AN EXAMPLE OF A CONTINUOUS FIELD OF ROE ALGEBRAS

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Abstract. The Roe algebra $C^*(X)$ is a non-commutative C^*-algebra reflecting metric properties of a space X, and it is interesting to understand relation between the Roe algebra of X and the (uniform) Roe algebra of its discretization. Here we do a minor step in this direction in the simplest non-trivial example $X = \mathbb{R}$ by constructing a continuous field of C^*-algebras over $[0, 1]$ with the fibers over non-zero points the uniform C^*-algebra of the integers, and the fiber over 0 a C^*-algebra related to $\mathbb{R}$.

1. Introduction

Roe algebras play an increasingly important role in the index theory of elliptic operators on noncompact manifolds and their generalizations [9, 4, 5, 6]. Following the ideology of noncommutative geometry [2], they provide an interplay between metric spaces (e.g. manifolds) and (noncommutative) C^*-algebras.

Let X be a proper metric measure space, that is, X is a set, which is equipped with a metric d and a measure m defined on the Borel σ-algebra defined by the topology on X induced by the metric, and all balls are compact. For a Hilbert space H we write $\mathbb{B}(H)$ (resp., $\mathbb{K}(H)$) for the algebra of all bounded (resp., all compact) operators on H.

Recall the definition of the Roe algebra of X [9]. Let H_X be a Hilbert space with an action of the algebra $C_0(X)$ of continuous functions on X vanishing at infinity (i.e. a $*$-homomorphism $\pi : C_0(X) \to \mathbb{B}(H_X)$). We will assume that

$$\{\pi(f)\xi : f \in C_0(X), \xi \in H_X\} \quad \text{is dense in} \quad H_X \tag{1}$$

and that

$$\pi(f) \in \mathbb{K}(H_X) \quad \text{implies that} \quad f = 0. \tag{2}$$

An operator $T \in \mathbb{B}(H_X)$ is *locally compact* if the operators $T\pi(f)$ and $\pi(f)T$ are compact for any $f \in C_0(X)$. It has *finite propagation* if there exists some $R > 0$ such that $\pi(f)T\pi(g) = 0$ whenever the distance between the supports of $f, g \in C_0(X)$ is greater than R. The *Roe algebra* $C^*(X, H_X)$ is the norm completion of the $*$-algebra of locally compact, finite propagation operators on H_X. As it does not depend on the choice of H_X satisfying (1) and (2), it is usually denoted by $C^*(X)$. If $X = \mathbb{R}$ with the standard metric and the standard measure (our main example) then we may (and will, for simplicity) take $H_X = L^2(X)$.

When X is discrete, the choice $H_X = l^2(X)$ does not satisfy the condition (2). In order fix this, one may take $H_X = l^2(X) \otimes H$ for an infinitedimensional Hilbert space H. But there is also another option: still to use $H_X = l^2(X)$. The resulting C^*-algebra is called the *uniform Roe algebra* of X, and is denoted by $C^*_u(X)$. This C^*-algebra is more tractable, but has less relations with elliptic theory.

Manifolds and some other spaces X are often endowed with discrete subspaces $D \subset X$ that are ε-dense for some ε, e.g. $\mathbb{Z} \subset \mathbb{R}$, or, more generally, lattices in Lie groups, or, even more generally, Delone sets in metric spaces [1]. Some problems related to X may become simpler when reduced to D (discretization). In particular, it would be interesting to understand relation between a Roe-type algebra of X and the uniform Roe algebra of

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its discretization D. As the first step, we consider here one of the simplest non-trivial cases, $X=\mathbb{R}$, $D=\mathbb{Z}$, and construct a continuous field of C^*-algebras over the segment $[0,1]$ such that the fiber over 0 is a certain C^*-algebra related to $\mathbb{R}$, while the fiber over any other point is the uniform Roe algebra of $\mathbb{Z}$. Such non locally trivial continuous fields of C^*-algebras are interesting because they provide relations between fibers over different points. In particular, they provide a map from the K-theory group of the fiber over 0 to the K-theory group of the fiber over non-zero points. A similar continuous field with the fiber over 0 the algebra of functions on a sphere and the fibers over non-zero points the algebra of compact operators was used in [7] to give a proof of Bott periodicity in K-theory.

