mathbf{A}a, \mathbf{A}b, \mathbf{A}c \,\big] = \mathbf{A}\big[\, a, b, c \,\big],$$

which yields

$$\nabla_z \mathbf{u} = \mathbf{U}\big[\, \partial_r \mathbf{v}, \partial_\theta \mathbf{v}, \partial_{x_3} \mathbf{v} \,\big] + \mathbf{U}\big[\, 0, \mathbf{S}\mathbf{v}, 0 \big].$$

Substituting this identity into (7.6) we obtain

$$\nabla_x \mathbf{u} = \mathbf{U}\Big(\big[\, \partial_r \mathbf{v}, \partial_\theta \mathbf{v}, \partial_{x_3} \mathbf{v} \,\big]\mathbf{R} + \big[\, 0, \mathbf{S}\mathbf{v}, 0\big]\mathbf{R} \Big) \mathbf{U}^\top.$$

It remains to note that

$$\big[\, \partial_r \mathbf{v}, \partial_\theta \mathbf{v}, \partial_{x_3} \mathbf{v} \,\big]\mathbf{R} = \boldsymbol{\lambda}(\mathbf{v}), \quad \big[\, 0, \mathbf{S}\mathbf{v}, 0\big]\mathbf{R} = \boldsymbol{\nu}(\mathbf{v})$$

and the lemma follows. □

The next technical lemma gives the representation of $\mathbf{u} \otimes \mathbf{u}$ and div$\mathbf{u}$.

Lemma 7.2. *Let* $\mathbf{u} \in W^{1,2}(\Omega)$ *admits the representation* $\mathbf{u} = \mathbf{U}(\theta)\mathbf{v}(z)$. *Then*

$$\mathbf{u} \otimes \mathbf{u} = \mathbf{U}(\mathbf{v} \otimes \mathbf{v})\mathbf{U}^\top, \quad \text{div } \mathbf{u} = \text{tr } \big(\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v}) \big). \tag{7.8}$$

Proof. The first equality in (7.8) easily follows from direct calculations. In order to prove the second, it suffices to note that by virtue of Lemma 7.1 we have

$$\text{div } \mathbf{u} = \text{tr } \nabla_x \mathbf{u} = \text{tr } \big(\mathbf{U}(\boldsymbol{\lambda} + \boldsymbol{\nu})\mathbf{U}^T \big) = \text{tr } \big(\boldsymbol{\lambda} + \boldsymbol{\nu} \big).$$

□

The following lemma is a straightforward consequence of Lemma 7.1.

Lemma 7.3. *Let* $\mathbf{u} : \Omega \to \mathbb{R}^3$ *be a rotationally symmetric vector field such that* $\mathbf{u} = \mathbf{U}(\theta)\,\mathbf{v}(y)$, $y = (r, x_3)$. *Then*

$$\int_\Omega |\nabla_x \mathbf{u}|^2\,dx = 2\pi \int_G |\nabla_y \mathbf{v}|^2\,r dr dx_3 + 2\pi \int_G (v_1^2 + v_2^2)\frac{1}{r} dr dx_3. \tag{7.9}$$

Proof. By virtue of Lemma 7.1, we have

$$|\nabla_x \mathbf{u}|^2 = |\mathbf{U}(\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v}))\mathbf{U}^\top|^2 = |\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v})|^2 =$$
$$(\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v})) : (\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v})) = |\boldsymbol{\lambda}(\mathbf{v})|^2 + |\boldsymbol{\nu}(\mathbf{v})|^2 + 2\boldsymbol{\lambda}(\mathbf{v}) : \boldsymbol{\nu}(\mathbf{v}).$$

Note that

$$\boldsymbol{\lambda}(\mathbf{v}) = \begin{pmatrix} \partial_r v_1 & 0 & \partial_{x_3} v_1 \\ \partial_r v_2 & 0 & \partial_{x_3} v_2 \\ \partial_r v_3 & 0 & \partial_{x_3} v_3 \end{pmatrix}, \quad \boldsymbol{\nu}(\mathbf{v}) = \begin{pmatrix} 0 & -\dfrac{1}{r} v_2 & 0 \\ 0 & \dfrac{1}{r} v_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that $\boldsymbol{\lambda}(\mathbf{v})$ and $\boldsymbol{\nu}(\mathbf{v})$ are orthogonal which yields

$$|\nabla_x \mathbf{u}|^2 = |\boldsymbol{\lambda}(\mathbf{v})|^2 + |\boldsymbol{\nu}(v)|^2.$$

Since $\mathbf{v}$ is independent of θ, we have

$$\int_\Omega |\nabla_x \mathbf{u}|^2\,dx = 2\pi \int_G (|\boldsymbol{\lambda}(\mathbf{v})|^2 + |\boldsymbol{\nu}(\mathbf{v})|^2)\,r\,dr dx_3.$$

It remains to note that

$$|\boldsymbol{\lambda}(\mathbf{v})|^2 = |\nabla_y \mathbf{v}|^2, \quad |\boldsymbol{\nu}(\mathbf{v})|^2 = \frac{1}{r^2}(v_1^2 + v_2^2).$$

□

We are now in a position to formulate the compressible Navier-Stokes equations in the cylindrical coordinates. Let us consider the totality of test vector fields $\boldsymbol{\xi} : Q_T \to \mathbb{R}^3$ satisfying the following conditions.

Condition 7.4.

$$\boldsymbol{\xi} \in C_0^1(Q_T), \quad \boldsymbol{\xi} = \mathbf{U}(\theta)\boldsymbol{\varsigma}(z, t). \tag{7.10}$$

The vector field $\boldsymbol{\varsigma} \in C^1(G \times \mathbb{S}^1 \times (0, T))$ *is compactly supported in* $G \times \mathbb{S}^1 \times (0, T)$.

Proposition 7.5. *Let* $\varrho(x, t)$, $\mathbf{u}(x, t)$ *be a rotationally symmetric solution to problem 1.15 given by Proposition 1.3. Then the vector field* $\mathbf{v}(y, t)$ *defined by the equality* $\mathbf{u} = \mathbf{U}(\theta)\mathbf{v}(y, t)$, *and the density* $\varrho = \varrho(y, t)$ *admit the estimate*

$$\begin{aligned} &\int_0^T \int_G \Big(|\nabla \mathbf{v}|^2 + \frac{1}{y_1^2}(v_1^2 + v_2^2)\Big)\,y_1 dy dt \le c, \\ &\operatorname{ess\,sup}_{t \in (0,T)} \int_G \Big(\varrho \mathbf{v}|^2 + e(\varrho)\Big)\,y_1 dy \le c, \end{aligned} \tag{7.11}$$

where the internal energy e is defined by (1.12), the constant c depends only on the constant E in (1.16). The integral identity

$$\int_0^T\int_{G\times\mathbb{S}^1}\Big(\varrho\mathbf{v}\cdot\partial_t\boldsymbol{\varsigma}+\varrho\mathbf{v}\otimes\mathbf{v}:\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)+p(\varrho)\mathrm{tr}\,\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)\Big)y_1\,dyd\theta dt$$
$$=\int_0^T\int_{G\times\mathbb{S}^1}\Big(\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})+\boldsymbol{\lambda}(\mathbf{v})^\top+\boldsymbol{\nu}(\mathbf{v})^\top\big):\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)\Big)y_1\,dy\,d\theta\,dt$$
$$-\frac{2}{3}\int_0^T\int_{G\times\mathbb{S}^1}\Big(\big(\mathrm{tr}\,\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})\big)\,\mathrm{tr}\,\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)\Big)y_1\,dyd\theta dt$$
$$-\int_0^T\int_{G\times\mathbb{S}^1}\varrho\mathbf{f}^*\cdot\boldsymbol{\varsigma}\,y_1\,dyd\theta dt \tag{7.12}$$

holds for all test functions $\boldsymbol{\varsigma}$ satisfying Condition 7.4. The integral identity

$$\int_0^T\int_{G\times\mathbb{S}^1}\Big(\varrho\partial_t\psi+\varrho\mathbf{v}\cdot(\mathbf{R}\nabla_z\psi)\Big)y_1\,dyd\theta dt=0 \tag{7.13}$$

holds for all functions $\psi\in C^1(G\times\mathbb{S}^1\times(0,T))$ compactly supported in $G\times\mathbb{S}^1\times(0,T)$. Here $\boldsymbol{\lambda}(\mathbf{v})$ and $\boldsymbol{\nu}(\mathbf{v})$ are defined by equalities (7.5) with $\partial_\theta\mathbf{v}=0$, the matrices $\boldsymbol{\lambda}(\boldsymbol{\varsigma})$ and $\boldsymbol{\nu}(\boldsymbol{\varsigma})$ are defined by equalities (7.5) with $\mathbf{v}$ replaced by $\boldsymbol{\varsigma}$, the vector field $\mathbf{f}^*(y,t)$ is defined by the equality $\mathbf{f}=\mathbf{U}(\theta)\mathbf{f}^*$.

Proof. Estimate (7.11) obviously follows from estimate (1.16) in Proposition 1.3 and Lemma7.3.

