COMPUTATION OF INTEGRAL TRANSFORMS IN TERMS OF DOUBLE HYPERGEOMETRIC SERIES ψ_2 WITH APPLICATIONS

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Abstract. The principal aim of the present paper is to evaluate a new integral transform involving the product of two Bessel Functions of the first kind expressed in terms of the Humbert function ψ_2. We also derive some interesting special cases from the main result of this paper. The presented result will be applied to generate some new laser beams called as the Humbert beams of type-II and study there propagation behaviour. From graphical simulations this type of wave seems interesting and useful in optical trapping which is used in some applications for pushing and moving particles such as: Biological cells, neutral atoms, Fluorescent, DNA molecules and trapping productivity on nano-sized spheres.

1. INTRODUCTION

The work of Belafhal and Hennani [2] is the pioneer of the present investigation. We have evaluated a closed form expression of the integral $I^{\lambda}_{\mu,\nu}$ with an integrand involving the product of two Bessel functions of the first kind with a quadratic argument. This integral has been derived in terms of the Humbert's confluent hypergeometric function of two variables. The literature has an astonishingly large number of integral transforms involving a range of special functions. In 2016, Khan *et al.* [9] have studied the integral intransform with the product of Bessel function and Whittaker function with a quadratic argument and obtained an expression in terms of Humbert function. In this paper, we establish a novel closed form of the integral transform $I^{\lambda}_{\mu,\nu}$ whenever the improper converges.

We start by recalling some important definitions which are uses in the paper. We recall the generalized hypergeometric function ${}_pF_q$ defined as (see [7])

$$ {}_pF_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\frac{z^n}{n!}, \tag{1.1}$$

where $(\lambda)_n$ is the Pochhammer symbol given, for $\lambda\in\mathbb{C}$, by (see [14])

$$\begin{aligned}(\lambda)_n&=\begin{cases}1, & n=0\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & n\in\mathbb{N}^*\end{cases}\\ &=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)},\ \text{for } \lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-,\end{aligned} \tag{1.2}$$

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and Γ is the familiar Gamma function.

The Bessel function J_μ of the first kind is defined by the series expansion (see [7])

$$J_\mu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\mu+2m}}{m!\Gamma(\mu+m+1)}, \tag{1.3}$$

with μ is the order and $z \in \mathbb{C} \setminus \{0\}$, $\mu \in \mathbb{C}$ and $\Re(\mu) > -1$.
The Humbert's confluent hypergeometric functions (see [14]) introduced by Humbert are defined as

$$\Psi_2\left[a; b, c; x, y\right] = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a)_{n+k}}{(b)_n (c)_k} \frac{x^n}{n!} \frac{y^k}{k!}, \tag{1.4}$$

and

$$\Phi_3\left[a; b; x, y\right] = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a)_n}{(b)_{n+k}} \frac{x^n}{n!} \frac{y^k}{k!}. \tag{1.5}$$

These functions converge absolutely $\forall x, y \in \mathbb{C}$.

We also recall the expression of the fourth Appell function of two variables (see [14]) expressed as

$$F_4\left[a, b; c, c'; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{1.6}$$

This function converges absolutely for $\sqrt{|x|} + \sqrt{|y|} < 1$.

2. MAIN RESULT

In this section, we evaluate the integral transform $I^{\lambda}_{\mu,\nu}$ with an integrand involving the product of two Bessel function of the first kind with a quadratic argument. This integral transforms has been derived in terms of the Humbert's confluent hypergeometric function of two variables.

Theorem 2.1. *: The following integral transform holds true:*

$$\int_0^{\infty} x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\gamma x)\, dx = \frac{\left(\frac{\beta}{2}\right)^\mu \left(\frac{\gamma}{2}\right)^\nu \Gamma\left(\frac{\mu+\nu+\lambda}{2}+1\right)}{2\alpha^{\frac{\mu+\nu+\lambda}{2}+1}\mu!\nu!}$$
$$\times \Psi_2\left[\frac{\mu+\nu+\lambda}{2}+1; \mu+1, \nu+1; -\frac{\beta^2}{4\alpha}, -\frac{\gamma^2}{4\alpha}\right], \tag{2.1}$$

where $\Re(\alpha) > 0$, $\Re(\mu+\nu+\lambda) > -2$, $\beta > 0$ *and* $\gamma > 0$.