2. Two maps

Let $D_t=t\mathbb{Z}\subset\mathbb{R}$. In this section we construct the maps $\alpha_t: C^*_u(D_t)\to C^*(\mathbb{R})$ and $\beta_t: C^*(\mathbb{R})\to C^*_u(D_t)$, $t\in(0,1]$.

Let $\varphi_0(x)=\begin{cases}1+x, & x\in[-1,0];\\ 1-x, & x\in[0,1];\\ 0, & \text{otherwise,}\end{cases}$ $\varphi_n(x)=\varphi_0(x-n)$, $\varphi^t_n(x)=\frac{1}{\sqrt{t}}\varphi_n(x/t)$. Then $\operatorname{supp}\varphi^t_n=[t(n-1),t(n+1)]$, and $\|\varphi^t_n\|_{L^2}=\sqrt{2/3}$ for any $n\in\mathbb{Z}$ and any $t\in(0,1]$ (here $\|\cdot\|_{L^2}$ denotes the norm in $L^2(\mathbb{R})$). In particular, φ^t_n and φ^t_m are orthogonal when $|m-n|\geq 2$. Let p_t denote the projection, in $L^2(\mathbb{R})$, onto the closure H_t of the linear span of φ^t_n, $n\in\mathbb{Z}$.

Let $(G_{nm})_{n,m\in\mathbb{Z}}$ be the Gram matrix for $\{\varphi^t_n\}$, $n\in\mathbb{Z}$, $G_{nm}=\langle\varphi_n,\varphi_m\rangle$, (note that G does not depend on t) and let $G\in\mathbb{B}(l^2(\mathbb{Z}))$ be the operator with the Gram matrix with respect to the standard basis of $l^2(\mathbb{Z})$.

Lemma 1. *The operator G is bounded, invertible and has finite propagation.*

Proof. Direct calculation shows that $G_{n,n}=\frac{2}{3}$, $G_{n,n\pm1}=\frac{1}{6}$, and $G_{n,m}=0$ when $|m-n|\geq 2$. Therefore, $\|G\|\leq\frac{2}{3}+\frac{1}{3}$ and $\|\frac{2}{3}-G\|=\frac{1}{3}<1$, hence G is invertible. □

Set $C=G^{-1/2}$. By functional calculus, C can be approximated by polynomials in G, hence C lies in the norm closure of operators of finite propagation, i.e. $C\in C^*_u(\mathbb{Z})$.

Let $A\in\mathbb{B}(l^2(\mathbb{Z}))$, and let A_{nm} be its matrix elements with respect to the standard basis. Define $\gamma_t(A)\in\mathbb{B}(H_t)$ by $\gamma_t(A)\varphi^t_m=\sum_{n,m\in\mathbb{Z}}A_{nm}\varphi^t_n$. Note that γ_t is a homomorphism, but not a $*$-homomorphism.

Lemma 2. *There exist $k_1,k_2>0$ such that $k_1\|A\|<\|\gamma_t(A)\|<k_2\|A\|$.*

Proof. Let S denote the right shift on $l^2(\mathbb{Z})$, $x=\sum_{i\in\mathbb{Z}}x_i\varphi^t_i$. Then

$$\begin{aligned}\|\gamma_t(A)x\|^2 &= \sum_{i,j,k,l\in\mathbb{Z}}\bar{x}_ix_j\bar{A}_{ki}A_{lj}\langle\varphi^t_k,\varphi^t_l\rangle\\ &= \sum_{i,j,k\in\mathbb{Z}}\bar{x}_ix_j\bar{A}_{ki}A_{kj}+\frac{1}{6}\sum_{i,j,k,l\in\mathbb{Z}}\bar{x}_ix_j\bar{A}_{ki}A_{k\pm1,j}\\ &= \frac{2}{3}\|A\tilde{x}\|+\frac{1}{6}\langle A\tilde{x},(S+S^*)A\tilde{x}\rangle,\end{aligned}$$

where $\tilde{x}\in l^2(\mathbb{Z})$ has coordinates x_i with respect to the standard basis of $l^2(\mathbb{Z})$. As $|\frac{1}{6}\langle A\tilde{x},(S+S^*A\tilde{x}\rangle|\leq\frac{1}{3}\|A\|\|\tilde{x}\|$, the conclusion follows. □