Let us turn to the proof if identities (7.12) and (7.13). It follows from Proposition 1.3 that the functions ϱ, $\mathbf{u}=\mathbf{U}\mathbf{v}$, and $\boldsymbol{\xi}=\mathbf{U}\boldsymbol{\varsigma}$ satisfy the integral identity

$$\int_{Q_T}\big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\xi}+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla\boldsymbol{\xi}+p(\varrho)\mathrm{div}\,\boldsymbol{\xi}-\mathbb{S}(\mathbf{u}):\nabla\boldsymbol{\xi}+\varrho\mathbf{f}\cdot\boldsymbol{\xi}\big)\,dxdt=0. \tag{7.14}$$

Note that

$$\mathbf{u}\cdot\partial_t\boldsymbol{\xi}=\mathbf{U}\mathbf{v}\cdot\mathbf{U}\partial_t\boldsymbol{\varsigma}=\mathbf{v}\cdot\partial_t\boldsymbol{\varsigma},\quad \mathbf{f}\cdot\boldsymbol{\xi}=\mathbf{U}\mathbf{f}^*\cdot\mathbf{U}\boldsymbol{\varsigma}=\mathbf{f}^*\cdot\boldsymbol{\varsigma}. \tag{7.15}$$

Next, we have

$$\mathbf{u}\otimes\mathbf{u}=(\mathbf{U}\mathbf{v})\otimes(\mathbf{U}\mathbf{v})=\mathbf{U}(\mathbf{v}\otimes\mathbf{v})\mathbf{U}^\top,$$

which yields

$$\mathbf{u}\otimes\mathbf{u}:\nabla\boldsymbol{\xi}=\mathbf{U}(\mathbf{v}\otimes\mathbf{v})\mathbf{U}^\top:\nabla\boldsymbol{\xi}.$$

It follows from Lemma 7.1 that

$$\nabla_x\boldsymbol{\xi}=\mathbf{U}\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)\mathbf{U}^\top.$$

Combining the obtained results we arrive at the identity

$$\begin{aligned}\mathbf{u}\otimes\mathbf{u}:\nabla\boldsymbol{\xi}&=\big(\mathbf{U}(\mathbf{v}\otimes\mathbf{v})\mathbf{U}^\top\big):\big(\mathbf{U}\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)\mathbf{U}^\top\big)\\&=\big(\mathbf{v}\otimes\mathbf{v}\big):\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big).\end{aligned} \tag{7.16}$$

Note that

$$\mathbb{S}(\mathbf{u}) = (\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^\top) - \frac{2}{3}\text{div } \mathbf{u} \ \mathbf{I},$$

which yields

$$\mathbb{S}(\mathbf{u}) : \nabla \boldsymbol{\xi} = (\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^\top) : \nabla \boldsymbol{\xi} - \frac{2}{3}\text{div } \mathbf{u}\,\text{div } \boldsymbol{\xi}. \tag{7.17}$$

It follows from Lemma 7.1 and Lemma 7.2 that

$$\begin{aligned} &(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^\top) : \nabla \boldsymbol{\xi} = \\ &\big(\mathbf{U}\,\big(\,\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v}) + \lambda(\mathbf{v})^\top + \boldsymbol{\nu}(\mathbf{v})^\top\,\big)\mathbf{U}^\top\big) : \big(\mathbf{U}\,\big(\,\boldsymbol{\lambda}(\boldsymbol{\varsigma}) + \boldsymbol{\nu}(\boldsymbol{\varsigma})\,\big)\mathbf{U}^\top\big) \\ &= \big(\,\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v}) + \boldsymbol{\lambda}(\mathbf{v})^\top + \boldsymbol{\nu}(\mathbf{v})^\top\,\big) : \big(\,\boldsymbol{\lambda}(\boldsymbol{\varsigma}) + \boldsymbol{\nu}(\boldsymbol{\varsigma})\,\big) \end{aligned}$$

and

$$\text{div } \mathbf{u}\,\text{div } \boldsymbol{\xi} = \text{tr }\big(\,\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v})\,\big)\,\text{tr }\big(\,\boldsymbol{\lambda}(\boldsymbol{\varsigma}) + \boldsymbol{\nu}(\boldsymbol{\varsigma})\big)$$

Substituting these results into (7.17) we arrive at the identity

$$\begin{aligned} &\mathbb{S}(\mathbf{u}) : \nabla \boldsymbol{\xi} = (\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^\top) : \nabla \boldsymbol{\xi} - \frac{2}{3}\text{div } \mathbf{u}\,\text{div } \boldsymbol{\xi} = \\ &\big(\,\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v}) + \boldsymbol{\lambda}(\mathbf{v})^\top + \boldsymbol{\nu}(\mathbf{v})^\top\,\big) : \big(\,\boldsymbol{\lambda}(\boldsymbol{\varsigma}) + \boldsymbol{\nu}(\boldsymbol{\varsigma})\,\big) - \\ &\frac{2}{3}\text{tr }\big(\,\boldsymbol{\lambda}(\mathbf{v}) + \boldsymbol{\nu}(\mathbf{v})\,\big)\,\text{tr }\big(\,\boldsymbol{\lambda}(\boldsymbol{\varsigma}) + \boldsymbol{\nu}(\boldsymbol{\varsigma})\big). \end{aligned} \tag{7.18}$$

Finally, substituting (7.15), (7.16), and (7.18) into (7.14) and noting that $dxdt = y_1 dy\, d\theta\, dt$ we arrive at the desired identity (7.12).

In order to prove (7.12), notice that

$$\int_0^T \int_\Omega (\varrho \partial_t \psi + \varrho \mathbf{u} \cdot \nabla_x \psi)\, dxdt = 0. \tag{7.19}$$

It is easy to see that

$$\varrho \mathbf{u} \cdot \nabla_x \psi = \varrho \mathbf{U} \mathbf{v} \cdot \mathbf{U}(\mathbf{R} \nabla_z \psi) = \varrho \mathbf{v} \cdot (\mathbf{R} \nabla_z \psi).$$

Substituting these equalities into (7.19) we obtain (7.13). □

In the paper we use the simplified version of identities (7.12) and (7.13) with special test function $\boldsymbol{\varsigma}$. The result is given by the following proposition.

Proposition 7.6. *Let ϱ, $\mathbf{v}$, $\boldsymbol{\varsigma}$, and ψ meet all requirements of Proposition 7.5. Furthermore assume that*

$$\partial_\theta \boldsymbol{\varsigma} = 0, \quad \varsigma_2 = 0, \quad \partial_\theta \psi = 0. \tag{7.20}$$

Then the vector fields

$$\overline{\mathbf{v}} = (v_1(y,t), v_3(y,t)), \quad \overline{\boldsymbol{\varsigma}} = (v_1(y,t), v_3(y,t)), \quad \overline{\mathbf{f}} = (f_1^*(y,t), f_3^*(y,t)) \tag{7.21}$$

satisfy the integral identities

$$\int_{G\times(0,T)}\Big(\varrho\overline{\mathbf{v}}\cdot\partial_t\overline{\boldsymbol{\varsigma}}+\varrho\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}:\nabla\overline{\boldsymbol{\varsigma}}+p(\varrho)\mathrm{div}\,\overline{\boldsymbol{\varsigma}}-\mathbf{B}(\overline{\mathbf{v}}):\nabla\overline{\boldsymbol{\varsigma}}$$
$$+\mathbf{b}(\overline{\mathbf{v}},\varrho)\,\varsigma_1+\varrho\overline{\mathbf{f}}\cdot\overline{\boldsymbol{\varsigma}}\Big)\,y_1dy\,dt=0, \quad (7.22)$$

$$\int_{G\times(0,T)}\Big(\,\varrho\partial_t\psi+\varrho\overline{\mathbf{v}}\cdot\nabla\psi\,\Big)\,y_1\,dy\,dt=0. \quad (7.23)$$

Here $\nabla\mathbf{a}=\nabla_y\mathbf{a}$ *for each vector field* $\mathbf{a}(y,t)$*. The matrix* $\mathbf{B}$ *and vector field* $\mathbf{b}$ *are defined by the equalities*

$$\begin{aligned}\mathbf{B}&=(\nabla\overline{\mathbf{v}}+\nabla\overline{\mathbf{v}}^\top)-\frac{2}{3}(\mathrm{div}\,\overline{\mathbf{v}}+\frac{1}{r}v_1),\\ \mathbf{b}&=\frac{1}{y_1}\varrho v_2^2+\frac{1}{y_1}p-\frac{4}{3y_1^2}v_1+\frac{2}{3y_1}\mathrm{div}\,\overline{\mathbf{v}}\end{aligned} \quad (7.24)$$

For every compact set $K\Subset G$*, the functions* $\mathbb{B}$ *and* $\mathbf{b}$ *admit the estimate*

$$\|\mathbf{B}\|_{L^2(K\times(0,T))}+\|\mathbf{b}\|_{L^1(K\times(0,T))}\le c, \quad (7.25)$$

where the constant c *depends only on* K *and the constant in estimate* (7.11)*.*

