S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 20, No. 2, pp. 144–145 (2023) УДК 517.95
https://doi.org/10.33048/semi.2023.20.??? MSC 35A35

ON RUNGE TYPE THEOREMS FOR SOLUTIONS TO STRONGLY UNIFORMLY PARABOLIC OPERATORS

A.A. SHLAPUNOV AND **P.YU. VILKOV**

Communicated by P.P. Petrov

Abstract: Let G_1, G_2 be domains with rather regular boundaries in $\mathbb{R}^{n+1}$, $n \geq 2$, such that $G_1 \subset G_2$. We investigate the problem of approximation of solutions to strongly uniformly $2m$-parabolic system $\mathcal{L}$ in the domain G_1 by solutions to the same system in the domain G_2. First, we prove that the space $S_{\mathcal{L}}(G_2)$ of solutions to the system $\mathcal{L}$ in the domain G_2 is dense in the space $S_{\mathcal{L}}(G_1)$, endowed with the standard Fréchet topology of uniform convergence on compact subsets in G_1, if and only if the sets $G_2(t) \setminus G_1(t)$ have no non-empty compact components in $G_2(t)$ for each $t \in \mathbb{R}$, where $G_j(t) = \{x \in \mathbb{R}^n : (x,t) \in G_j\}$. Next, under additional assumptions on the regularity of the bounded domains G_1 and $G_1(t)$, we prove that solutions from the Lebesgue class $L^2(G_1) \cap S_{\mathcal{L}}(G_1)$ can be approximated by solutions from $S_{\mathcal{L}}(G_2)$ if and only if the same assumption on the sets $G_2(t) \setminus G_1(t)$, $t \in \mathbb{R}$, is fulfilled.

Keywords: approximation theorems, Frećhet topologies, strongly uniformly parabolic operators.

Shlapunov, A.A., Vilkov P.Yu. On Runge type theorems for solutions to strongly uniformly parabolic operators.

The second author was supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement No. 075-02-2024-1429). The first author was supported by the Ministry of Science and Higher Education of the Russian Federation (Agreement 075-10-2021-093, Project MTH-RND-2124).

Received January, 1, 2023, Published December, 31, 2023.

Apparently, the Approximation Theory in Analysis begins with the famous Weierstraß theorem for continuous functions on segments of the real line, see [43], where polynomials were used as the approximating set. It appears that the matter has become significantly more complicated for complex functions of complex variable if one wants to approximate them by *holomorphic* polynomials, see [32]. Actually, C. Runge [32] proposed to use the Cauchy kernel for the so-called rational approximation of holomorphic functions on open connected sets (plane domains) in $\mathbb{C}$ in the topology related to uniform convergence on compact subsets of the domain. The notion of Runge's pair $\omega \subset \Omega$ of plane domains, for which the space $\mathcal{O}(\Omega)$ of holomorphic functions in Ω is dense in the space $\mathcal{O}(\omega)$ (endowed with the topology discussed above) gives rise to the investigation of topological/geometrical conditions on the pair. The analysis of continuous functions approximable on compact sets led to the notion of the analytic capacity and the creation of the related theory, see, for instance, the paper by A.G. Vitushkin [42]. The matters were successfully extended to the theory of harmonic functions, see [27], and even to solutions to general elliptic operators with the uniqueness condition in small, see for instance, [3], [21], [23], [39, ch. 4, 5], where fundamental solutions to the related operators were used instead of the Cauchy kernel. Of course, one should mention the theorem for (non-necessarily elliptic) operators with constant coefficients related to the uniform approximation by exponential-polynomial solutions in convex domains, see [24], [29, Ch. VI]. By the way, F.E. Browder [3] extended Runge type results to some non-elliptic, e.g., hyperbolic, operators.

As for the parabolic equations, the problem of the uniform approximation in the context of Runge's pairs was successfully solved for the heat equation in the papers [15], [5] (see also [13] for "the rational approximation"by the functions constructed with the use of the heat kernel).