Set $\psi^t_n=\gamma_t(C)\varphi^t_n=\sum_{m\in\mathbb{Z}}C_{mn}\varphi^t_m$. Then $\langle\psi^t_n,\psi^t_m\rangle=\langle\gamma_t(G^{-1})\varphi_n,\varphi_m\rangle=\delta_{n,m}$, hence $\{\psi^t_n\}_{n\in\mathbb{Z}}$ is an orthonormal system. Invertibility of C implies that the closures of the linear spans of $\{\varphi^t_n\}_{n\in\mathbb{Z}}$ and of $\{\psi^t_n\}_{n\in\mathbb{Z}}$ coincide. The advantage of this orthonormal system with respect to the system obtained from $\{\varphi_n^t\}_{n\in\mathbb{Z}}$ by Gram–Schmidt orthogonalization is that it is obtained from the original non-orthogonal system by an operator from $C_u^*(\mathbb{Z})$.

Define a map $\alpha_t : C_u^*(\mathbb{Z}) \to C^*(\mathbb{R})$. Let $T \in C_u^*(\mathbb{Z})$, $T = (T_{nm})_{n,m\in\mathbb{Z}}$. Set

$$\alpha_t(T)(f) = \sum_{n,m\in\mathbb{Z}} T_{nm}\psi_n^t\langle\psi_m^t, f\rangle, \quad f \in L^2(\mathbb{R}).$$

Let $U_t : l^2(\mathbb{Z}) \to L^2(\mathbb{R})$ be the isometry defined by $U_t(\delta_n) = \psi_n^t$. Then it is easy to see that $\alpha_t(T) = U_t T U_t^*$. Hence α_t is a $*$-homomorphism, in particular, it is isometric. As T is bounded, $\alpha_t(T)$ is bounded as well.

As $\gamma_t(C)$ can be considered as the transition matrix from the basis $\{\varphi_n^t\}$ to the basis $\{\psi_n^t\}$, we can write $\alpha_t(T) = \gamma_t(C)^{-1}\gamma_t(T)\gamma_t(C)$.

It remains to check that $\alpha_t(T) \in C^*(\mathbb{R}) \subset \mathbb{B}(L^2(\mathbb{R}))$. To this end, consider one more basis for H_t. By construction of C, for any $\varepsilon > 0$ there exists an operator $C_\varepsilon \in \mathbb{B}(l^2(\mathbb{Z}))$ of finite propagation M_ε such that $\|C - C_\varepsilon\| < \varepsilon$. Set $\psi_n^{t,\varepsilon} = \gamma_t(C_\varepsilon)\varphi_n^t$. Set $\tilde{T}_\varepsilon = \gamma_t(C_\varepsilon)^{-1}\gamma(T)\gamma(C_\varepsilon)$.

Lemma 3. *For sufficiently small ε there exists $K > 0$ such that $\|\alpha_t(T) - \tilde{T}_\varepsilon\| < K\varepsilon$ for any $t \in (0, 1]$.*

Proof. One should take ε small enough to provide invertibility of $\gamma(C_\varepsilon)$. Then

$$\begin{aligned}
\|\alpha_t(T) - \tilde{T}_\varepsilon\| &\le \|\gamma_t(C)^{-1} - \gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(T)\| \cdot \|\gamma_t(C)\| \\
&\quad + \|\gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(T)\| \cdot \|\gamma_t(C) - \gamma_t(C_\varepsilon)\| \\
&\le \|\gamma_t(C - C_\varepsilon)\| \cdot \|\gamma_t(C)^{-1}\| \cdot \|\gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(C)\| \\
&\quad + \|\gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(T)\| \cdot \|\gamma_t(C - C_\varepsilon)\| \\
&< k_2\varepsilon(\|\gamma_t(C)^{-1}\| \cdot \|\gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(C)\| + \|\gamma_t(C_\varepsilon)^{-1}\| \cdot \|\gamma_t(T)\|).
\end{aligned}$$

□

Lemma 4. $\alpha_t(T) \in C^*(\mathbb{R})$ *for any* $T \in C_u^*(\mathbb{Z})$.