Proof. The proof is based on the identities (7.12) and (7.13). It follows from the conditions of the proposition and relation (7.5) that

$$\boldsymbol{\lambda}(\mathbf{v})=\begin{pmatrix}\partial_r v_1 & 0 & \partial_{x_3}v_1\\ \partial_r v_2 & 0 & \partial_{x_3}v_2\\ \partial_r v_3 & 0 & \partial_{x_3}v_3\end{pmatrix},\quad \boldsymbol{\nu}(\mathbf{v})=\begin{pmatrix}0 & -\frac{1}{r}v_2 & 0\\ 0 & \frac{1}{r}v_1 & 0\\ 0 & 0 & 0\end{pmatrix}. \quad (7.26)$$

and

$$\boldsymbol{\lambda}(\boldsymbol{\varsigma})=\begin{pmatrix}\partial_r\varsigma_1 & 0 & \partial_{x_3}\varsigma_1\\ 0 & 0 & 0\\ \partial_r\varsigma_3 & 0 & \partial_{x_3}\varsigma_3\end{pmatrix},\quad \boldsymbol{\nu}(\boldsymbol{\varsigma})=\begin{pmatrix}0 & 0 & 0\\ 0 & \frac{1}{r}\varsigma_1 & 0\\ 0 & 0 & 0\end{pmatrix}. \quad (7.27)$$

In particular, $\boldsymbol{\lambda}(\boldsymbol{\varsigma})$ is orthogonal to $\boldsymbol{\nu}(\mathbf{v})$ and to $\boldsymbol{\nu}(\mathbf{v})^\top$. It follows that

$$\begin{aligned}&\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})+\boldsymbol{\lambda}(\mathbf{v})^\top+\boldsymbol{\nu}(\mathbf{v})^\top\big):\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)=\\ &\quad\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\lambda}(\mathbf{v})^\top\big):\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\big(\boldsymbol{\nu}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})^\top\big):\boldsymbol{\nu}(\boldsymbol{\varsigma}).\end{aligned}$$

It is easy to check that

$$\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\lambda}(\mathbf{v})^\top\big):\boldsymbol{\lambda}(\boldsymbol{\varsigma})=(\nabla\overline{\mathbf{v}}+\nabla\overline{\mathbf{v}}^\top):\nabla\overline{\boldsymbol{\varsigma}}$$

and

$$\big(\boldsymbol{\nu}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})^\top\big):\boldsymbol{\nu}(\boldsymbol{\varsigma}))=\frac{2}{r^2}\,v_1\,\varsigma_1.$$

We thus get

$$\begin{aligned}&\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})+\boldsymbol{\lambda}(\mathbf{v})^\top+\boldsymbol{\nu}(\mathbf{v})^\top\big):\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)=\\ &\quad(\nabla\overline{\mathbf{v}}+\nabla\overline{\mathbf{v}}^\top):\nabla\overline{\boldsymbol{\varsigma}}+\frac{2}{r^2}\,v_1\,\varsigma_1.\end{aligned} \quad (7.28)$$

Next, we have

$$\mathrm{tr}\,\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})\big)=\mathrm{div}\,\overline{\mathbf{v}}+\frac{1}{r}v_1,\quad \mathrm{tr}\,\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)=\mathrm{div}\,\overline{\boldsymbol{\varsigma}}+\frac{1}{r}\varsigma_1,$$

which yields

$$\begin{gathered}\frac{2}{3}\mathrm{tr}\,\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})\big)\mathrm{tr}\,\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)=\\ \frac{2}{3}(\mathrm{div}\,\overline{\mathbf{v}}+\frac{1}{r}v_1)(\mathrm{div}\,\overline{\boldsymbol{\varsigma}}+\frac{1}{r}\varsigma_1).\end{gathered}\tag{7.29}$$

Combining (7.28) and (7.29) we arrive at the identity

$$\begin{aligned}\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})+\boldsymbol{\lambda}(\mathbf{v})^\top+\boldsymbol{\nu}(\mathbf{v})^\top\big):\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)-&\\ \frac{2}{3}\mathrm{tr}\,\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})\big)\mathrm{tr}\,\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)=&\\ (\nabla\overline{\mathbf{v}}+\nabla\overline{\mathbf{v}}^\top):\nabla\overline{\boldsymbol{\varsigma}}+\frac{2}{r^2}\,v_1\,\varsigma_1-&\\ \frac{2}{3}(\mathrm{div}\,\overline{\mathbf{v}}+\frac{1}{r}v_1)(\mathrm{div}\,\overline{\boldsymbol{\varsigma}}+\frac{1}{r}\varsigma_1)=&\\ \big(\nabla\overline{\mathbf{v}}+\nabla\overline{\mathbf{v}}^\top-\frac{2}{3}(\mathrm{div}\,\overline{\mathbf{v}}+\frac{1}{r}v_1)\,\mathbf{I}\big):\nabla\overline{\boldsymbol{\varsigma}}+\Big(-\frac{4}{3r^2}\,v_1+\frac{2}{3r}\mathrm{div}\,\overline{\mathbf{v}}\Big)\varsigma_1.&\end{aligned}$$

From this and expression (7.24) for $\mathbf{B}$, we conclude that

$$\begin{gathered}\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})+\boldsymbol{\lambda}(\mathbf{v})^\top+\boldsymbol{\nu}(\mathbf{v})^\top\big):\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)-\\ \frac{2}{3}\mathrm{tr}\,\big(\boldsymbol{\lambda}(\mathbf{v})+\boldsymbol{\nu}(\mathbf{v})\big)\mathrm{tr}\,\big(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma})\big)=\\ \mathbf{B}:\nabla\overline{\boldsymbol{\varsigma}}+\Big(-\frac{4}{3r^2}\,v_1+\frac{2}{3r}\mathrm{div}\,\overline{\mathbf{v}}\Big)\,\varsigma_1.\end{gathered}\tag{7.30}$$

Next, notice that relations (7.27) imply

$$\varrho\mathbf{v}\otimes\mathbf{v}:(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma}))=\varrho\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}:\nabla\overline{\boldsymbol{\varsigma}}+\frac{1}{r}\varrho v_2^2\,\varsigma_1.\tag{7.31}$$

Relation (7.27) also implies

$$p(\varrho)\mathrm{tr}\,(\boldsymbol{\lambda}(\boldsymbol{\varsigma})+\boldsymbol{\nu}(\boldsymbol{\varsigma}))=p(\varrho)\mathrm{div}\,\overline{\boldsymbol{\varsigma}}+p(\varrho)\frac{1}{r}\varsigma_1.\tag{7.32}$$

Obviously we have

$$\varrho\mathbf{v}\cdot\partial_t\boldsymbol{\varsigma}=\varrho\overline{\mathbf{v}}\cdot\partial_t\overline{\boldsymbol{\varsigma}},\quad \varrho\mathbf{f}^*\cdot\boldsymbol{\varsigma}=\varrho\overline{\mathbf{f}}\cdot\overline{\boldsymbol{\varsigma}}.\tag{7.33}$$

Substituting (7.30)-(7.33) into integral identity (7.12), recalling formula (7.24) for $\mathbf{b}$, and noting that the integrand in (7.12) is independent of θ we arrive at the desired identity (7.22).

Estimate (7.25) for $\mathbf{B}$ and $\mathbf{b}$ is a straightforward consequence of estimate (7.11).

It remains to prove integral identity (7.24). To this end, it suffices to note that the test function ψ is independent of θ and hence

$$\varrho\mathbf{v}\cdot(\mathbf{R}\nabla_z\psi)=\varrho\overline{\mathbf{v}}\cdot\nabla_y\psi.$$

Substituting this equality into integral identity (7.13) we arrive at (7.24). □

8 Appendix. Potential estimates. Singular operators

In this section we collect the basic facts from the theory of potential operators and singular integrals.

8.1. Estimates of potentials. We begin with the representation of the Sobolev functions via their gradients.

Lemma 8.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a Lipschitz boundary. Then the equality*

$$u(x) = \frac{1}{n\omega_n} \int_\Omega \frac{(x_i - y_i)\partial_i u(y)}{|x - y|^n}\, dy \quad a.e.\ in \quad \mathbb{R}^n \tag{8.1}$$

holds for ever function $u \in W^{1,1}(\mathbb{R}^n)$ compactly supported in Ω. Here ω_n is a volume of $\mathbb{S}^{n-1}$

Proof. This statement is a particular case of Lemma 7.14 in [9]. □

Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set of positive measure. For every $\mu \in (0, 1)$, define the integral operator

$$\mathcal{V}_\mu u(x) = \int_\Omega |x - y|^{-n(1-\mu)} u(y)\, dy. \tag{8.2}$$

Lemma 8.2. *Let $1 < s \leq r < \infty$ and*

$$0 \leq \kappa = s^{-1} - r^{-1} < \mu.$$

Then the operator $\mathcal{V}_\mu : L^s(\Omega) \to L^r(\Omega)$ is bounded and

$$\|\mathcal{V}_\mu u\|_{L^r(\Omega)} \leq \Big(\frac{1-\kappa}{\mu - \kappa}\Big)^{1-\kappa} \omega_n^{1-\mu} (\text{meas } \Omega)^{\mu-\kappa}\, \|u\|_{L^s(\Omega)} \tag{8.3}$$

for all $u \in L^s(\Omega)$.