On the other hand, approximation theorems in various function spaces, where behaviour of the elements are controlled up to the boundary of the considered sets, appeared to be more important for applications, see, for instance, [3] in the context of abstract locally convex topological vector spaces, [14] for the analytic functions or the monograph [38, Ch. 5–8] for the Sobolev solutions to systems of differential equations with surjective/injective symbols. But at the end of the XX-th century interest to this topic in the framework of general theory of differential equations faded. In particular, for parabolic equations there are two principal reasons for this fact. The first one is the above mentioned theorem on the uniform approximation of solutions to operators with the constant coefficients in convex domains by the exponential-polynomial solutions covering many needs of approximation results in the Theory of Partial Differential Equations. The second reason is that typical problems for parabolic operators (such as the Cauchy problem in $\mathbb{R}^n \times [0, T)$ or the problems in the cylinder domains related to the Dirichlet or Neumann data on the lateral side, combined with the Cauchy data on the base of the domain) were handled mostly with the Fourier method of separation of variables in the Bochner-Sobolev spaces, where approximation with respect to the functions of the space variables is crucial, see, for example, [22], [40], or by the integral representation method in the Hölder spaces where the approximation is not usually used, see, for instance, [11], [20].

However in recent years Runge's type approximation results were established for some classes of non-elliptic operators, including the parabolic ones. For instance, we mention [9] on approximation in Hölder spaces of solutions on closed sets to second order scalar parabolic equations with Hölder coefficients, [17] on approximation in Fréchet spaces of C^∞-smooth solutions for constant coefficients partial differential operators with a single characteristic direction, complemented with a theorem on the so-called "quantitative approximatation"[4], and [10] about approximation of solutions to Schrödinger equation in Bochner spaces. Also, for solutions from the Lebesgue spaces to the heat equation or to the parabolic Lamé type operator on cylinder domains in $\mathbb{R}^{n+1}$, we refer to recent papers [33] and [41], respectively.

Significantly, many important applications of Runge type results for solutions to parabolic operators were discovered at the beginning of the XXI-st century, especially to Euler and Navier-Stokes equations of Hydrodynamics and to some inverse problems of Mathematical Physics, see [8], [12], [31].

Our primary interest is the non-standard Cauchy problem for solutions to the parabolic equations in cylinder domains with the Cauchy data on a part of the lateral side of the cylinder that behaves much more like the ill-posed Cauchy problem for solutions to elliptic systems, see, for instance, [34], [38, Ch. 10] for the elliptic theory and [19], [26], [41], [30], for the parabolic theory in the Sobolev type spaces. The approximation theorems for Runge's pairs in the Lebesgue type spaces are crucial for this type of problems because they provide both dense solvability and a possibility to construct the approximate solutions to the problems, see [27], [25], [34, Theorem 7.6], and [41, Corollary 3.5], respectively. Actually, the non-standard ill-posed Cauchy problem for the parabolic operators plays essential role in the development of non-invasive methods of Cardiology, see, for instance, [16, §4]. Please, note that formulations of such problems do not usually use initial data at a suitable time $t = t_0$ and hence we do not need to pose them in the cylinder domains.

Thus, in the present paper we consider approximation theorems for solutions to the strongly uniformly parabolic matrix differential operator $\mathcal{L} = \partial_t - L$ on a strip $\mathbb{R}^n \times \mathcal{I}$, where $\mathcal{I}$ is an (open) interval on the time axis, $(-L)$ is a strongly elliptic operator with bounded regular coefficients on the strip. We additionally assume that both $\mathcal{L}$ and its formal adjoint operator $\mathcal{L}^*$ admit bilateral fundamental solutions and possess the Unique Continuation Property with respect to the space variables. The normality property for the fundamental solution plays an essential role in the considerations. As far as the topic targets the problems without initial data, the presented approach is fit for both the parabolic and backwards parabolic operators.

Section §1 is devoted to the uniform approximation on compact subsets of the domain $G_1 \subset \mathbb{R}^{n+1}$ of the elements of the space $S_{\mathcal{L}}(G_1)$ consisting of continuous solutions to the equation $\mathcal{L}u = 0$ in G_1 by the elements of the space $S_{\mathcal{L}}(G_2)$ where the domain G_2 contains G_1. We prove that, under reasonable assumptions on the regularity of the coefficients of the operator L, the space $S_{\mathcal{L}}(G_2)$ is dense in $S_{\mathcal{L}}(G_1)$ if and only if for each $t \in \mathcal{I}$ the complement of the set $G_1(t) = \{x \in \mathbb{R}^n : (x,t) \in G_1\}$ in the set $G_2(t)$ has no compact (non-empty) components; this is quite similar to the case of the heat equation. Actually, taking in accounts a small gap in the proof of the approximation theorem for the heat equation in [5] (that was discovered by [13]) concerned with rather general assumptions on the structure of the domains considered as Runge's pairs in this particular situation, we additionally assume some regularity of the domains' boundaries, see assumption (A), in section §1.