Proof. As $T \in C_u^*(\mathbb{Z})$, it can be approximated by finite propagation operators T^N, $N \in \mathbb{N}$, with propagation N. This means that the matrix of T^N has the band structure ($T_{nm}^N = 0$ when $|m - n| > c$ for some $c > 0$). Then we may write T^N as a matrix with $2N + 1$ diagonals: $T^N\delta_n = \sum_{k=-N}^{N} \lambda_{n,k}\delta_{n+k}$, where the numbers $\lambda_{n,k}$ are uniformly bounded by $\|T\|$.

As $\alpha_t(T^N)$ can be approximated by operators of the form $\tilde{T}_\varepsilon^N$, it suffices to show that $\tilde{T}_\varepsilon^N \in C^*(\mathbb{R})$.

Let $f \in C_0(\mathbb{R})$ has compact support, say $[a, b] \subset \mathbb{R}$. Then

$$\begin{aligned}
\tilde{T}_\varepsilon^N\pi(f)(g) &= \sum_{n,m\in\mathbb{Z}} T_{nm}^N\langle\psi_m^{t,\varepsilon}, fg\rangle\psi_n^{t,\varepsilon} = \sum_{n,m\in\mathbb{Z}} T_{nm}^N\langle\gamma_t(C\varepsilon)\varphi_m^t, fg\rangle\gamma_t(C_\varepsilon)\varphi_n^t \\
&= \sum_{n\in\mathbb{Z}}\sum_{k=-N}^{N} \lambda_{n,k}\langle\gamma_t(C_\varepsilon)\varphi_{n+k}^t, fg\rangle\gamma_t(C_\varepsilon)\varphi_n^t. \qquad (3)
\end{aligned}$$

As $\mathrm{supp}(fg) \subset [a, b]$ and as propagation of $C_\varepsilon \le M_\varepsilon$, we have $\mathrm{supp}(\gamma_t(C_\varepsilon)\varphi_n^t) \subset [t(n - 1 - M_\varepsilon), t(n+1+M_\varepsilon)]$. Therefore, $\langle\gamma_t(C_\varepsilon)\varphi_{n+k}^t, fg\rangle \neq 0$ only when $[a, b]\cap[t(n-1-M_\varepsilon), t(n+1 + M_\varepsilon)] \neq \emptyset$, thus the sum (3) contains only finite number of non-zero summands, i.e. $\mathrm{Ran}\,\tilde{T}_\varepsilon^N\pi(f)$ is finitedimensional. Similarly, $\mathrm{Ran}\,\pi(f)\tilde{T}_\varepsilon^N$ is finitedimensional. Thus $\alpha_t(T)\pi(f)$ and $\pi(f)\alpha_t(T)$ are compact. Approximation of functions in $C_0(\mathbb{R})$ by functions f with finite support proves that $\alpha_t(T)$ is locally compact.

Similarly one can show that $\alpha_t(T)$ is of finite propagation. Indeed, let $f, g \in C_0(\mathbb{R})$ are such that the distance between their supports is greater than R. Then

$$\pi(f)\alpha_t(T_\varepsilon^N)\pi(g)(h) = \sum_{n\in\mathbb{Z}}\sum_{k=-N}^{N} \lambda_{n,k}\langle\gamma_t(C_\varepsilon)\varphi^t_{n+k}, gh\rangle f\gamma_t(C_\varepsilon)\varphi^t_n.$$

We have $\gamma_t(C_\varepsilon)\varphi^t_{n+k}, gh\rangle = 0$ when $\operatorname{supp} g \cap [t(n-k-1-N), t(n+k+1+N)] = \emptyset$, while $f\gamma_t(C_\varepsilon)\varphi^t_n = 0$ when $\operatorname{supp} f \cap [t(n-1-M_\varepsilon), t(n+1+M_\varepsilon)] = \emptyset$, so if R is sufficiently great then their product vanishes. □