Proof. This statement is a particular case of Lemma 7.12 in [9]. □

8.2. Singular integrals. The Calderon-Zygmund singular integral operators are defined as follows

$$\mathbf{S}u(x) = \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} K(x, x - y) u(y)\, dy, \quad x, y \in \mathbb{R}^d, \quad d \geq 1.$$

Here the Calderon-Zygmund kernel K satisfies the conditions

$$K(x, z) = \frac{1}{|z|^d}\, \omega\big(x, \frac{z}{|z|}\big), \quad \omega \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1}), \quad \int_{\mathbb{S}^{d-1}} \omega(x, \xi)\, dS^{d-1}(\xi) = 0.$$

It is known that the the operator $\mathbf{S} : L^r(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ is bounded for all $1 < r < \infty$. Moreover, it admits the estimate, see [2],

$$\|\mathbf{S}u\|_{L^r(\mathbb{R}^d)} \leq c \frac{r^2}{r-1} \|u\|_{L^r(\mathbb{R}^d)}, \quad u \in L^r(\mathbb{R}^d),$$

where c is independent of r and u. The typical example of the Calderon-Zygmund singular integral is the Riesz operator

$$\mathbf{R}_j u = \partial_j(-\Delta)^{-1/2}u.$$

In particular it satisfies the inequality

$$\|\mathbf{R}_i u\|_{L^r(\mathbb{R}^d)} \le c\frac{r^2}{r-1}\|u\|_{L^r(\mathbb{R}^d)}, \quad u \in L^r(\mathbb{R}^d). \tag{8.4}$$

The operators $\mathbf{R}_{ij} = \mathbf{R}_i\mathbf{R}_j$ are not Calderon-Zygmund operators. However, they are products of such operators and admit the estimates

$$\|\mathbf{R}_{ij} u\|_{L^r(\mathbb{R}^d)} \le c\frac{r^4}{(r-1)^2}\|u\|_{L^r(\mathbb{R}^d)}, \quad u \in L^r(\mathbb{R}^d), \quad 1 \le i,j \le d. \tag{8.5}$$

9 Appendix. Proof of Proposition 4.2

Note that conditions **C.1-C.3** and equality $\rho = \zeta\varrho$ imply the estimates

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\mathbb{R}^2} \rho \ln(2+\rho)\, dy \le c_0. \tag{9.6}$$

$$\int_{G\times(0,T)} \big((-\Delta)^{-1/4}\rho\big)^2 dydt \le c. \tag{9.7}$$

Here

$$(-\Delta)^{-1/4}\rho(y,t) = c_{1/4}\int_G \frac{1}{|y-z|^{3/2}}\,\rho(z,t)\,dz.$$

The further considerations are based on the following lemma.

Lemma 9.1. *Let ρ and $\mathbf{W}$ satisfy all conditions of Proposition 4.2. Then for every integer $n \ge 2$, we have*

$$\int_{\mathbb{R}^3} \rho|\mathbf{W}|^n\, dydt \le c, \tag{9.8}$$

where c depends on K, n, and constants in (9.6), (9.7).

Proof. Since $\mathbf{W}(\cdot,t)$ is compactly supported in G, it follows from the embedding theorem that

$$\int_{\mathbb{R}^2} |\mathbf{W}(y,t)|^r\, dy = \int_G |\mathbf{W}(y,t)|^r\, dy \le c(r)\|\mathbf{W}(\cdot,t)\|^r_{W^{1,2}(G)} \le c(r) \tag{9.9}$$

for every $r \in (1,\infty)$ and a.e. $t \in \mathbb{R}$. Now set

$$\varphi = |\mathbf{W}|^{n-1}|\nabla\mathbf{W}|, \quad |\nabla|\mathbf{W}|^n| = n|\mathbf{W}|^{n-2}|\nabla\mathbf{W}^\top\mathbf{W}| \le n\varphi. \tag{9.10}$$

Estimate (9.9) and the Hölder inequality imply the estimate

$$\begin{aligned}\|\varphi(t)\|_{L^s(\mathbb{R}^2)} = \||\mathbf{W}|^{n-1}(t)|\nabla\mathbf{W}|(t)\|_{L^s(\mathbb{R}^2)} \le \\ \||\mathbf{W}(t)|^{n-1}\|_{L^{\frac{2s}{2-s}}(\mathbb{R}^2)}\||\nabla\mathbf{W}(t)|\|_{L^2(\mathbb{R}^2)} \le c,\end{aligned} \tag{9.11}$$

which holds for every $s \in [1,2)$ and a.e. $t \in \mathbb{R}$. It follows from identity (8.1) in Lemma 8.1 that

$$|\mathbf{W}(y,t)|^n = \frac{1}{4\pi}\int_G \frac{\nabla|\mathbf{W}|^n(z,t)\cdot(y-z)}{|y-z|^2}\,dz, \quad y \in \mathbb{R}^2$$

We thus get

$$|\mathbf{W}(y,t)|^n \le c\int_{\mathbb{R}^2}\frac{1}{|y-z|}\,\varphi(z,t)\,dz = c(-\Delta)^{-1/2}\varphi(y,t). \tag{9.12}$$

Recall that φ is compactly supported in $G\times(0,T)$. We have

$$(-\Delta)^{-1/2}\varphi = (-\Delta)^{-1/4}(-\Delta)^{-1/4}\varphi, \quad (-\Delta)^{-1/4}\varphi = c_{1/4}\int_{\mathbb{R}^2}\frac{\varphi(z,t)}{|y-z|^{3/2}}\,dz.$$

Since G is bounded, there is a disc $B_R = \{|y| \le R\}$ such that $G \Subset B_R$. Note that the integral operator

$$(-\Delta)^{-1/4}\varphi = c_{1/4}\int_{\mathbb{R}^2}\frac{\varphi(z,t)}{|y-z|^{3/2}}\,dz = c_{1/4}\int_{B_{2R}}\frac{\varphi(z,t)}{|y-z|^{3/2}}\,dz$$

meets all requirements of Lemma 8.2 with $\Omega = B_{2R}$, $\mu = 1/4$. Now set $r = 3$, $s = 2$. We have $\kappa = 2^{-1} - 3^{-1} < 4^{-1}$ Applying inequality (8.3) in Lemma 8.2 we arrive at the estimate

$$\|(-\Delta)^{-1/4}\varphi(t)\|_{L^3(B_{2R})} \le c \ \text{ for a.e. } \ t \in \mathbb{R}. \tag{9.13}$$

Since $\varphi(t)$ is supported in $G \subset B_R$, we have

$$|(-\Delta)^{-1/4}\varphi(t)| \le c|y|^{-3/2}\|\varphi\|_{L^1(G)} \le c|y|^{-3/2} \ \text{ in } \ \mathbb{R}^2 \setminus B_{2R}.$$

It follows from this and (9.13) that

$$\|(-\Delta)^{-1/4}\varphi(t)\|_{L^2(\mathbb{R}^2)} \le c \ \text{ for a.e. } \ t \in \mathbb{R}.$$

Since $\varphi(t)$ vanishes outside of the interval $(0,T)$, we have

$$\|(-\Delta)^{-1/4}\varphi\|_{L^2(\mathbb{R}^3)} \le c.$$

Combining this result with estimates (9.12) and estimate (9.7) we finally obtain the desired estimate

$$\begin{aligned}\int_{\mathbb{R}^3}\rho|\mathbf{W}|^n\,dydt &\le c\int_{\mathbb{R}^3}\rho(-\Delta)^{-1/2}\varphi\,dydt\\ &= c\int_{\mathbb{R}^3}\rho(-\Delta)^{-1/4}(-\Delta)^{-1/4}\varphi\,dydt\\ &= c\int_{\mathbb{R}^3}(-\Delta)^{-1/4}\rho(-\Delta)^{-1/4}\varphi\,dydt \le c.\end{aligned}$$

□

We are now in a position to complete the proof of Proposition 4.2. We have

$$\begin{aligned}\rho|\mathbf{W}|^2\ln(2+\rho|\mathbf{W}|^2)^{1/2} \le\ & c\rho\ln(2+\rho)^{1/2}|\mathbf{W}|^2 + \\ & c\rho|\mathbf{W}|^2\ln(2+|\mathbf{W}|^2)^{1/2}.\end{aligned} \tag{9.14}$$

It is easily seen that

$$\begin{aligned}\rho\ln(2+\rho)^{1/2}|\mathbf{W}|^2 &\leq \rho\ln(2+\rho)+\rho|\mathbf{W}|^4,\\ \rho|\mathbf{W}|^2\ln(2+|\mathbf{W}|^2) &\leq c\rho|\mathbf{W}|^2+c\rho|\mathbf{W}|^3.\end{aligned}$$

Substituting these inequalities into (9.14) we arrive at the estimate

$$\rho|\mathbf{W}|^2\ln(2+\rho|\mathbf{W}|^2)^{1/2}\leq c\rho\ln(2+\rho)+c\rho\sum_{k=2}^4|\mathbf{W}|^3.$$

Integrating both the sides of this inequality over $\mathbb{R}^3$ and applying estimates (9.6) and (9.8) we obtain the desired estimate (4.12). This completes the proof of Proposition 4.2.