In the section §2 we consider a more subtle problem of approximation of solutions to parabolic operator $\mathcal{L}$ from the Lebesgue class $L^2(G_1)$ by more regular solutions in a bigger domain G_2. We present a solution to this problem in the case where G_1 is a bounded domain with piece-wise smooth boundary with additional geometric restrictions (see (A1), (A2)). Finally, as a by-product, we obtain the theorem on existence of bases with double orthogonality property in Sobolev type spaces of solutions to parabolic systems.

1 The uniform approximation

Let $\mathbb{R}^n$, $n \geq 1$, be the n-dimensional Euclidean space with the coordinates $x = (x_1, \dots, x_n)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain (open connected set). As usual, denote by $\overline{\Omega}$ the closure of Ω, and by $\partial\Omega$ its boundary.

We consider functions over $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. As usual, for $s \in \mathbb{Z}_+$ we denote by $C^s(\Omega)$ and $C^s(\overline{\Omega})$ the spaces of all s times continuously differentiable functions on Ω and $\overline{\Omega}$, respectively. Next, for $p \in [0,1)$, we denote by $C^{s,p}(\overline{\Omega})$ the standard Hölder spaces. The spaces $C^{s,p}(\overline{\Omega})$ are known to be Banach spaces with the standard norms and the $C^{s,p}(\Omega)$ are the Fréchet spaces with the standard semi-norms, see, for instance, [20].

Let also $L^2(\Omega)$ be the Lebesgue space over Ω with the standard inner product $(u,v)_{L^2(\Omega)}$ and let $H^s(\Omega)$, $s \in \mathbb{N}$, be the Sobolev space with the standard inner product $(u,v)_{H^s(\Omega)}$. As usual, we consider the Sobolev space $H^{-s}(\Omega)$, $s \in \mathbb{N}$, as the dual space of $H^s_0(\Omega)$ where $H^s_0(\Omega)$ is the closure of the space $C^\infty_{\mathrm{comp}}(\Omega)$ consisting of smooth functions with compact supports in Ω.

For a natural number k, it is convenient to denote by $\mathbf{C}^{s,p}_k(\Omega)$ the space of k-vector functions with components of the class $C^{s,p}(\Omega)$ and, similarly, for the space $\mathbf{L}^2_k(\Omega)$, etc.

Let $\mathcal{I}$ be a finite or infinite (open) interval on the time axis and let L be a $(k \times k)$-matrix differential operator with continuous coefficients in the strip

$\mathbb{R}^n \times \mathcal{I}$ of an even order $2m$:

$$L = \sum_{|\alpha|\leq 2m} L_\alpha(x,t)\partial_x^\alpha$$

where $L_\alpha(x,t)$ are $(k\times k)$-matrices with entries as described above and such that $L^*_\alpha(x,t) = L_\alpha(x,t)$ for all multi-indexes $\alpha \in \mathbb{N}^n$ with $|\alpha| = 2m$, all $t \in \mathcal{I}$ and all $x \in \mathbb{R}^n$. Consider the strongly uniformly (Petrovsky) $2m$-parabolic operator

$$\mathcal{L} = \partial_t - L,$$

see, for instance, [7], [36]. More precisely, this additionally means that the operator $(-L)$ is strongly elliptic, i.e. there is a positive constant c_0 such that

$$(-1)^{m+1}w^*\Big(\sum_{|\alpha|=2m} L_\alpha(x,t)\zeta^\alpha\Big)w \geq c_0|w|^2|\zeta|^{2mk}$$

for all $t \in \mathcal{I}$, all $x \in \mathbb{R}^n$, all $\zeta \in \mathbb{R}^n \setminus \{0\}$ and all $w \in \mathbb{C}^k \setminus \{0\}$; here w^* is the transposed and complex adjoint vector for the complex vector $w \in \mathbb{C}^k$. In particular, for each fixed $t_0 \in \mathcal{I}$ the operator L is (Petrovsky) elliptic with respect to the space variables x, i.e.

$$\det\Big(\sum_{|\alpha|=2m} L_\alpha(x,t_0)\zeta^\alpha\Big) \neq 0$$

for all $x \in \mathbb{R}^n$ and all $\zeta \in \mathbb{R}^n \setminus \{0\}$.