The second map, $\beta_t : C^*(\mathbb{R}) \to C^*_u(\mathbb{Z})$, goes in the opposite direction and is not a homomorphism (but linear and even completely positive). In fact, it extends to a completely positive map from a greater C^*-algebra $C^*_p(\mathbb{R}) \supset C^*(\mathbb{R})$, which is the norm closure of all bounded operators of finite propagation. For $S \in C^*_p(\mathbb{R})$ set $(\beta_t(S))_{nm} = \langle\psi^t_n, S\psi^t_m\rangle$. Then the operator $\beta_t(S)$ can be written as $\beta_t(S)(\delta_m) = \sum_{n\in\mathbb{Z}}\langle\psi^t_n, S\psi^t_m\rangle\delta_n$. Recall that we denote by $U_t : l^2(\mathbb{Z}) \to L^2(\mathbb{R})$ the isometry that maps the standard basis $\{\delta_n\}_{n\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$ to the basis $\{\psi^t_n\}_{n\in\mathbb{Z}}$ of $H_t \subset L^2(\mathbb{R})$. Then $\beta_t(S) = U^*_t SU$. In particular, this implies that $\beta_t(S)$ is bounded for any bounded operator S.

Lemma 5. *Let $S \in C^*_p(\mathbb{R})$. Then $\beta_t(S) \in C^*_u(\mathbb{Z})$ for any $t > 0$.*

Proof. It suffices to show that $\beta_t(S) \in C^*_u(\mathbb{Z})$ for operators of finite propagation. For an operator S of finite propagation set $\tilde{S} = U^*_t D^* SDU_t$, where $D = C_\varepsilon C^{-1}$. As $\|1 - D\| < \varepsilon\|C^{-1}\|$, $\beta_t(S)$ can be approximated by operators of the form $\tilde{S}$. Let us show that $\tilde{S}$ has finite propagation, which means, for discrete spaces, that the matrix of this operator is a band matrix. We have

$$\tilde{S}_{nm} = \langle\psi^t_n, D^*SD\psi^t_m\rangle = \langle D\psi^t_n, SD\psi^t_m\rangle = \langle\psi^{t,\varepsilon}_n, S\psi^{t,\varepsilon}_m\rangle.$$

As $\operatorname{supp}\psi^{t,\varepsilon}_n \in [t(n-1-M_\varepsilon), t(n+1+M_\varepsilon)]$ and as S has finite propagation, $\tilde{S}_{nm} = 0$ when $|n-m|$ is sufficiently great. □

Note that $\beta_t \circ \alpha_t(S) = p_t S|_{H_t}$, in particular, this means that $p_t S|_{H_t}$ is locally compact for any $S \in C^*_p(\mathbb{R})$.

3. The fiber over 0

Let $L^\infty_0(\mathbb{R})$ denote the norm closure of $\cup_N L^\infty([-N, N]) \subset L^\infty(\mathbb{R})$. The group $\mathbb{R}$ acts on $L^\infty_0(\mathbb{R})$ by translations. Set $A_0 = L^\infty_0(\mathbb{R}) \rtimes \mathbb{R}$.

Lemma 6. *$A_0 \subset C^*_p(\mathbb{R})$.*

Proof. Let $f \in L^\infty([-N, N])$, and let $g \in C_0(\mathbb{R})$ be a continuous function with compact support. The linear combinations of operators of the form $S_{f,g}$, where

$$S_{f,g}(u)(x) = \int f(x)g(y)u(x-y)\,dy,$$

are dense in $L^\infty_0(\mathbb{R}) \rtimes \mathbb{R}$, so it suffices to show that $S_{f,g} \in C^*_p(\mathbb{R})$. Let $\operatorname{supp}(g) \subset [-M, M]$, and let $\varphi, \psi \in C_0(\mathbb{R})$ have supports at the distance greater than L. Then

$$(\pi(\varphi)S_{f,g}\pi(\psi)(u))(x) = \varphi(x)\int f(x)g(y)\psi(x-y)u(x-y)\,dy = 0$$

if $L > M$. □

Recall that C is the transition matrix that maps φ^t_n to ψ^t_n, i.e. $\psi^t_n = \sum_{m\in\mathbb{Z}} C_{mn}\varphi^t_m$. We have defined C by $C = G^{-1/2}$, where G is the Gram matrix for $\{\varphi^t_n\}_{n\in\mathbb{N}}$. We need the following technical result.