10 Appendix. Proof of Lemma 5.4

By virtue of (3.5), the function ρ satisfies the integral identity

$$\int_{\mathbb{R}^3}\big(\rho\partial_t\psi+\rho\overline{\mathbf{w}}\cdot\nabla\psi+\varrho\,\mathbf{h}\,\psi\big)\,y_1dydt=0$$

for every $\psi\in C^1_{\mathrm{loc}}(\mathbb{R}^3)$. It follows that it satisfies the first order differential equation

$$y_1\partial_t\rho+\ \mathrm{div}\ (\rho y_1\overline{\mathbf{w}})=y_1\rho\mathbf{h},$$

which is understood in the sense of distribution. Note that ρ and $\mathbf{w}$ are compactly supported in $G\times(0,T)$. Moreover, Proposition 3.1 implies that

$$0<R^{-1}\leq y_1\leq R,$$

where R depends only on the compact set K. Hence we may rewrite equation for ρ in the form

$$\partial_t\rho+\ \mathrm{div}\ (\rho\overline{\mathbf{w}})=\rho\big(\nabla\zeta\cdot\overline{\mathbf{w}}-\frac{1}{y_1}w_1\big). \tag{10.15}$$

Here we use the relation $\mathbf{h}=\nabla\zeta\cdot\overline{\mathbf{w}}$. Since $\rho\in L^\infty(\mathbb{R};L^3(\mathbb{R}^2))$ and $\overline{\mathbf{w}}\in L^2(\mathbb{R};L^r(\mathbb{R}^2))$, the right hand side of this equation belongs to the class $L^2(\mathbb{R};L^q(\mathbb{R}^2))$ for every $q\in[1,3)$. Next, the vector field $\overline{\mathbf{w}}$ belongs to the class $L^2(\mathbb{R};W^{1,2}(\mathbb{R}^2))$. Moreover, the functions ρ and $\overline{\mathbf{w}}$ are compactly supported in $\mathbb{R}^3$. Hence we may apply the renormalization procedure to to obtain the equation for $\varphi(\rho)$.

$$\partial_t\varphi+\ \mathrm{div}\ (\varphi\overline{\mathbf{w}})=\boldsymbol{\pi}. \tag{10.16}$$

where

$$\boldsymbol{\pi}=\frac{\delta\rho}{1+\delta\rho}\big(\nabla\zeta\cdot\overline{\mathbf{w}}-\frac{1}{y_1}w_1-\ \mathrm{div}\ \overline{\mathbf{w}}\big)+\varphi\ \mathrm{div}\ \overline{\mathbf{w}}.$$

Equation (10.16) is equivalent to the system of integral identities

$$\int_{\mathbb{R}^3}\big(\varphi\partial_t\psi+\varphi\overline{\mathbf{w}})\cdot\nabla\psi+\boldsymbol{\pi}\,\psi\big)\,dydt, \tag{10.17}$$

where $\psi \in C^1_{\rm loc}(\mathbb{R}^3)$ is an arbitrary function. Now choose an arbitrary vector field $\boldsymbol{v} \in C^\infty_0(\mathbb{R}^3)$. Substituting the test function $\psi = \Delta^{-1}$ div $\boldsymbol{v}$ into identity (10.17) and noting that

$$(\Delta^{-1} \text{ div })^\top = -\nabla\Delta^{-1}$$

we arrive at the integral identity

$$\int_{\mathbb{R}^3} \big((\nabla\Delta^{-1}\varphi)\cdot\partial_t\boldsymbol{v} + \nabla\Delta^{-1} \text{ div}(\varphi\overline{\mathbf{w}})\cdot\boldsymbol{v} + \nabla\Delta^{-1}\boldsymbol{\pi}\cdot\boldsymbol{v}\big)\, dydt = 0.$$

Since $\boldsymbol{v}$ is an arbitrary smooth compactly supported vector field, it follows from this that

$$\partial_t(\nabla\Delta^{-1}\varphi) = -\nabla\Delta^{-1} \text{ div}(\varphi\overline{\mathbf{w}}) + \nabla\Delta^{-1}\boldsymbol{\pi}. \tag{10.18}$$

Notice that

$$\varphi \in L^\infty(\mathbb{R}; L^r(\mathbb{R}^2)), \quad \overline{\mathbf{w}} \in L^2(\mathbb{R}; L^r(\mathbb{R}^2)), \quad 1 < r < \infty,$$

which yields

$$\varphi\overline{\mathbf{w}} \in L^2(\mathbb{R}; L^r(\mathbb{R}^2)), \quad \boldsymbol{\pi} \in L^2(\mathbb{R}; L^q(\mathbb{R}^2)).$$

for all $r \in (1,\infty)$ and $q \in (1,2)$. From this and estimates (8.3), (8.5) for potential and singular integrals in Section 8 we conclude that

$$\begin{aligned} \nabla\Delta^{-1} \text{ div}(\varphi\overline{\mathbf{w}}) \in L^2(\mathbb{R}; L^r(\mathbb{R}^2)), \qquad \nabla^2\Delta^{-1}\varphi \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^2)), \\ \nabla\Delta^{-1}\varphi \in L^\infty(\mathbb{R}; C(\mathbb{R}^2)), \qquad \nabla\Delta^{-1}\boldsymbol{\pi} \in L^2(\mathbb{R}; L^r(\mathbb{R}^2)), \end{aligned} \tag{10.19}$$

for all $r \in (1,\infty)$. In particular, we have

$$\partial_t(\nabla\Delta^{-1}\varphi) \in L^2(\mathbb{R}; L^r(\mathbb{R}^2)) \ \text{ for all } \ r \in (1,\infty). \tag{10.20}$$

On the other hand, we have

$$\begin{aligned} \rho \in L^\infty(\mathbb{R}, L^3(\mathbb{R}^2)), \quad \rho\overline{\mathbf{w}} \in L^2(\mathbb{R}; L^q(\mathbb{R}^2)), \\ \varrho\overline{\mathbf{w}}\otimes\overline{\mathbf{w}} \in L^1(\mathbb{R}; L^q(\mathbb{R}^2)) \end{aligned} \tag{10.21}$$

for all $q \in (1,3)$. Hence we may substitute the test function $\overline{\boldsymbol{\varsigma}} = \nabla\Delta^{-1}\varphi$ into integral identity (3.4). Using formula (10.18) for $\partial_t\overline{\boldsymbol{\varsigma}} = \partial_t\nabla\Delta^{-1}\varphi$ we arrive at the identity

$$\sum_{i=1}^5 I_i = \int_{\mathbf{R}^3} \zeta p(\varrho)\varphi\, y_1 dydt. \tag{10.22}$$

It remains to note that desired estimate (5.10) follows from this and the obvious relations

$$p_2(\rho) \equiv \zeta p_2(\zeta\varrho) \le \zeta p(\varrho).$$

This completes the proof of Lemma 5.4

11 Appendix. Proof of Proposition 5.6

Throughout of this section we deal with the functions $\rho : \mathbb{R}^2 \to \mathbb{R}^+$ and vector fields $\mathbf{W} = (W_1, W_3) : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following condition.

Condition 11.1. *The functions ρ and $\mathbf{W}$ are compactly supported in G. They admit the estimate*

$$\begin{gathered}\int_{\mathbb{R}^2} \rho |\ln \rho|\, dy + \int_{\mathbb{R}^2} e_1(\rho)\, dy \le c_1 < \infty, \quad \|\mathbf{W}\|_{W^{1,2}(\mathbb{R}^3)} \le 1, \\ 0 < c_1^{-1} \le y_1 \le c_1 \quad \text{on the supports of } \rho \text{ and } \mathbf{W}.\end{gathered} \tag{11.1}$$

Remark 11.2. *Under the above assumptions,*

$$\int_{\mathbb{R}^2} \rho^{1+\varepsilon}\, dy \le c\,\varepsilon^{-1}. \tag{11.2}$$

Here c depends only on c_1 and G. Indeed, set $\lambda = \varepsilon\rho$. We have

$$\lambda^{1+\varepsilon} \ln(1+\lambda) = \varepsilon^{1+\varepsilon} \rho^{1+\varepsilon} \ln(1+\varepsilon\rho) \le c e_1(\rho),$$

which yields

$$\int_{\mathbb{R}^2} \lambda^{1+\varepsilon} \ln(1+\lambda)\, dy \le c. \tag{11.3}$$

Since ρ and λ are supported in G, we also have

$$\lambda^{1+\varepsilon} \le c\big(\lambda^{1+\varepsilon} \ln(1+\lambda) + \mathbf{1}_G\big).$$

From this and (11.3) we obtain the inequality

$$\int_{\mathbb{R}^2} \lambda^{1+\varepsilon}\, dy \le c.$$

which obviously yields (11.2)

Recall the formulation of Proposition 5.6.