Proof. Using the following identity (see [11])

$$J_\nu = \frac{(z/2\,)^\nu}{\nu!}\, {}_0F_1\left(-; \nu+1; -\frac{z^2}{4}\right), \tag{2.2}$$

and introducing the expansion of J_μ given by (1.3), we obtain

$$I_{\mu,\nu}^{\lambda}=\frac{\left(\frac{\beta}{2}\right)^{\mu}\left(\frac{\gamma}{2}\right)^{\nu}}{\nu!}\sum_{m=0}^{\infty}\frac{\left(\frac{-\beta^2}{4}\right)^{m}}{m!\;\Gamma\left(\mu+m+1\right)}I_m, \tag{2.3}$$

where I_m is given by

$$I_m=\int_0^\infty x^{\mu+\nu+\lambda+2m+1}e^{-\alpha x^2}{}_0F_1\left(-;\nu+1;-\frac{\gamma^2}{4}x^2\right)dx. \tag{2.4}$$

By using (1.1), and replacing the variable x^2 by t, (2.4) can be written as

$$I_m=\frac{1}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{-\gamma^2}{4}\right)^{k}}{\left(\nu+1\right)_k k!}\int_0^\infty t^{\frac{\mu+\nu+\lambda}{2}+m+k}e^{-\alpha t}dt. \tag{2.5}$$

To evaluate this integral, we use the undermentioned identities (see [7, 14])

$$\int_0^\infty t^{\sigma-1}e^{-\alpha t}dt=\frac{\Gamma(\sigma)}{\alpha^\sigma}, \tag{2.6}$$

and

$$(\lambda)_{m+n}=(\lambda)_m(\lambda+m)_n, \tag{2.7}$$

which helps us in rearranging (2.5) to write

$$I_m=\frac{\Gamma\left(\frac{\mu+\nu+\lambda}{2}+m+1\right)}{2\alpha^{\frac{\mu+\nu+\lambda}{2}+m+1}}{}_1F_1\left(\frac{\mu+\nu+\lambda}{2}+m+1;\nu+1;-\frac{\gamma^2}{4\alpha}\right). \tag{2.8}$$

Now in view of (2.8), (2.3) becomes

$$\begin{aligned}I_{\mu,\nu}^{\lambda}=&\frac{(\beta/2)^{\mu}(\gamma/2)^{\nu}\Gamma\left(\frac{\mu+\nu+\lambda}{2}+1\right)}{2\mu!\nu!\alpha^{\frac{\mu+\nu+\lambda}{2}+1}}\sum_{m=0}^{\infty}\frac{\left(-\beta^2/4\alpha\;\right)^m}{m!}\frac{\left(\frac{\mu+\nu+\lambda}{2}+1\right)_m}{\left(\mu+1\right)_m}\\ &\times{}_1F_1\left(\frac{\mu+\nu+\lambda}{2}+m+1;\nu+1;\frac{-\gamma^2}{4\alpha}\right).\end{aligned} \tag{2.9}$$

Evaluating the above equation with the help of expression of Ψ_2 given as (see [14]),

$$\Psi_2\left[a;b,c;x,y\right]=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{x^n}{n!}\;{}_1F_1\left(a+n;c;y\right). \tag{2.10}$$

we easily arrive at our main result (2.1). This completes the proof.

Remark 2.1. *It is easy to show that the result in (2.1) is equivalent to (see Eq. 6.633.1 of* [7]*). For that, we use a result (see* [10, 13]*) given by*

$$\Psi_2\left[a;b,c;x,y\right]=\sum_{m=0}^{\infty}\frac{(a)_m}{(b)_m}\frac{x^m}{m!}\;{}_2F_1\left(-m,1-b-m;c;\frac{y}{x}\right), \tag{2.11}$$

for $c \neq 0, -1, -2, \ldots$ and $x \neq 0$. Consequently, (2.1) can be written as

$$\begin{aligned} I_{\mu,\nu}^{\lambda} = & \frac{(\beta/2)^{\mu}(\gamma/2)^{\nu}\Gamma\left(\frac{\mu+\nu+\lambda}{2}+1\right)}{2\mu!\nu!\alpha^{\frac{\mu+\nu+\lambda}{2}+1}} \sum_{m=0}^{\infty} \frac{(-\beta^2/4\alpha\ )^m}{m!} \frac{\left(\frac{\mu+\nu+\lambda}{2}+1\right)_m}{(\mu+1)_m} \\ & \times {}_2F_1\left(-m, -m-\mu; \nu+1; \frac{\gamma^2}{\beta^2}\right). \end{aligned} \tag{2.12}$$

Thus, our main result permit us to write (see Eq. 6.633.1 of [7]) which is an infinite series of the hypergeometric function ${}_2F_1$, in terms of the Humbert function ψ_2.

3. PARTICALAR CASES

In this section, as special cases of our main result, we present some new interesting integral transforms by specializing certain parameters.

Corollary 3.1. *Let $\Re(p) > -1$, $|\arg\sqrt{\alpha}| < \frac{\pi}{4}$, $\beta > 0$ and $\gamma > 0$. Then*

$$\int_0^{\infty} xe^{-\alpha x^2} J_p(\beta x)\, J_p(\gamma x)\, dx = \frac{1}{2\alpha} e^{-\frac{\beta^2+\gamma^2}{4\alpha}} I_p\left(\frac{\beta\gamma}{2\alpha}\right). \tag{3.1}$$

Proof. By substituting $\lambda = 0$ and $\mu = \nu = p$, (2.1) becomes

$$I_{p,p}^{0} = \frac{(\beta\gamma/4\alpha\ )^p}{2\alpha p!}\ \Psi_2\left[p+1; p+1, p+1; -\frac{\beta^2}{4\alpha}, -\frac{\gamma^2}{4\alpha}\right]. \tag{3.2}$$