As usual, we denote by $\mathcal{L}^*$ the formal adjoint operator for $\mathcal{L}$:

$$\mathcal{L}^* = -\partial_t - \sum_{|\alpha|\leq 2m} (-1)^{|\alpha|}\partial_x^\alpha(L^*_\alpha(x,t)\,\cdot).$$

As we always may proceed with a decomplexification, doubling the dimensions of vectors and matrices, but preserving the ellipticity properties of the operator L, we assume all the vector functions and coefficients of differential operators under the consideration are real-valued.

Following [7, pp. 245-246], we assume also that the operator $\mathcal{L}$ satisfies the following assumptions:

- (α_1) the coefficients $L_\alpha(x,t)$ are uniformly continuous on the strip $\mathbb{R}^n\times\mathcal{I}$ for $|\alpha| = 2m$;
- (α_2) the coefficients $L_\alpha(x,t)$ are uniformly bounded on the strip $\mathbb{R}^n \times \mathcal{I}$;
- (α_3) the coefficients $L_\alpha(x,t)$ satisfy a Hölder condition with respect to the space variables x uniformly on the strip $\mathbb{R}^n \times \mathcal{I}$;
- (α_4) each coefficient $L_\alpha(x,t)$ has partial derivatives $\partial_x^\beta L_\alpha(x,t)$ for all multi-indeces β with $\beta \leq \alpha$ on the strip $\mathbb{R}^n\times\mathcal{I}$ satisfying assumptions (α_1), (α_2), (α_3).

Under these assumptions on the coefficients of the operator it admits a unique fundamental solution $\Phi(x,y,t,\tau)$ possessing standard estimates [7, formulas (2.16), (2.17)]) and the normality property ([7, Property 2.2]), i.e.

$$\mathcal{L}_{x,t}\Phi(x,y,t,\tau) = I_k\,\delta(x-y,t-\tau), \tag{1}$$

i.e. the right hand side equals to the unit matrix I_k multiplied by the Dirac distribution at the point (x, t) which is commonly written as $\delta(x - y, t - \tau)$, where δ denotes the Dirac distribution at the origin, and

$$\mathcal{L}^*_{y,\tau} \Phi^*(x, y, t, \tau) = I_k \delta(x - y, t - \tau), \tag{2}$$

where $\Phi^* = (\Phi_{ji})$ is the adjoint matrix for $\Phi = (\Phi_{ij})$. Note that the normality property still holds for complex-valued operators with coefficients satisfying assumptions (α_1)–(α_4), but in this case $\Phi^* = (\overline{\Phi}_{ji})$ is the Hermitian adjoint.

We are going to investigate solutions to $2m$-parabolic equations in non-cylinder domains of special type, see for instance [36, §22]. Namely let G be a domain in the strip $\mathbb{R}^n \times \mathcal{I}$ and let

$$T_1(G) = T_1 = \inf_{(x,t)\in G} t,\ T_2(G) = T_2 = \sup_{(x,t)\in G} t.$$

Consider sets $G(t) = \{x \in \mathbb{R}^n : (x, t) \in G\}$, $t \in \mathcal{I}$, playing an essential role in the sequel. We assume that the boundary of G satisfies the following property.

(A) The set $G(t)$ is a Lipshitz domain in $\mathbb{R}^n$ for each $t \in (T_1, T_2)$ and for any numbers t_3, t_4 such that $T_1 < t_3 < t_4 < T_2$ the set $\Gamma_{t_3,t_4} = \cup_{t\in[t_3,t_4]} \partial G(t)$ is a Lipschitz surface in $\mathbb{R}^{n+1}$.

For us, the primary interest for studying parabolic equations in such domains is a possibility of applications in the Cardiology, where $G(t) \subset \mathbb{R}^3 \times (T_1, T_2)$ is the shape of the human myocardium at the time t (see, for instance, [1] for the related mathematical models or [16] for the particular bi-domain model).