Proposition 11.3. *Let ρ and $\mathbf{W}$ satisfy Condition 11.1. Then there is a constant c, depending on c_1 and G such that*

$$\|\rho \mathbf{W}\|_{L^{1+\frac{\varepsilon}{2}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^2}, \qquad \|\rho |\mathbf{W}|^2\|_{L^{1+\frac{\varepsilon}{2}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^3}, \tag{11.4}$$

$$\|\nabla \Delta^{-1} \operatorname{div}(\varphi \mathbf{W})\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^6}, \qquad \|\nabla^2 \Delta^{-1} \varphi\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^5}, \tag{11.5}$$

$$\|\nabla \Delta^{-1} \boldsymbol{\pi}^*\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)} \le \frac{c}{\varepsilon^{3/2}}, \qquad \|\nabla \Delta^{-1} \varphi\|_{L^\infty(\mathbb{R}^2)} \le c. \tag{11.6}$$

Here

$$\varphi = \ln(1+\delta\rho), \quad \boldsymbol{\pi}^* = \frac{\delta\rho}{1+\delta\rho}\big(\nabla\zeta \cdot \mathbf{W} - \frac{1}{y_1} W_1 - \operatorname{div} \mathbf{W}\big) + \varphi \operatorname{div} \mathbf{W}. \tag{11.7}$$

The proof is based on the two technical lemmas.

Lemma 11.4. *Under the assumptions of Proposition* (11.3)*, the function* φ *admits the estimate*

$$\|\varphi\|_{L^\gamma(\mathbb{R}^2)} \le c\gamma \quad \textit{for all} \quad \gamma \in [2, \infty). \tag{11.8}$$

Here c *depends only on* G *and the constant* c_1 *in Condition 11.1.*

Proof. Note that the function $\ln(1+s)^\gamma/(1+s)$, $s \in [0, \infty)$, takes the maximum at the point $s = e^\gamma - 1$, where it equals $\gamma^\gamma e^{-\gamma}$. Since $\varphi(\rho)$ and ρ are supported in G, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi^\gamma \, dy &= \int_G \varphi^\gamma \, dy \le \int_G \ln(1+\rho)^\gamma \, dy \\ &\le \gamma^\gamma e^{-\gamma} \int_G (1+\rho)\, dy \le c\,\gamma^\gamma e^{-\gamma},\end{aligned}$$

which obviously yields (11.8) $\square$

Lemma 11.5. *Under the assumptions of Proposition* (11.3)*, the vector* $\mathbf{W}$ *admits the estimate*

$$\|\mathbf{W}\|_{L^\gamma(\mathbb{R}^2)} \le c\gamma \quad \textit{for all} \quad \gamma \in [2, \infty). \tag{11.9}$$

Here c *depends only on* G.

Proof. It follows from identity (8.1) in Lemma 8.1 that

$$\mathbf{W} = \mathcal{V}_i \partial_i \mathbf{W}, \quad \text{where} \quad \mathcal{V}_i f = c \int_{\mathbb{R}^2} \frac{y_i - z_i}{|y-z|^2} f(z)\, dz.$$

The potential operator $\mathcal{V}_i$ meets all requirements of Lemma 8.2 with $\Omega = G$, $n = 2$, and $\mu = 1/2$. It follows from estimate (8.3) in Lemma 8.2 with $s = 2$ and $r = \gamma$ that

$$\|\mathcal{V}_i \nabla \mathbf{W}\|_{L^\gamma(G)} \le c\Big(\frac{1-\kappa}{\frac12 - \kappa}\Big)^{1-\kappa} \|\nabla \mathbf{W}\|_{L^2(G)} \le c\Big(\frac{1-\kappa}{\frac12 - \kappa}\Big)^{1-\kappa}, \quad \kappa = \frac12 - \frac1\gamma.$$

Thus we get

$$\begin{aligned}\|\mathbf{W}\|_{L^\gamma(G)} &\le \|\mathcal{V}_i \nabla \mathbf{W}\|_{L^\gamma(G)} \\ &\le c\Big(\frac{1-\kappa}{\frac12 - \kappa}\Big)^{1-\kappa} = c\Big(\frac{2+\gamma}{2}\Big)^{\frac12 + \frac1\gamma} \le c\,\gamma,\end{aligned} \tag{11.10}$$

and the lemma follows. $\square$

We split the rest of proof of Proposition 11.3 into three steps.

Step 1. Proof of estimates (11.4)**.** It follows from the Hölder inequality that the estimate

$$\int_{\mathbb{R}^2} \rho^{1+\frac{\varepsilon}{2}} |\mathbf{W}|^{1+\frac{\varepsilon}{2}} \, dy \le \Big(\int_{\mathbb{R}^2} \rho^{\alpha(1+\frac{\varepsilon}{2})} \, dy\Big)^{\frac1\alpha} \Big(\int_{\mathbb{R}^2} |\mathbf{W}|^{\frac{\alpha}{\alpha-1}(1+\frac{\varepsilon}{2})} \, dy\Big)^{\frac{\alpha-1}{\alpha}}$$

holds for every $1 < \alpha < \infty$. Now set

$$\alpha = \frac{1+\varepsilon}{1+\frac{\varepsilon}{2}} = \frac{2+2\varepsilon}{2+\varepsilon}, \quad \frac{\alpha}{\alpha-1} = \frac{2+2\varepsilon}{\varepsilon}.$$

By virtue of Remark (11.2), we have

$$\Big(\int_{\mathbb{R}^2}\rho^{\alpha(1+\frac{\varepsilon}{2})}\,dy\Big)^{\frac{1}{\alpha}}=\Big(\int_{\mathbb{R}^2}\rho^{1+\varepsilon}\,dydy\Big)^{\frac{1}{\alpha}}\le c\Big(\frac{1}{\varepsilon}\Big)^{\frac{2+\varepsilon}{2+2\varepsilon}}.$$

Noting that

$$\frac{\alpha}{\alpha-1}(1+\frac{\varepsilon}{2})=(2+\varepsilon)\frac{1+\varepsilon}{\varepsilon}:=\gamma>2,$$

we obtain

$$\|\rho\mathbf{W}\|_{L^{1+\frac{\varepsilon}{2}}(\mathbb{R}^2)}\le c\Big(\frac{1}{\varepsilon}\Big)^{\frac{2+\varepsilon}{2+2\varepsilon}}\ \|\mathbf{W}\|_{L^{\gamma}(\mathbb{R}^2)}.$$

From this and inequality (11.9) in Lemma 11.5 we get desired estimate (11.4) for $\rho\mathbf{W}$. Repeating these arguments we obtain

$$\|\rho|\mathbf{W}|^2\|_{L^{1+\frac{\varepsilon}{2}}(\mathbb{R}^2)}\le c\Big(\frac{1}{\varepsilon}\Big)^{\frac{2+\varepsilon}{2+2\varepsilon}}\ \||\mathbf{W}|^2\|_{L^{\gamma}(\mathbb{R}^2)}\le\frac{c}{\varepsilon}\,\|\mathbf{W}\|^2_{L^{2\gamma}(G)}.$$

Applying estimate (11.9) with γ replaced by 2γ we arrive at the estimate (11.4) for $\rho|\mathbf{W}|^2$. This completes the proof of inequalities (11.4).

Step 2. Proof of estimates (11.5). Note that the matrix-value operator $\nabla\Delta^{-1}$ div admits the representation

$$\big(\nabla\Delta^{-1}\ \mathrm{div}\ \big)_{ij}=\mathbf{R}_i\mathbf{R}_j=\mathbf{R}_{ij},$$

where $\mathbf{R}_i$ is the Riesz operator. It follows from the Calderon-Zygmund estimate (8.5) that

$$\|\mathbf{R}_{ij}f\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}\le\frac{c}{\varepsilon^4}\,\|f\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}\quad\text{for all}\quad f\in L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2).\tag{11.11}$$

In particular, we have

$$\|\nabla\Delta^{-1}\ \mathrm{div}\ (\varphi\mathbf{W})\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}\le\frac{c}{\varepsilon^4}\,\|\varphi\mathbf{W}\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}.\tag{11.12}$$

The Cauchy inequality implies

$$\|\varphi\mathbf{W}\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}\le\|\varphi\|_{L^{2\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}\ \|\mathbf{W}\|_{L^{2\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}.$$

From this and estimates (11.8), (11.9) with $\gamma=2(2+\varepsilon)\varepsilon^{-1}$ we obtain

$$\|\varphi\mathbf{W}\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}\le\frac{c}{\varepsilon^2}$$

Substituting this inequality into (11.12) we arrive at the first estimate in (11.5). In order to prove the second, note that $\nabla^2\Delta^{-1}\varphi$ is the matrix valued func tion with entries

$$(\nabla^2\Delta^{-1}\varphi)_{ij}=\mathbf{R}_j\mathbf{R}_i\varphi.$$

Applying inequality (11.11) we arrive at the estimate

$$\|\nabla^2\Delta^{-1}\varphi\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}\le\frac{c}{\varepsilon^4}\|\varphi\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2)}.$$

From this and estimate (11.8) with $\gamma=(2+\varepsilon)\varepsilon^{-1}$ we obtain the second estimate (11.5).