By using the following relationships (see [11])

$$\Psi_2[b; b, b; x, y] = e^{(x+y)}\ {}_0F_1(-; b; xy), \tag{3.3}$$

and

$$I_p(z) = \frac{(z/2\ )^p}{p!}\ {}_0F_1\left(-; p+1, \frac{z^2}{4}\right), \tag{3.4}$$

one can write the identity

$$\Psi_2\left[p+1; p+1, p+1; -\frac{\beta^2}{4\alpha}, -\frac{\gamma^2}{4\alpha}\right] = p!\left(\frac{4\alpha}{\beta\gamma}\right)^p e^{-\frac{\beta^2+\gamma^2}{4\alpha}} I_p\left(\frac{\beta\gamma}{2\alpha}\right). \tag{3.5}$$

Thus, we get the desired assertion (3.1) which is analogous to the result in (see Eq. 6.633.2 of [7]). This completes the proof of corollary (3.1).

Note that if we evaluate $I_{p,p}^{0}$ involving the product of $I_p(\beta x)$ with $J_p(\gamma x)$, we find the following result

$$\int_0^{\infty} xe^{-\alpha x^2} I_p(\beta x) J_p(\gamma x)\, dx = \frac{1}{2\alpha} e^{\frac{\beta^2-\gamma^2}{4\alpha}} J_p\left(\frac{\beta\gamma}{2\alpha}\right), \tag{3.6}$$

is equivalent to (see Eq. 6.633.4 of [7]).

Corollary 3.2. *Let* $\beta > 0$, $\Re(\mu+\nu+\lambda) > 2$ *and* $\Re(\alpha) > -2$. *Then*

$$\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x)\, J_\nu(\beta x)\, dx = \frac{(\beta/2\,)^{(\mu+\nu)}\,\Gamma\left(\frac{\mu+\nu+\lambda}{2}+1\right)}{2\mu!\,\nu!\,\alpha^{\frac{\mu+\nu+\lambda}{2}+1}}$$
$$\times\, {}_3F_3\left(\frac{\mu+\nu+\lambda}{2}+1, \frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1; \mu+1, \nu+1, \mu+\nu+1; -\frac{\beta^2}{\alpha}\right). \tag{3.7}$$

Proof. The following representation (see [10])

$$\Psi_2\left[a; b, c; x, x\right] = {}_3F_3\left(a, \frac{c+b-1}{2}, \frac{c+b}{2}; b, c, c+b-1; 4x\right) \tag{3.8}$$

allows us to easily obtain (3.7).

Note here that, in the case of $\lambda = \chi - 2$, (3.7) becomes

$$I^{\chi-2}_{\mu,\nu} = \int_0^\infty x^{\chi-1} e^{-\alpha x^2} J_\mu(\beta x)\, J_\nu(\beta x)\, dx = \frac{(\beta/2\,)^{\mu+\nu}}{2\mu!\nu!} \frac{\Gamma\left(\frac{\mu+\nu+\chi}{2}\right)}{\alpha^{\frac{\mu+\nu+\chi}{2}}}$$
$$\times\, {}_3F_3\left(\frac{\mu+\nu+\chi}{2}, \frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1; \mu+1, \nu+1, \mu+\nu+1; -\frac{\beta^2}{\alpha}\right). \tag{3.9}$$

This last equation is analogous to (see Eq. 6.633.5 of [7]).

Corollary 3.3. *Let* $\beta > 0$, $\gamma > 0$, $\Re(\alpha) > 0$ *and* $\Re(\mu) > -1$. *Then*

$$\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x)\, J_{\mu-\lambda}(\gamma x)\, dx$$
$$= \frac{(\beta\gamma/4\alpha\,)^{\mu}}{2\alpha\,(\mu-\lambda)!(\gamma/2\,)^{\lambda}}\; e^{-\frac{\beta^2+\gamma^2}{4\alpha}}\; \Phi_3\left(-\lambda; \mu-\lambda+1; \frac{\gamma^2}{4\alpha}, \frac{\beta^2\gamma^2}{16\alpha^2}\right). \tag{3.10}$$

Proof. If we take $\nu = \mu - \lambda$ and apply the following identity (see [10])

$$\Psi_2\left[b; b, c; x, y\right] = e^{(x+y)}\Phi_3\left(c-b; c; -y, xy\right), \tag{3.11}$$

where Φ_3 is given by (1.5), we get the desired assertion (3.10).

Lemma 3.1. *Let* $\Re(\alpha) > 0$, $\beta > 0$, *and* $\gamma > 0$. *Then*

$$\Phi_3\left[-; p+1; \frac{\gamma^2}{4\alpha}, \frac{\beta^2\gamma^2}{16\alpha^2}\right] = p!\left(\frac{4\alpha}{\beta\gamma}\right)^p I_p\left(\frac{\beta\gamma}{2\alpha}\right). \tag{3.12}$$

Proof. If $\lambda = 0$ and $\mu = p$ (3.10) becomes

$$I^0_{p,p} = \frac{(\beta\gamma/4\alpha\,)^p}{2\alpha p!}\; e^{-\frac{\beta^2+\gamma^2}{4\alpha}}\; \Phi_3\left(-; p+1; \frac{\gamma^2}{4\alpha}, \frac{\beta^2\gamma^2}{16\alpha^2}\right). \tag{3.13}$$

With the help of (3.1), one can deduce a particular expression of the Humbert function Φ_3 given by (3.12).