With this purpose we need the standard Banach anisotropic spaces $C^{2sm,s}(\overline{G})$, see for instance, [36, §22], with the norm:

$$\|u\|_{C^{2ms,s}(\overline{G})} = \sum_{|\alpha|+2mj\le 2ms} \|\partial^\alpha_x \partial^j_t u\|_{C(\overline{G})}.$$

The corresponding anisotropic spaces $C^{2sm,s}(G)$ are Fréchet spaces with the topology of uniform convergence on compact subsets of G with all the derivatives $\partial^\alpha_x \partial^j_t$, $|\alpha| + 2mj \le 2ms$, see, for instance, [35].

Now, $S_\mathcal{L}(G)$ be the set of all the generalized k-vector functions on G, satisfying the (homogeneous) equation

$$\mathcal{L}u = 0 \text{ in } G \tag{3}$$

in the sense of distributions. We endow this space with the standard topology of the uniform convergence on compact subsets of G. Next, we note that estimates [7, formulas (2.16), (2.17)]) for the fundamental solution imply the standard interior a priori estimates for solutions to (3), see, for instance, [36, §19], or [11, Ch. 4, §2] for the second order operators. This means that all the distributional solutions to equation (3) are $(2m, 1)$-differentiable on their domain, i.e. the following continuous embedding holds true:

$$S_\mathcal{L}(G) \subset \mathbf{C}^{2m,1}_k(G).$$

In particular, this means that $S_{\mathcal{L}}(G)$ is a closed subspace in $\mathbf{C}_k(G)$ and it is a Fréchet space itself (with the standard Fréchet topology inducing the standard uniform convergence together with all the derivatives on compact subsets of G).

We also need more assumptions on the operator $\mathcal{L}$: the Unique Continuation Property with respect to the space variables for $\mathcal{L}$ and $\mathcal{L}^*$. Namely, we recall that the Unique Continuation Property with respect to the space variables for a differential operator $\tilde{L}$ on a domain $G \subset \mathbb{R}^{n+1}$ consists in the following:

(UCP) for any solution $u \in S_{\tilde{\mathcal{L}}}(G)$ and any $t_0 \in (T_1, T_2)$, if $u(x, t_0) = 0$ for all x from an open subset $\omega \subset G(t_0)$ then $u \equiv 0$ in the open connected component of $G(t_0)$, containing ω.

Of course, if $\mathcal{L}$ is the operator with constant coefficients then all the assumptions (α_1), (α_2), (α_3), (α_4) and (UCP) for both $\mathcal{L}$ and $\mathcal{L}^*$ are fulfilled. Besides, they hold true if the coefficients of the operator $\mathcal{L}$ are smooth, bounded and real analytic with respect to the variables x for each $t \in \mathcal{I}$. In this case, the elements of $S_{\mathcal{L}}(G)$ are actually smooth in G and they are real analytic with respect to the space variable $x \in G(t)$ for all $t \in (T_1, T_2)$, see, for instance [7]. Moreover, the following continuous embedding holds true in this particular situation:

$$S_{\mathcal{L}}(G) \subset \mathbf{C}^{\infty}_k(G).$$

Next, following C. Runge [32] and [5], we call domains $G_1 \subset G_2 \subset \mathbb{R}^{n+1}$ $\mathcal{L}$-Runge's pair if $S_{\mathcal{L}}(G_2)$ is everywhere dense in $S_{\mathcal{L}}(G_1)$. The following theorems on Runge's type approximations are quite similar to the corresponding statement for the heat equation, see [15] for the case $G_2 = \mathbb{R}^{n+1}$ or [5] for the case of general domains. The typical assumption in this type of theorems is the following:

(B) for each $t \in \mathcal{I}$ the set $G_2(t) \setminus G_1(t)$ has no compact (non-empty) components in the set $G_2(t)$.