Lemma 7. *The series* $\sum_{n\in\mathbb{Z}}|C_{nm}|$ *and* $\sum_{m\in\mathbb{Z}}|C_{nm}|$ *converge. The sums* $\sum_{n\in\mathbb{Z}}|C_{nm}|$ *(resp.,* $\sum_{m\in\mathbb{Z}}|C_{nm}|$*) are bounded uniformly with respect to* m *(resp., to* n*).*

Proof. When working with matrices with the same entries along any diagonal it is convenient to identify $l^2(\mathbb{Z})$ with the square-integrable functions on the circle, and the basis $\{\delta_n\}_{n\in\mathbb{Z}}$ with the basis $\{e^{inx}\}$. Under this identification, the matrix $B_{nm} = b_{n-m}$ can be identified with the operator of multiplication by the function $\sum_{n\in\mathbb{N}} b_n e^{inx}$. Thus, the Gram matrix G corresponds to the invertible function $\frac{2}{3}+\frac{1}{3}\cos x$, and the matrix C corresponds to the function $(\frac{2}{3}+\frac{1}{3}\cos x)^{-1/2}$. As this function is smooth, its Fourier coefficients a_n, $n=0,1,\ldots$, are of rapid decay, i.e. $a_n = o(n^{-k})$ for any $k\in\mathbb{N}$. Therefore, the series $\sum_{n\in\mathbb{N}}|a_n|$ is convergent. As $C_{nm} = a_{|n-m|}$, the series $\sum_{n\in\mathbb{Z}}|C_{nm}|$ and $\sum_{m\in\mathbb{Z}}|C_{nm}|$ converge. Uniform boundedness is obvious. □

Denote the map $t\mapsto \beta_t(S)$ by $\beta^S : (0,\infty)\to C^*_u(\mathbb{Z})$.

Theorem 8. *The map* β^S *is norm-continuous on* $(0,\infty)$ *for any* $S\in A_0$.

Proof. Note that the linear combinations of operators $S_{f,g}$, $S_{f,g}(u)(x) = \int f(x)g(y)u(x-y)\,dy$, with $f\in L^\infty(\mathbb{R})$, $g\in C_0(\mathbb{R})$ of finite support are dense in $A_0 = L^\infty_0(\mathbb{R})\rtimes\mathbb{R}$, so it suffices to show continuity of the map $t\mapsto\beta_t(S)$ for $S = S_{f,g}$ for f and g with compact support.

Let $\|S_{f,g}\| = 1$, $\operatorname{supp}(f), \operatorname{supp}(g)\subset[-N,N]$, $a = \sum_{n\in\mathbb{Z}} a_n\delta_n \in l^2(\mathbb{Z})$, $\|a\| = 1$. Then

$$\begin{aligned}
\|(\beta_t(S_{f,g}) - \beta_{t_0}(S_{f,g}))a\|^2 &= \sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}(\langle\psi^t_n, S_{f,g}\psi^t_m\rangle - \langle\psi^{t_0}_n, S_{f,g}\psi^{t_0}_m\rangle)a_m\Big)^2\\
&\le \sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\langle\psi^t_n - \psi^{t_0}_n, S_{f,g}\psi^t_m\rangle a_m\Big)^2 \qquad (4)\\
&+ \sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}(\langle\psi^{t_0}_n, S_{f,g}(\psi^t_m - \psi^{t_0}_m)\rangle a_m\Big)^2. \qquad (5)
\end{aligned}$$

We shall estimate the first summand (4). The second summand (5) can be estimated in the same way (or, passing to the adjoint of $S_{f,g}$).

Recall that $\psi^t_n = \sum_{k\in\mathbb{Z}} C_{kn}\varphi^t_k$. Then

$$\sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\langle\psi^t_n - \psi^{t_0}_n, S_{f,g}\psi^t_m\rangle a_m\Big)^2 = \sum_{n\in\mathbb{Z}}\Big(\sum_{m,k,l\in\mathbb{Z}} C_{kn}C_{lm}\langle\varphi^t_k - \varphi^{t_0}_k, S_{f,g}\varphi^t_l\rangle a_m\Big)^2. \qquad (6)$$