Corollary 3.4. *Let* $\Re(\alpha) > 0$, $\beta > 0$, $\gamma > 0$ *and* $\Re(\nu) > \Re(\mu)$. *Then*

$$\int_0^\infty x^{\nu-3\mu-2} e^{-\alpha x^2} J\mu\,(\beta x) J_\nu\,(\gamma x)\,dx = \left(\frac{\alpha\beta}{2}\right)^\mu \left(\frac{\gamma}{2\alpha}\right)^\nu \frac{\Gamma(\nu-\mu)}{2\mu!\nu!} e^{-\gamma^2/4\alpha}\ {}_1F_1\left(\nu-\mu;\nu+1;\frac{\gamma^2-\beta^2}{4\alpha}\right). \tag{3.14}$$

Proof. If we take $\lambda = \nu - 3\mu - 2$ and by using the identity (see [11])

$$\Psi_2\left[b;b',b+b';x,y\right] = e^y{}_1F_1\left(b;b+b';x-y\right), \tag{3.15}$$

we get (3.14).

Corollary 3.5. *Let* $\Re(\alpha) > 0$, $\beta > 0$ *and* $\Re(\mu) > \frac{1}{2}$. *Then*

$$\int_0^\infty x^{-\mu} e^{-\alpha x^2} J_\mu\,(\beta x)\,J_{2\mu+1}\,(\beta x)\,dx = \frac{\beta}{4\alpha\,(2\mu+1)!}\left(\frac{\beta^3}{8\alpha}\right)^\mu {}_2F_2\left(\frac{3\,(\mu+1)}{2},\frac{3\mu}{2}+1;2\,(\mu+1)\,,3\mu+2;-\frac{\beta^2}{\alpha}\right). \tag{3.16}$$

Proof. It's easy to prove (3.16) by taking $\lambda = -(\mu+1)$, $\nu = 2\mu+1$ and by using the following identity (see [13])

$$\Psi_2\left[b;b,2b;x,x\right] = {}_2F_2\left(\frac{3b}{2},\frac{3b-1}{2};2b,3b-1;4x\right). \tag{3.17}$$

Corollary 3.6. *Let* $\Re(\alpha) > 0$, $\beta > 0$ *and* $\Re(2\mu+\lambda) > -2$. *Then*

$$\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu\,(\beta x)\,I_\mu\,(\beta x)\,dx = \frac{\Gamma\left(\lambda/2\ +\mu+1\right)}{2(\mu!)^2\alpha^{\lambda/2\ +1}}\left(\frac{\beta}{4\alpha}\right)^\mu \times {}_2F_3\left(\frac{\lambda+2\,(\mu+1)}{4},\frac{\lambda+2\,(\mu+2)}{4};\mu+1,\frac{\mu+1}{2},\frac{\mu}{2}+1;-\frac{\beta^4}{16\alpha^2}\right) \tag{3.18}$$

Proof. If $\mu = \nu$, $\gamma = i\beta$ and by using the identites (see [7], [10])

$$I_\nu\,(z) = i^{-\gamma} J_\nu\,(iz)\,, \tag{3.19}$$

and

$$\Psi_2\left[a;c,c;x,-x\right] = {}_2F_3\left(\frac{a}{2},\frac{a+1}{2};c,\frac{c}{2},\frac{c+1}{2};-x^2\right), \tag{3.20}$$

(3.18) is deduced.

Corollary 3.7. *Let* $\Re(\alpha) > 0$, $\beta > 0$ *and* $\Re(\mu) > \frac{1}{2}$. *Then*

$$\int_0^\infty e^{-\alpha x^2} J_\mu\,(\beta x)\,I_\mu\,(\beta x) dx = \left(\frac{\beta^2}{4\alpha}\right)^\mu \frac{\Gamma\left(\mu+1/2\ \right)}{2\sqrt{\alpha}(\mu!)^2} \times {}_1F_1\left(\frac{2\mu+1}{4};\mu+1;-\frac{i\beta^2}{2\alpha}\right){}_1F_1\left(\frac{2\mu+1}{4};\mu+1;\frac{i\beta^2}{2\alpha}\right). \tag{3.21}$$

Proof. This case is obtained by taking $\lambda = -1$, $\mu = \nu$, $\gamma = i\beta$ and using the identity (see [10])

$$\Psi_2\left[2a; 2a+\frac{1}{2}, 2a+\frac{1}{2}; x, -x\right] = {}_1F_1\left(a; 2a+\frac{1}{2}; 2ix\right){}_1F_1\left(a; 2a+\frac{1}{2}; -2ix\right). \tag{3.22}$$

So, our main finding given by (2.1). This completes the proof of the corollary.