Step 3. The proof of estimates (11.6)**.** We begin with the observation that

$$\begin{aligned} |\nabla\Delta^{-1}\boldsymbol{\pi}^*| &= c\Big|\int_G \frac{y-z}{|y-z|^2}\boldsymbol{\pi}^*(z)\,dz\Big| \\ &\leq c\int_G \frac{1}{|y-z|}|\boldsymbol{\pi}^*(z)|\,dz. \end{aligned}$$

On the other hand, relation (11.7) implies

$$|\boldsymbol{\pi}^*| \leq c|\mathbf{W}| + |1+\varphi|\,|\operatorname{div}\,\mathbf{W}|.$$

We get

$$\begin{aligned} |\nabla\Delta^{-1}\boldsymbol{\pi}^*| &\leq c(-\Delta)^{-1/2}(|\mathbf{W}|) + c(-\Delta)^{-1/2}\big(|1+\varphi|\,|\operatorname{div}\,\mathbf{W}|\big) \qquad (11.13) \\ c\int_G \frac{1}{|y-z|}|\mathbf{W}|\,dz &+ c\int_G \frac{1}{|y-z|}|1+\varphi|\,|\operatorname{div}\,\mathbf{W}|\,dz. \end{aligned}$$

It is easy to see that the potential $(-\Delta)^{-1/2}$ meets all requirements of Lemma 8.2 with $n=2$, $\mu=1/2$, and $\Omega=G$. Now set

$$s=2,\quad r=\frac{2+\varepsilon}{\varepsilon},\quad \kappa=\frac{1}{2}-\frac{\varepsilon}{2+\varepsilon}.$$

Applying estimate (8.3) in Lemma 8.2 to the potential $\Delta^{-1/2}(|\mathbf{W}|)$ and noting that

$$\Big(\frac{1-\kappa}{\mu-\kappa}\Big)^{1-\kappa} = \Big(\frac{2+3\varepsilon}{2\varepsilon}\Big)^{\frac{2+3\varepsilon}{2\varepsilon}} \leq \frac{c}{\varepsilon},$$

we arrive at the estimate for the first integral in the right hand side of (11.13).

$$\|(-\Delta)^{-1/2}(|\mathbf{W}|)\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(G)} \leq \frac{c}{\varepsilon}\|\mathbf{W}\|_{L^2(G)} \leq \frac{c}{\varepsilon}. \qquad (11.14)$$

The estimate of the second integral in (11.13) is more complicated. Choose the number s satisfying the equalities

$$\frac{2s}{2-s} = \frac{2+\varepsilon}{\varepsilon(1-\varepsilon)},\quad s=\frac{2(2+\varepsilon)}{2+3\varepsilon-2\varepsilon^2}\in(1,2). \qquad (11.15)$$

Now set

$$r=\frac{2+\varepsilon}{\varepsilon},\quad \kappa=\frac{1}{s}-\frac{1}{r}=\frac{2+\varepsilon-2\varepsilon^2}{2(2+\varepsilon)}<\frac{1}{2}.$$

Recall that the potential operator $(-\Delta)^{-1/2}$ satisfies all conditions of Lemma 8.2 with $n=2$, $\mu=1/2$, and $\Omega=G$. For our choice s and r, the quantity $\kappa<1/2$. Hence we may apply estimate (8.3) to obtain

$$\begin{gathered} \|(-\Delta)^{-1/2}\big(|1+\varphi|\,|\operatorname{div}\,\mathbf{W}|\big)\|_{L^r(G)} \leq \\ c\Big(\frac{1-\kappa}{1/2-\kappa}\Big)^{1-\kappa}\,\||1+\varphi|\,|\operatorname{div}\,\mathbf{W}|\|_{L^s(G)}. \end{gathered}$$

It follows from the expression for κ that

$$\Big(\frac{1-\kappa}{1/2-\kappa}\Big)^{1-\kappa} = \Big(\frac{2+\varepsilon+2\varepsilon^2}{\varepsilon^2}\Big)^{\frac{2+\varepsilon+2\varepsilon^2}{2(2+\varepsilon)}} \leq \frac{c}{\varepsilon^2}.$$

Thus we get

$$\|(-\Delta)^{-1/2}\big(|1+\varphi|\,|\ \mathrm{div}\ \mathbf{W}|\big)\|_{L^r(G)} \le \frac{c}{\varepsilon^2}\,\||1+\varphi|\,|\ \mathrm{div}\ \mathbf{W}|\|_{L^s(G)}. \quad (11.16)$$

Let us estimate the right hand side of this inequality. Set

$$\alpha = \frac{2}{s}, \quad \frac{\alpha}{\alpha-1} = \frac{2}{2-s}$$

It follows from the Hölder inequality that

$$\||1+\varphi|\,|\ \mathrm{div}\ \mathbf{W}|\|_{L^s(G)} \le \|1+\varphi\|_{L^{\frac{\alpha s}{\alpha-1}}(G)}\|\ \mathrm{div}\ \mathbf{W}|\|_{L^{\alpha s}(G)},$$

or equivalently

$$\begin{aligned}\||1+\varphi|\,|\ \mathrm{div}\ \mathbf{W}|\|_{L^s(G)} &\le \|1+\varphi\|_{L^{\frac{2s}{2-s}}(G)}\|\ \mathrm{div}\ \mathbf{W}|\|_{L^2(G)} \quad (11.17)\\ &\le c\|1+\varphi\|_{L^{\frac{2s}{2-s}}(G)}.\end{aligned}$$

Estimate (11.15) yields

$$\begin{aligned}\|1+\varphi\|_{L^{\frac{2s}{2-s}}(G)} &\le c\big(1+\|\varphi\|_{L^{\frac{2s}{2-s}}(G)}\big)\\ &= c\big(1+\|\varphi\|_{L^{\frac{2+\varepsilon}{\varepsilon(1-\varepsilon)}}(G)}\big).\end{aligned}$$

Applying Lemma 11.4 with $\gamma = (2+\varepsilon)/(\varepsilon(1-\varepsilon))$ we arrive at the estimate

$$\|1+\varphi\|_{L^{\frac{2s}{2-s}}(G)} \le \frac{c}{\varepsilon}.$$

From this and inequality (11.17) we get the estimate

$$\||1+\varphi|\,|\ \mathrm{div}\ \mathbf{W}|\|_{L^s(G)} \le \frac{c}{\varepsilon}.$$

Substituting this inequality into (11.16) and noting that $r = (2+\varepsilon)/\varepsilon$ we get the estimate

$$\|(-\Delta)^{-1/2}\big(|1+\varphi|\,|\ \mathrm{div}\ \mathbf{W}|\big)\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(G)} \le \frac{c}{\varepsilon^3}.$$

Combining this result with estimate (11.14) and recalling inequality (11.13) we finally obtain the desired estimate

$$\begin{aligned}\|\nabla\Delta^{-1}\boldsymbol{\pi}^*\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(G)} &\le c\|(-\Delta)^{-1/2}\big(|(1+\varphi)\ \mathrm{div}\ \mathbf{W}|\big)\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(G)}\\ &+ c\|(-\Delta)^{-1/2}\big(\,|\mathbf{W}|\big)\|_{L^{\frac{2+\varepsilon}{\varepsilon}}(G)} \le \frac{c}{\varepsilon^3}.\end{aligned}$$

It remains to prove the second inequality in (11.6). Arguing as before we obtain

$$|\nabla\Delta^{-1}\varphi| \le c(-\Delta)^{-1/2}\varphi = c\int_G \frac{1}{|y-z|}\,\varphi(z)\,dz.$$

Obviously we have

$$(-\Delta)^{-1/2}\varphi \le c\|\varphi\|_{L^3(G)}$$

Applying Lemma 11.4 with $\gamma = 3$ we arrive at the desired estimate

$$|\nabla\Delta^{-1}\varphi| \le c(-\Delta)^{-1/2}\varphi \le c\|\varphi\|_{L^3(G)} \le c.$$

This completes the proof of estimate (11.6) and Proposition 11.3.

12 Appendix. Proof of Lemma 6.1

First we prove the equi-integrability of the sequence $\varrho_\varepsilon \mathbf{u}_\varepsilon$. To this end, choose an arbitrary set $A \subset \Omega$ and N such that

$$|A| := \text{ meas } A < 1/3, \quad N > 3. \tag{12.1}$$

Fix $t \in (0, T)$ and set

$$A_N = \{x \in A : \varrho_\varepsilon(x,t) \geq N\} \subset A.$$

For every $\epsilon > 0$ we have

$$\begin{aligned} \int_A \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|\,dx &\leq \frac{1}{2\epsilon}\int_A \varrho_\varepsilon(t)\,dx + \frac{\epsilon}{2}\int_A \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|^2\,dx \\ &= \frac{1}{2\epsilon}\Big(\int_{A\setminus A_N} \varrho_\varepsilon(t)\,dx + \int_{A_N}\varrho_\varepsilon(t)\,dx\Big) + \frac{\epsilon}{2}\int_A \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|^2\,dx. \end{aligned}$$

It follows from estimates (6.2) and (6.3) that

$$\int_{A_N} \varrho_\varepsilon(t)\,dx \leq \frac{c}{\ln N}, \quad \int_A \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|^2\,dx \leq c.$$

Thus we get

$$\int_A \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|\,dx \leq c\frac{1}{2\epsilon}\big(N|A| + \frac{1}{\ln N}\big) + c\epsilon. \tag{12.2}$$

Introduce the function $\lambda : [3, \infty) \to \mathbb{R}^+$ defined by the equality $\lambda(s) = s\ln s$. It has the smooth inverse $\lambda^{-1} : [3\ln 3, \infty) \to \mathbb{R}$ with the properties

$$\frac{\ln s}{s}\,\lambda^{-1}(s) \to 1, \quad \ln(\lambda^{-1}(s))/\ln s \to 1 \ \text{ as } \ s \to \infty.$$