Let $t\in[\frac{t_0}{2}, 2t_0]$. As the supports of f and g lie in $[-N,N]$, $S_{f,g}\varphi^t_l = 0$ for $|l| > (N+2)/t$, hence the sum over l is finite, over $|l|\le 2(N+2)/t_0$. Also the support of $S_{f,g}\varphi^t_l$ lies in $[(l-1)t-N, (l+1)t+N)]$, hence there are only finitely many k such that $\langle\varphi^t_k - \varphi^{t_0}_k, S_{f,g}\varphi^t_l\rangle \ne 0$. In other words, the sum in (6) can be written as

$$\sum_{n\in\mathbb{Z}}\Big(\sum_{|k|,|l|\le M}\sum_{m\in\mathbb{Z}} C_{kn}C_{lm}\langle\varphi^t_k - \varphi^{t_0}_k, S_{f,g}\varphi^t_l\rangle a_m\Big)^2 \qquad (7)$$

for some M.

For any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\varphi^t_k - \varphi^{t_0}_k\|_{L^2} < \frac{\varepsilon}{M^2}$ for any $|k|\le M$ when $|t-t_0| < \delta$.

Fix k and l, and estimate

$$\sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}} C_{kn}C_{lm}\langle\varphi_k^t-\varphi_k^{t_0},S_{f,g}\varphi_l^t\rangle a_m\Big)^2$$

$$\leq \sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}|C_{kn}C_{lm}|\frac{\varepsilon}{M^2}\|S_{f,g}\|\|\varphi_l^t\|_{L^2}\|a\|\Big)^2=\sum_{n\in\mathbb{Z}}|C_{kn}|\Big(\sum_{m\in\mathbb{Z}}|C_{lm}|\Big)^2\frac{\varepsilon^2}{M^4}.$$

By Lemma 7 the series $\sum_{m\in\mathbb{Z}}|C_{lm}|$ converges, hence is bounded by some L, hence

$$\sum_{n\in\mathbb{Z}}|C_{kn}|\Big(\sum_{m\in\mathbb{Z}}|C_{lm}|\Big)^2\frac{\varepsilon^2}{M^4}\leq L^3\frac{\varepsilon^2}{M^4}.$$

For shortness' sake set $x_{nmkl}=C_{kn}C_{lm}\langle\varphi_k^t-\varphi_k^{t_0},S_{f,g}\varphi_l^t\rangle a_m$. We have shown that

$$\sum_{n\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}x_{nmkl}\Big)^2\leq L^3\varepsilon^2/M^2$$

for any k,l.

Coming back to (7) and using $2xx'\leq x^2+(x')^2$, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\Big(\sum_{|k|,|l|\leq M}\sum_{m\in\mathbb{Z}}x_{nmkl}\Big)^2 &= \sum_{n,m,m',k,k',l,l'}x_{nmkl}x_{nm'k'l'}\\ &\leq M^2\sum_{n,m,m',k,l}x_{nmkl}x_{nm'kl}\\ &= M^2\sum_{k,l}\sum_{n}\Big(\sum_{m}x_{nmkl}\Big)^2\\ &\leq M^4\cdot L^3\varepsilon^2/M^4=L^3\varepsilon^2.\end{aligned}$$

Thus, for $|t-t_0|<\delta$ we have $\|\beta_t(S_{f,g})-\beta_{t_0}(S_{f,g})\|^2<2L^3\varepsilon^2$ which proves continuity. □

4. Continuous field of Roe algebras

Continuous fields of C^*-algebras (aka bundles of C^*-algebras or $C(T)$-C^*-algebras) were introduced by Fell [8] and Dixmier ([3], Section 10). Recall that a continuous field of C^*-algebras over a locally compact Hausdorff space T is a triple $(T,A,\pi_t:A\to A_t)$, where A and A_t, $t\in A$, are C^*-algebras, the $*$-homomorphisms π_t are surjective, the family $\{\pi_t\}_{t\in T}$ is faithful, and the map $t\mapsto\|\pi_t(a)\|$ is continuous for any $a\in A$.