Theorem 1. *Let $\mathcal{L}$ satisfy assumptions (α_1), (α_2), (α_3), (α_4) and $G_1 \subset G_2$ be domains in the strip $\mathbb{R}^n \times \mathcal{I}$ such that $G_2 \neq \mathbb{R}^n \times \mathcal{I}$. If* (UCP) *is fulfilled for $\mathcal{L}^*$, and the pair G_1, G_2 satisfies assumptions* (A) *and* (B)*, then $S_{\mathcal{L}}(G_2)$ is everywhere dense in $S_{\mathcal{L}}(G_1)$.*

Theorem 2. *Let $\mathcal{L}$ satisfy assumptions (α_1), (α_2), (α_3), (α_4) and $G_1 \subset G_2$ be domains in the strip $\mathbb{R}^n \times \mathcal{I}$ satisfying assumption* (A)*. If* (UCP) *is fulfilled for $\mathcal{L}$ and for the coefficients L_α, $0 < |\alpha| \leq 2m$, the partial derivatives $\partial_t L_\alpha$ are continuous on $\mathbb{R}^n \times \mathcal{I}$, then assumption* (B) *is neccessary for $S_{\mathcal{L}}(G_2)$ to be everywhere dense in $S_{\mathcal{L}}(G_1)$.*

Let us proceed with the related proofs.

Proof of Theorem 1. The proof follows the scheme of typical Runge's type theorems related to elliptic operators, cf. [23], [3], and their generalizations to parabolic ones. To be more precise, we slightly modify the proof from

[5] for the solutions to the heat equation, using the duality theorems from modern functional analysis, see, for instance, [6] and the parametrix method for differential equations, see, for example, [7], [11], [37].

As we have noted above, the space $S_{\mathcal{L}}(G_1)$ is a closed subspace of the space $\mathbf{C}_k(G_1)$, endowed with the standard topology of the uniform convergence on compact subsets of G_1. Then the Hahn-Banach Theorem implies that G_1, G_2 is a $\mathcal{L}$-Runge's pair if and only if any continuous functional F on $\mathbf{C}_k(G_1)$ annihilating the space $S_{\mathcal{L}}(G_2)$ also annihilates the space $S_{\mathcal{L}}(G_1)$.

On the other hand, according the Riesz Theorem, see, for instance, [6, Theorem 4.10.1], any element F of the dual space $\mathbf{C}^*_k(G_1)$ for $\mathbf{C}_k(G_1)$ can be presented with the use of a (k-vector valued) Radon measure μ_F with compact support $K(\mu_F)$ in G_1 , i.e.

$$F(u) = \langle u, d\mu_F\rangle \text{ for all } u \in S_{\mathcal{L}}(G_1). \tag{4}$$

Let W be the vector function, with components obtained by applying the functional F to the corresponding columns of the matrix $(x,t) \to \Phi(x,t,y,\tau)$. For obvious reasons we write

$$W(y,\tau) = \langle \Phi^*(x,y,t,\tau), d\mu_F(x,t)\rangle;$$

of course, it is well-defined outside the support of $d\mu_F$. By (1), for any vector $\varphi \in C^\infty_{k,comp}(\mathbb{R}^n \times \mathcal{I})$ we have

$$\mathcal{L}_{x,t}\,\langle \Phi^*(x,y,t,\tau), \varphi(y,\tau)\rangle = \varphi(x,t),$$

and then, the interior a priori estimates for parabolic systems, see [36, §19], imply that the vector function $V_\varphi(x,t) = \langle \Phi^*(x,y,t,\tau), \varphi(y,\tau)\rangle$ belongs to $\mathbf{C}^{2m,1}_k(\mathbb{R}^n \times \mathcal{I})$. In particular, the vector function W can be extended as a distribution to $\mathbb{R}^n \times \mathcal{I}$ via

$$\langle W, \varphi\rangle = \langle V_\varphi, d\mu_F\rangle,$$

because $d\mu_F$ is a (k-vector valued) Radon measure with compact support.

Besides, according to (1), columns of the matrix $\Phi(x,t,y,\tau)$ belong to $S_{\mathcal{L}}(G_2)$ with respect to variables (x,t) for each fixed $(y,\tau) \notin G_2$. Hence, if $F \in \mathbf{C}^*_k(G_1)$ annihilates the space $S_{\mathcal{L}}(G_2)$ then we have

$$W(y,\tau) = 0 \text{ for all } (y,\tau) \notin G_2. \tag{5}$$

But (2) implies that

$$\mathcal{L}^* W = d\mu_F \text{ in } \mathbb{R}^n \times \mathcal{I}, \tag{6}$$

in the sense of distributions and, in particular,

$$\mathcal{L}^* W = 0 \text{ in } \Big(\mathbb{R}^n \times \mathcal{I}\Big) \setminus K(\mu_F). \tag{7}$$