Set

$$N = \lambda^{-1}\big(\frac{1}{|A|}\big), \quad \epsilon = \frac{1}{\sqrt{\ln N}}.$$

From this and inequality (12.2) we obtain the estimate

$$\int_A \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|\,dx \leq c\ln\big(|A|^{-1}\big)^{-1/2} \tag{12.3}$$

for $|A| \leq 1/3$ and for a.e. $t \in (0, T)$. Note that the constant c is independent of t. Inequality (12.3) implies the weak type estimate for the momentum $\varrho_\varepsilon \mathbf{u}_\varepsilon$. By virtue of estimates (6.2) and (6.3), we have

$$\int_\Omega \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|\,dx \leq \frac{1}{2}\int_\Omega \varrho_\varepsilon(t)\,dx + \frac{1}{2}\int_\Omega \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|^2\,dx \leq c_1,$$

where $c_1 \geq 1$ depends only of the constants in (6.2), (6.3) and is independent of t. From this and the Chebyshev inequality we obtain that for every $m > 0$,

$$\text{meas } \{\varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)| \geq m\} \leq \frac{c_1}{m}.$$

Now set

$$B_m = \{\varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)| \geq m\}. \tag{12.4}$$

Inequality (12.3) with $A = B_m$ implies the weak type estimate

$$\int_{B_m} \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|\,dx \leq c\ln(m/c_1)^{-1/2} \quad \text{for} \quad m \geq 3c_1. \tag{12.5}$$

Now we proceed as in the proof of Theorem 4.1. Fix $t \in (0,T)$. Denote by $E(\lambda)$, $\lambda \geq 0$ the monotone function defined by the relation

$$E(\lambda) = \text{meas } \big\{\, x \in \Omega: \ \varrho_\varepsilon(x,t)|\mathbf{u}_\varepsilon(x,t)| \geq \lambda \,\big\}.$$

It is the standard fact of calculus, that the equality

$$\int_\Omega f(\varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|)\,dx = -\int_{[0,\infty)} f(\lambda)\ dE(\lambda)$$

holds for every piecewise continuous function f such that $f(\varrho_\varepsilon(t)|\mathbf{u}_\varepsilon|) \in L^1(\Omega)$. Here $-dE(\lambda)$ is the Borel-Stieltjes measure associated with the function $-E(\lambda)$. In particular, we have

$$\int_{B_m} \varrho_\varepsilon|\mathbf{u}_\varepsilon|\,dydt = -\int_{[m,\infty)} \lambda\,dE(\lambda). \tag{12.6}$$

With this notation, the desired inequality

$$\|\varrho_\varepsilon|\mathbf{u}_\varepsilon|\ln(2+\varrho_\varepsilon|\mathbf{u}_\varepsilon|)^\alpha\|_{L^\infty(0,T;L^1(\Omega))} \leq c \tag{12.7}$$

reads

$$\int_\Omega \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|\ln(2+\varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|)^\alpha\,dydt \equiv -\int_{[0,\infty)} \lambda\ln(2+\lambda)^\alpha\,dE(\lambda) \leq c. \tag{12.8}$$

Here $\alpha \in [0,1/4)$, c is independent of t.

Introduce the monotone function F and the corresponding Borel-Stieltjes measure $-dF$ defined by the equalities

$$F(s) = -\int_{[s,\infty)} \lambda\,dE(\lambda), \quad dF(\lambda) = \lambda\,dE(\lambda).$$

Identity (12.8) and estimate (12.5) imply

$$F(s) \leq c\ln(s/c_1)^{-1/2} \quad \text{for} \quad s \geq 3c_1. \tag{12.9}$$

Next note that

$$\ln(1+\frac{2}{s}) \leq \frac{2}{s} \leq \frac{2}{3} \leq \ln 3 \leq \ln\big(\frac{s}{c_1^2}\big) = \ln s - \ln c_1 \quad \text{for} \quad s \geq 3c_1^2,$$

which yields

$$\ln(2+s) = \ln s + \ln(1+\frac{2}{s}) \leq 2\ln s - 2\ln c_1 = 2\ln\big(\frac{s}{c_1}\big) \quad \text{for} \quad s \geq 3c_1.$$

From this and (12.9) we conclude that

$$F(s) \leq c\ln(2+s)^{-1/2} \quad \text{for} \quad s \geq 3c_1. \tag{12.10}$$

Since

$$F(s) = -\int_{0,\infty} \lambda dE(\lambda) = \int_\Omega \varrho_\varepsilon(t)|\mathbf{u}_\varepsilon(t)|\,dx \le c,$$

we obviously have

$$F(s) \le c\ln(2+s)^{-1/2} \quad \text{for} \quad s \le 3c_1^2.$$

Combining this result with (12.10) we finally obtain

$$F(s) \le c\ln(2+s)^{-1/2} \quad \text{for} \quad s \ge 0. \tag{12.11}$$

Here the constant c depends only on the constant in energy estimates (6.2), (6.3) and is independent of t. It is easy to see that

$$-\int_{[0,\infty)} \lambda \ln(2+\lambda)^\alpha\, dE(\lambda) = -\int_{[0,\infty)} \ln(2+\lambda)^\alpha\, dF(\lambda)$$

Therefore, our task is to prove that the integral in the right hand side exists and admits the effective estimate. Let us consider the regularized integral

$$I(\epsilon, N) = -\int_{[0,N)} g(\lambda) dF(\lambda), \quad g(\lambda) = \frac{\ln(2+\lambda)^\alpha}{1+\epsilon^2\lambda^2}$$

Note that g is bounded and continuously differentiable. On the other hand, the total variation of F is finite. Hence, we may integrate by parts to obtain

$$I(\epsilon, N) = \int_{[0,N)} g'(\lambda)\, F(\lambda)\, d\lambda + g(0)F(0) - g(N) \lim_{\lambda\to N-0} F(\lambda). \tag{12.12}$$

We have

$$\begin{aligned} g'(\lambda) &= g_0(\lambda) - \mathbf{R}(\lambda), \qquad\qquad (12.13)\\ g_0(\lambda) &= \alpha \ln(2+\lambda)^{\alpha-1}(2+\lambda)^{-1}(1+\epsilon^2\lambda^2)^{-1}\\ \mathbf{R}(\lambda) &= 2\epsilon^2\lambda\ln(2+\lambda)^\alpha(1+\epsilon^2\lambda^2)^{-2} \end{aligned}$$

Since $\alpha \in (0, 1/2)$, estimate (12.11) implies that

$$|g'(\lambda)F(\lambda)| \le c(1+\epsilon^2\lambda^2)^{-1}$$

and hence $g'F \in L^1(\mathbb{R}^+)$. On the other hand, we have

$$0 \le g(N)F(N) \le c\ln(2+N)^{\alpha-1}(1+\epsilon N)^{-1} \to 0 \quad \text{as} \quad N \to \infty.$$

Letting $N \to \infty$ in equality (12.12) we conclude that there exists the integral

$$\begin{aligned} -\int_{[0,\infty)} g(\lambda)\, dF(\lambda) &= \int_0^\infty g_0(\lambda)\, F(\lambda)\, d\lambda \\ &\quad - \int_0^\infty \mathbf{R}(\lambda)\, F(\lambda)\, d\lambda + g(0)F(0). \end{aligned}$$

Since $\mathbf{R}$ and F are nonnegative, we have

$$-\int_{[0,\infty)} g(\lambda)\, dF(\lambda) \le \int_0^\infty g_0(\lambda)\, F(\lambda)\, d\lambda + g(0)F(0). \tag{12.14}$$

It follows from (12.11) that $F(0)\leq c$. Next, estimate (12.11) yields

$$\begin{aligned}\int_0^\infty g_0F\,d\lambda &\leq \alpha\int_0^\infty \ln(2+\lambda)^{\alpha-1}(2+\lambda)^{-1}\,F(\lambda)d\lambda\\ &\leq \int_0^\infty \ln(2+\lambda)^{\alpha-1-1/2}(2+\lambda)^{-1}\,d\lambda\leq c(\alpha).\end{aligned}$$

It follows from this and (12.14) that

$$-\int_{[0,\infty)} g_0(\lambda)\,dF(\lambda)\leq c(\alpha)+\ln 2^\alpha c\leq c. \tag{12.15}$$

Recall that $-dF$ is a nonnegative Borel measure. Note that g_0 is an increasing function of ϵ and

$$g_0(\lambda)\nearrow \ln(2+\lambda)^\alpha \quad\text{as}\quad \epsilon\searrow 0.$$

Letting $\epsilon\to 0$ in (4.29) and applying the Fatou theorem we conclude that there exists the integral

$$-\int_{[0,\infty)}\ln(2+\lambda)^\alpha\,dF(\lambda)\equiv-\int_{[0,\infty)}\lambda\ln(2+\lambda)dE(\lambda)\leq c.$$

This completes the proof of Lemma 6.1.

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Pavel Igorevich Plotnikov
Voronej State University
University square, 1,
394018, Voronej, Russia
Email address: piplotnikov@mail.ru