Set $T=[0,1]$, $A_t=C^*_u(\mathbb{Z})$ for $t\neq 0$. The fiber A_0 over 0 was defined in the previous section. Set

$$A=C_0((0,1];C^*_u(\mathbb{Z}))+\{\beta^S:S\in A_0\}\subset\prod_{t\in T}A_t.$$

Lemma 9. *The set A is norm closed.*

Proof. First, let us show that

$$\sup_{t\in(0,1]}\|\beta_t(S)\|=\|S\|=\lim_{t\to 0}\|\beta_t(S)\|. \qquad (8)$$

Consider the projections $p_t=U_tU_t^*$ in $L^2(\mathbb{R})$. Note that $\|U_t^*SU_t\|=\|p_tSp_t\|$. Let $f\in C_0(\mathbb{R})$ be a Lipschitz function with finite support $[a,b]$, and let L be the Lipschitz constant for f. Let $g_t=\sum_{n\in\mathbb{Z}}\sqrt{t}f(tn)\varphi_n^t$ be a piecewise linear function such that $g_t(tn)=f(tn)$ for any $n\in\mathbb{N}$. Then $|f(x)-g_t(x)|_{L^2}\leq Lt$ for any $x\in\mathbb{R}$, hence $\|f-g_t\|_{L^2}\leq Lt\sqrt{b-a+2}$. As g_t lies in the linear span of the functions φ_n^t, $n\in\mathbb{N}$, we have $g_t=p_tg_t$. As $\|f-p_tf\|_{L^2}\leq\|f-p_tg_t\|_{L^2}$, we have $\lim_{t\to0}f-p_tf=0$. As Lipschitz functions with finite support are dense in $L^2(\mathbb{R})$, we conclude that the $*$-strong limit of p_t is the identity operator. Note also that $\varphi_n^{2t} = \sqrt{2}\varphi_{2n}^t + \frac{\sqrt{2}}{2}\varphi_{2n-1}^t + \frac{\sqrt{2}}{2}\varphi_{2n+1}^t$, hence the linear span of $\{\varphi_n^t\}$ lies in the linear span of $\{\varphi_n^{t/2}\}$, therefore $p_t \leq p_{t/2}$ for any $t \in (0,1]$, hence the sequence $\|p_{t/2^k} S p_{t/2^k}\|$ is increasing.

Consider the norm closure $\overline{A}$ of A. Then $I = C_0((0,1]; C_u^*(\mathbb{Z}))$ is a closed ideal in $\overline{A}$. Let $\{f_n + \beta^{S_n}\}$ be a Cauchy sequence in A. Passing to the quotient C^*-algebra $\overline{A}/I$, the sequence $\{f_n + \beta^{S_n} + I\} = \{\beta^{S_n} + I\}$ is also a Cauchy sequence, as the quotient $*$-homomorphisms have norm 1 ([3], Section 1.8). Note that

$$\|\beta^S + I\| = \inf_{f \in I} \|f + \beta^S\| \geq \lim_{t \to 0} \|f(t) + \beta_t(S)\| = \lim_{t \to 0} \|\beta_t(S)\| = \|S\|,$$

hence $\{S_n\}$ is also a Cauchy sequence. As A_0 is norm closed, it has a limit in A_0. But then $\{f_n\}$ is also a Cauchy sequence, and as I is norm closed, its limit lies in I. Thus $\{f_n + \beta^{S_n}\}$ converges in A. □

Define $\pi_t : A \to A_t$ by $\pi_t(f + \beta^S) = f(t) + \beta_t(S)$ for $t > 0$, and $\pi_0(f + \beta^S) = S$. These maps are well defined as $f_1 + \beta^{S_1} = f_2 + \beta^{S_2}$ implies that $f_1 = f_2$ and $S_1 = S_2$.

Theorem 10. *The triple $(T, A, \pi_t : A \to A_t)$ is a continuous field of C^*-algebras.*

Proof. Each π_t is clearly surjective. If $\pi_t(f_1 + \beta^{S_1}) = \pi_t(f_2 + \beta^{S_2})$ for any $t \in T$ then, taking $t = 0$, we conclude that $S_1 = S_2$. Then we see that $f_1(t) = f_2(t)$ for any $t \in (0,1]$, hence $f_1 = f_2$. Finally, we have to check that the map $t \mapsto \pi_t(a)$ is continuous. Let $a = f + \beta^S$. Continuity at $t > 0$ follows from continuity of f (by definition) and continuity of β^S (Theorem 8). Continuity at 0 follows from (8). □

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