Note that the operator $\mathcal{L}^*$ is backwards-parabolic and, for any solution $v(y,\tau)$ to the equation $\mathcal{L}^* v = 0$, the vector $w(y,\tau) = v(y,-\tau)$ is a solution to the strongly parabolic system of equations $(\partial_\tau - L^*_y)w = 0$. Therefore $W(y,\tau) \in \mathbf{C}^{2m,1}_k\big(\big(\mathbb{R}^n \times \mathcal{I}\big) \setminus K(\mu_F)\big)$ and, in particular, it is C^{2m}-smooth with respect to y in $\mathbb{R}^n \setminus K(\mu_F)(\tau)$ for each $\tau \in \mathcal{I}$ where $K(\mu_F)(\tau) = \{x \in \mathbb{R}^n : (x, \tau) \in K(\mu_F)\}$.

As both $G_1 \subset G_2$ satisfy assumption (A) and $G_2 \neq \mathbb{R}^n \times \mathcal{I}$, the componets of sets $\mathbb{R}^n \setminus G_2(t) \subset \mathbb{R}^n \setminus G_1(t)$ are either empty sets or closures of Lipschitz domains. Since the set $G_2(t) \setminus G_1(t)$ has no compact components in $G_2(t)$, we see that each bounded component of $\mathbb{R}^n \setminus \overline{G_1(t)}$ intersects with $\mathbb{R}^n \setminus \overline{G_2(t)}$ by a non-empty open set for each $t \in (T_1, T_2)$. Hence, by (UCP) for $\mathcal{L}^*$, the vector W vanishes on every bounded component of $\mathbb{R}^n \setminus \overline{G_1(t)}$ for each $t \in (T_1, T_2)$. Next, let $\hat{G}_j(t)$ be the union of $G_j(t)$ with all the components of the set $G_j(t)$ that are relatively compact in $\mathbb{R}^n$. By the discussion above, the closure of $\hat{G}_1(t)$ lies in the closure of $\hat{G}_2(t)$. Then, by De Morgan's Law in the Sets' Theory we have

$$\left(\mathbb{R}^n \setminus \overline{\hat{G}_1(t)}\right) \cap \left(\mathbb{R}^n \setminus \overline{\hat{G}_2(t)}\right) = \mathbb{R}^n \setminus \left(\overline{\hat{G}_2(t)} \cup \overline{\hat{G}_1(t)}\right) = \mathbb{R}^n \setminus \overline{\hat{G}_2(t)}. \quad (8)$$

In particular, this means that the vector W vanishes on unbounded components of the set $\mathbb{R}^n \setminus \overline{G_1(t)}$ for each $t \in (T_1(G_1), T_2(G_1))$, too. Thus, (5) and the Unique Continuation Property (UCP) for the operator $\mathcal{L}^*$ imply that

$$W(y, \tau) = 0 \text{ in } \mathbb{R}^n \setminus \overline{G_1(\tau)} \text{ for all } \tau \in \mathcal{I},$$

i.e. the vector W is supported in $\overline{G}_1$.

As Diaz [5, p. 644] noted, complications may arise if W is not compactly supported in G_1 even in the case where $\mathcal{L}$ is the heat operator. Nevertheless he considered more general type of domains and for this reason the related proof was rather complicated.

Clearly, for $(x, t) \in \partial G_1$ with $t \in (T_1(G_1), T_2(G_1))$ it holds $x \in \partial G_1(t)$. By hypothesis (A) and $\text{dist}(\partial G_1, K(\mu(F))) > 0$ it follows immediately that W vanishes in an open subset of the connected component of $(\mathbb{R}^n \times \mathcal{I}) \setminus K(\mu(F)$ in $\mathbb{R}^n \times \mathcal{I}$ which contains (x, t). Therefore, by (UCP) for $\mathcal{L}^*$ and (7) it follows that W vanishes in a neghbourhood of the point (x, t). Moreover for $(x, t) \in \partial G_1$ with $t \in \{T_1(G_1), T_2(G_1)\}$ it is even simpler to conclude that W vanishes in a neghbourhood of the point (x, t). Hence the support $\text{supp}\,(W)$ of W is contained in G_1.