Corollary 3.8. *Let* $\Re(\alpha) > 0$, $\beta > 0$, $\gamma > 0$ *and* $\Re(\lambda+\mu+\gamma) > -2$. *Then*

$$\begin{aligned}\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_{-\frac{1}{2}}(\beta x) J_{-\frac{1}{2}}(\gamma x)\, dx &= \frac{\Gamma\left((\lambda+1)/2\,\right)\Gamma\left(\lambda/2+1\,\right) 2^{\frac{\lambda-3}{2}}}{\sqrt{\pi\beta\gamma}(-{}^1/{}_2!)^2\alpha^{\frac{\lambda+1}{2}}}\, e^{-\frac{\beta^2+\gamma^2}{8\alpha}} \\ &\times\left\{e^{-\frac{\beta\gamma}{4\alpha}}\left[D_{-(\lambda+1)}\left(-\frac{i}{\sqrt{2\alpha}}(\beta+\gamma)\right) + D_{-(\lambda+1)}\left(\frac{i}{\sqrt{2\alpha}}(\beta+\gamma)\right)\right]\right. \\ &\left.+\, e^{\frac{\beta\gamma}{4\alpha}}\left[D_{-(\lambda+1)}\left(-\frac{i}{\sqrt{2\alpha}}(\beta-\gamma)\right) + D_{-(\lambda+1)}\left(\frac{i}{\sqrt{2\alpha}}(\beta-\gamma)\right)\right]\right\}.\end{aligned} \tag{3.23}$$

Proof. If we take $\mu = \nu = -\frac{1}{2}$, (2.1) can be written as

$$I^{\lambda}_{-{}^1/{}_2 -{}^1/{}_2} = \frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{2\left(\frac{\beta\gamma}{4}\right)^{{}^1/{}_2}\left(-\frac{1}{2}!\right)^2\alpha^{\left(\frac{\lambda+1}{2}\right)}}\;\Psi_2\left[\frac{\lambda+1}{2}; \frac{1}{2}, \frac{1}{2}; -\frac{\beta^2}{4\alpha}, -\frac{\gamma^2}{4\alpha}\right]. \tag{3.24}$$

We know that (see [4])

$$\begin{aligned}\Psi_2\left[a; \frac{1}{2}, \frac{1}{2}; x, y\right] &= \frac{2^{a-2}}{\sqrt{\pi}}\, e^{(x+y)/2}\;\Gamma\left(a+\frac{1}{2}\right) \\ &\times\left\{e^{\sqrt{xy}}\left[D_{-2a}(-\sqrt{2}(\sqrt{x}+\sqrt{y})) + D_{-2a}(\sqrt{2}(\sqrt{x}+\sqrt{y}))\right]\right. \\ &\left.+\, e^{-\sqrt{xy}}\left[D_{-2a}(-\sqrt{2}(\sqrt{x}-\sqrt{y})) + D_{-2a}(\sqrt{2}(\sqrt{x}-\sqrt{y}))\right]\right\},\end{aligned} \tag{3.25}$$

where D_ν is the parabolic cylinder function. With this representation and by taking $a = \frac{\lambda+1}{2}$, $x = -\frac{\beta^2}{4\alpha}$ and $y = -\frac{\gamma^2}{4\alpha}$, the corollary is proved.

Corollary 3.9. *Let $\Re(\alpha) > 0$, $\beta > 0$, $\gamma > 0$, $\Re(\lambda) > -2$ and $\nu = -\mu = \frac{1}{2}$. Then*

$$\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_{-\frac{1}{2}}(\beta x)\, J_{\frac{1}{2}}(\gamma x)\, dx$$
$$= \sqrt{\frac{\gamma}{\pi\beta\alpha^\lambda}} \frac{\Gamma(\lambda/2+1)}{(-{}^1/{}_2!)\,({}^1/{}_2!)} \frac{2^{\frac{\lambda-5}{2}}}{i\gamma} \Gamma\left(\frac{\lambda+1}{2}\right) e^{-\frac{\beta^2+\gamma^2}{8\alpha}}$$
$$\times \left\{ e^{-\frac{\beta\gamma}{4\alpha}} \left[D_{-(\lambda+1)}\left(-\frac{i}{\sqrt{2\alpha}}(\beta+\gamma)\right) - D_{-(\lambda+1)}\left(\frac{i}{\sqrt{2\alpha}}(\beta+\gamma)\right)\right]\right. \tag{3.26}$$
$$\left. - e^{\frac{\beta\gamma}{4\alpha}} \left[D_{-(\lambda+1)}\left(-\frac{i}{\sqrt{2\alpha}}(\beta-\gamma)\right) - D_{-(\lambda+1)}\left(\frac{i}{\sqrt{2\alpha}}(\beta-\gamma)\right)\right]\right\}.$$

Proof. It is easy to prove (3.26) by introducing in (3.12) the identity (see [4])

$$\Psi_2\left[a; \frac{1}{2}, \frac{3}{2}; x, y\right] = \frac{2^{a-{}^7/{}_2}}{\sqrt{\pi y}}\, e^{(x+y)/2}\, \Gamma\left(a - \frac{1}{2}\right)$$
$$\times \left\{ e^{\sqrt{xy}} \left[D_{1-2a}(-\sqrt{2}(\sqrt{x}+\sqrt{y})) - D_{1-2a}(\sqrt{2}(\sqrt{x}+\sqrt{y}))\right]\right. \tag{3.27}$$
$$\left. - e^{-\sqrt{xy}} \left[D_{1-2a}(-\sqrt{2}(\sqrt{x}-\sqrt{y})) - D_{1-2a}(\sqrt{2}(\sqrt{x}+\sqrt{y}))\right]\right\}.$$

4. APPLICATION: GENERATION OF A DONUT HUMBERT BEAM OF TYPE-II

In this section, we investigate the generation of new beams family called Humbert beams of type-II by a spiral phase plate (SPP).

For this, we consider the propagation of a Bessel-Gauss beam through this element which its field is given at $z' = 0$ in the cylindrical coordinates (ρ', ϕ', z') by (see [6])

$$E_m(\rho', \phi', z' = 0) = E_f e^{im\phi'} J_m(\alpha\rho')\, e^{-\rho'^2/\omega_0^2}, \tag{4.1}$$

where $\alpha = k \sin\theta$, m is the beam order, w_0 is the waist of the Gaussian enveloppe, E_f is the constant parameter and J_m is the mth order Bessel function of the first kind. Note that for $\alpha \to \infty$, this field can be reduced to a Gaussian beam, or for $m \neq 0$ to a higher order Gauss-Laguerre mode, and for $\omega_0 \to \infty$ to a Bessel beam which is a nondiffracting beam.

In the present case, we examine the conversion of the field given by (4.1), by a SPP with negligible thickness to generate a novel donut family propagating through an ABCD optical system (see Fig. 1). We know that the SPP imposes azimuth dependent phase radiation given by $e^{i\chi\phi'}$, where χ is the topological charge given by $\chi = \frac{h.\Delta n}{\lambda}$, with Δn is the difference of refractive index between the SPP and its surrounding, h is the step height and λ is the wavelength of the input field.

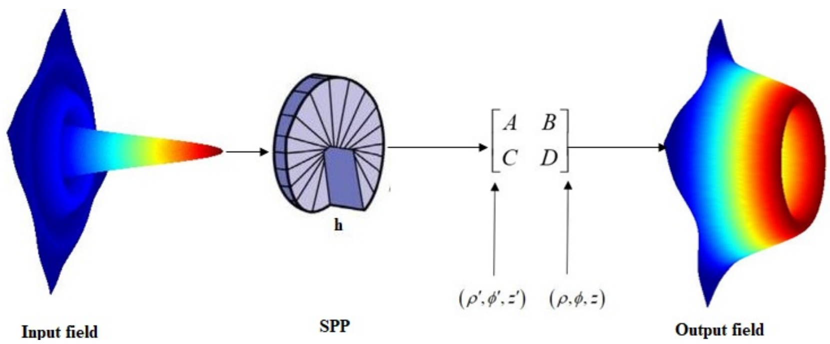

FIGURE 1. Generation of a donut Humbert beam of type-II by propagating a pseudo-nondiffracting beam through a SPP.

In the following, we apply the well-known Collins-Huygens integral (see [5]) on the incident beam centred at position z' on the SPP, which can be expressed as

$$E_m^{\chi}(\rho,\phi,z) = -\frac{i}{\lambda B}e^{ikz}\int_0^{+\infty}\int_0^{2\pi} E_m(\rho',\phi',z')exp\left\{\frac{ik}{2B}\left(A\rho'^2 + D\rho^2\right)\right\} \times \exp\left[-i\frac{k\rho\rho'\cos\left(\phi-\phi'\right)}{B}\right]\rho'.d\rho'.d\phi', \tag{4.2}$$

where

$$E_m^{\chi}\left(\rho',\phi',z'\right) = E_f\frac{\omega_0}{\omega\left(z'\right)}J_m\left(\alpha'\rho'\right)e^{-\beta\rho'^2}e^{ikz'}e^{im\phi'}e^{i\chi\phi'}, \tag{4.3}$$

with

$$\alpha' = \frac{\alpha}{1+{}^{iz'}/_{z_R}},\ \beta = \frac{1}{\omega^2\left(z'\right)} - i\frac{k}{2R\left(z'\right)},\ \omega\left(z'\right) = \omega_0\sqrt{1+{}^{z'^2}/_{z_R^2}}, \quad R\left(z'\right) = \frac{z_R^2}{z'} + z' \text{ and } \phi'\left(z'\right) = tg^{-1}\left(\frac{z'}{z_R}\right). \tag{4.4}$$

In the above equations, $k = {}^{2\pi}/_{\lambda}$ is the wave number and z_R is the Rayleigh range. For the topological charge, we take $\chi = l$ with l is an integer (see [3]).

By substituting (4.3) in (4.2) and with the help of the following identity (see [7])

$$\int_0^{2\pi} e^{i(m+l)\phi'}e^{-i\frac{k\rho\rho'\cos(\phi-\phi')}{B}}d\phi' = 2\pi i^{(m+l)}e^{i(m+l)\phi}J_{(m+l)}\left(\frac{k\rho}{B}\rho'\right), \tag{4.5}$$

and after tedious algebraic calculations, we obtain

$$E_m^l\left(\rho,\phi,z\right) = -2\pi\frac{E_f\omega_0}{\lambda B\omega\left(z'\right)}i^{m+l+1}e^{ik\left(z+z'+{}^{D\rho^2}/_{2B}\right)}e^{i(m+l)\phi}.K_m^l, \tag{4.6}$$

where

$$K_m^l = \int_0^{+\infty} e^{-\left(\beta - i\frac{kA}{2B}\right)\rho'^2} J_m\left(\alpha'\rho'\right) J_{m+l}\left(\frac{k\rho}{B}\rho'\right)\rho' d\rho'. \tag{4.7}$$

This last equation can be evaluated by using our Theorem 2.1, and the result can be rearranged as

$$E_m^l\left(\rho,\phi,z\right) = A_m^l \times \Psi_2\left[m + \frac{l}{2} + 1; m+1, m+l+1; -\frac{\alpha'^2}{4\xi}, -\frac{k^2}{4\xi B^2}\rho^2\right], \tag{4.8}$$

where

$$\xi = \beta - i\frac{kA}{2B},$$

and

$$A_m^l = -\frac{\pi E_f}{\lambda B\xi}\frac{\omega_0}{\omega\left(z'\right)} i^{m+l+1}\left(\frac{k\alpha'}{4B\xi}\right)^m\left(\frac{k}{2B\sqrt{\xi}}\right)^l \frac{\Gamma\left(m+\frac{l}{2}+1\right)}{m!\left(m+l\right)!}$$
$$\times e^{ik\left(z+z'+D\rho^2/2B\right)} e^{i(m+l)\phi}\rho^{(m+l)}.$$

The expression in (4.8) is the output beam generated behind the SPP expressed in terms of Humbert's confluent hypergeometric function Ψ_2 defined by (1.4). For this reason, we have called this beams family as: Donut Humbert beam of type-II. Note that the term $\rho^{(m+l)}$ in (4.8) yields a donut field amplitude propagating through any ABCD optical system. These type of beams are very useful and suitable to trap or manipulate the nano-particles (see [1]).

Our new field is a generation of some used beams. Then, by using the following relation

$${}_1F_1\left(a+n; 2\mu+1, y\right) = \frac{e^{y/2}}{y^{\mu+1/2}} M_{\mu+\frac{1}{2}-a-n,\mu}\left(y\right), \tag{4.9}$$

where M is the Whittaker function, (2.10) can be expressed as

$$\Psi_2\left[a; b, c; x, y\right] = \frac{e^{y/2}}{y^{c/2}}\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{x^n}{n!} M_{\frac{c}{2}-a-n,\frac{c-1}{2}}\left(y\right). \tag{4.10}$$

Substituting (4.10) into (4.8), we obtain

$$E_m^l\left(\rho,\phi,z\right) = A\frac{\exp\left(-k^2\rho^2/8B^2\xi\right)}{\left(-k^2\rho^2/4B^2\xi\right)^{\frac{m+l+1}{2}}}\sum_{n=0}^{\infty}\frac{\left(m+\frac{l}{2}+1\right)_n}{(m+1)_n}\frac{\left(-\alpha'^2/4\xi\right)^n}{n!}$$
$$\times M_{-\frac{(m+1)}{2}-n,\frac{m+l}{2}}\left(-k^2\rho^2/4B^2\xi\right). \tag{4.11}$$

This last output field is a superposition of Whittaker beams and a generalization of hypergeometric fields proposed by Karimi *et al.* [8].

One can note that, for some special values of the index of the Whittaker function, we can prove that the donut Humbert beam of type-II is a superposition of Bessel-modulated Gaussian, Laguerre-Gaussian and Hermite-Gaussian lights. In the following, some numerical simulations are depicted to study the propagation through an ABCD optical system of this new beam. The calculation parameters are: $z = 3000mm$, $z' = 300mm$, $\omega_0 = 1mm$, $\lambda = 632.8nm$ and $l = 1$. Fig. (2) presents the intensity profile of Humbert beams of type-II for two values of α. We observe from this figure, when the beam order m is equals to zero and by increasing the parameter α, the intensity increases.

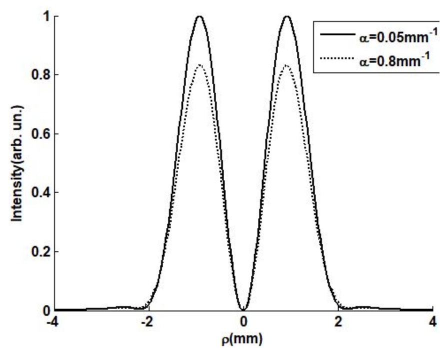

FIGURE 2. Intensity profile of Humbert beams of type-II with $m = 0$ and two values of α.

The 3D simulation and its corresponding contour graph is given in Fig. 3, to show the influence of the beam order m on the comportment of the donut Humbert beam of type-II propagating in a free-space.

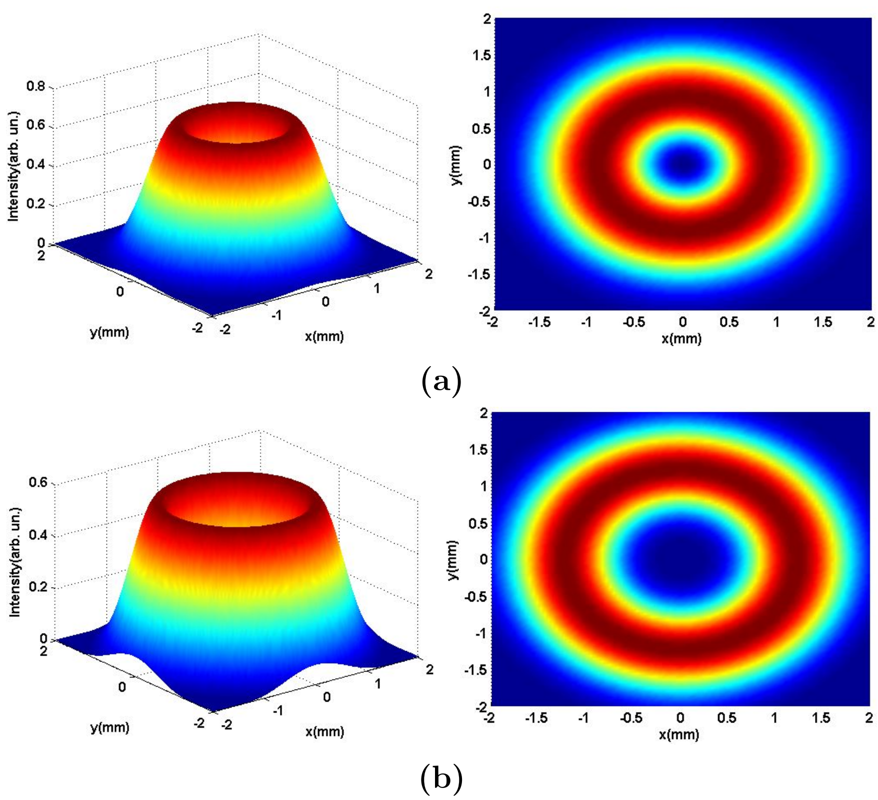

FIGURE 3. Intensity profile of Humbert beams of type-II in 3D and 2D, with **(a)** m=0 and **(b)** m=1, for $\alpha = 0.8mm^{-1}$.

From these plots, it is shown that the considered beam has a dark centered distribution, which increases with the increment of the beam order m.

Figure. 4 shows the effect of the variation of the waist width ω_0 on the propagation of Humbert beams of type-II with $l = 2$, $m = 0$, $\alpha = 0.8mm^{-1}$ and the other parameters are the same of those taken in Figs. (2) and (3). As shown in the below figure, the maximum of the intensity of the main lobe and also the side lobes increase when ω_0 increases.

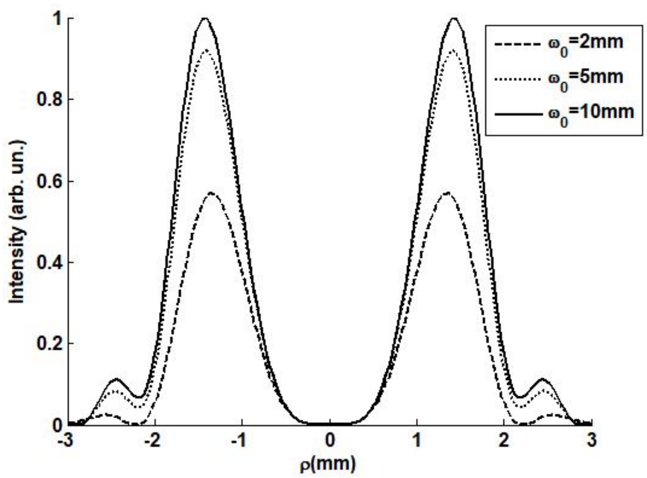

FIGURE 4. Intensity profile of Humbert beams of type-II with l=2, m=0 and for three values of ω_0.

5. Conclusion

The integral transforms involving the product of Bessel functions is computed in terms of the Humbert Function ψ_2. This integral has a distinct advantage in generating a family of new beams oftenly encountered in the problems of physics. Some interesting cases have also been investigated as special case of our main result. As an application of our main findings, we generate a new kind of donut waves called Humbert beams of type-II. Several numerical simulations are also established to show the influence of some parameters on the behaviour of this new beams family.

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