Following [15], we may even say that W is supported in

$$\hat{K}(\mu_F) = \cup_{\tau \in (T_1, T_2)} \hat{K}_{G_1}(\mu_F)(\tau),$$

where $\hat{K}_{G_1}(\mu_F)(\tau)$ is the union of $K(\mu_F)(\tau)$ with all the components of the set $G_1(\tau) \setminus K(\mu_F)(\tau)$ that are relatively compact in $G_1(\tau)$. Then, to prove that $\text{supp}\,(W)$ is compact, we may almost literally repeat the arguments from the proof of the crucial lemma in [15, §2]. Indeed, each section $K_{G_1}(\mu_F)(\tau) = \emptyset$ for all sufficiently large $|\tau|$. Therefore, $\hat{K}(\mu_F)$ is bounded. To prove that $\hat{K}(\mu_F)$ is closed, suppose that $(x, \tau) \notin \hat{K}(\mu_F)$. This means $x \notin \hat{K}(\mu_F)(\tau)$, and this implies that there exists a continuous curve γ in $\mathbb{R}^n$ which starts at x and tends to ∞ lying in the open set $\mathbb{R}^n \setminus \hat{K}(\mu_F)(\tau)$. Choose a closed ball $B \subset \mathbb{R}^n \setminus \hat{K}(\mu_F)(\tau)$ centered at x. Then there exists $\varepsilon > 0$ such that for $|\tau-\tau'|<$the set $K_{G_1}(\mu_F)(\tau')$ is disjoint from B and the image of γ. But then for $x'\in B$ the point $x'\notin \hat{K}(\mu_F)(\tau')$. Thus, $B(x)\times(\tau-\varepsilon,\tau+\varepsilon)$ is disjoint from $\hat{K}(\mu_F)$. This proves $\mathbf{R}^{n+1}\setminus \hat{K}(\mu_F)$ is open and then $\hat{K}(\mu_F)$ is closed, i.e. $\hat{K}(\mu_F)$ is a compact.

Thus, as W is compactly supported in G_1 then we may take a function $\varphi\in C^\infty(G_2)$ compactly supported in G_1 such that $\varphi\equiv 1$ on a neighbourhood of supp (W). Then, by the Leibniz rule, for each $u\in S_{\mathcal{L}}(G_1)$ we have

$$\mathcal{L}(\varphi u)=\varphi\mathcal{L}u+(\partial_t\varphi)u+\sum_{|\alpha|\le 2m}L_\alpha(x)\sum_{\beta+\gamma=\alpha,|\gamma|\ge 1}C_{\beta,\gamma}(\partial^\gamma\varphi)(\partial^\beta u)=$$

$$(\partial_t\varphi)u+\sum_{|\alpha|\le 2m}L_\alpha(x)\sum_{\beta+\gamma=\alpha,|\gamma|\ge 1}C_{\beta,\gamma}(\partial^\gamma\varphi)(\partial^\beta u),$$

where $C_{\beta,\gamma}$ are binomial type coefficients; in particular,

$$\operatorname{supp}(\mathcal{L}(\varphi u))\cap\operatorname{supp}(W)=\emptyset.$$

Therefore (4) and (6) yield

$$F(u)=\langle u,d\mu_F\rangle=\langle u,\mathcal{L}^*W\rangle=\langle(\varphi u),\mathcal{L}^*W\rangle=\langle\mathcal{L}(\varphi u),W\rangle=0, \quad (9)$$

i.e. F annihilates $S_{\mathcal{L}}(G_1)$, too. □

Now we may proceed with the proof of the second theorem.

Proof of Theorem 2. Let $K(t)\Subset G_2(t)$ stand for compact (possibly, empty) component of $G_2(t)\backslash G_1(t)$ in $G_2(t)$. Taking into the account assumption (A), we see that $K(t)\cup G_1(t)$ is an open subset in $G_2(t)$ and $K(t)\Subset K(t)\cup G_1(t)$ for $t\in(T_1(G_1),T_2(G_1))$. Let there be a number $t_0\in\mathbb{R}$ such that the set $G_2(t_0)\backslash G_1(t_0)$ have a compact non-empty component $K(t_0)$ in the set $G_2(t_0)$. Let us prove that there is a vector $u\in S_{\mathcal{L}}(G_1)$ that can not be approximated by elements of $S_{\mathcal{L}}(G_2)$.

With this purpose, fix the oprentation of $\mathbb{R}^{n+1}$ by choosing orders of coordinate axes $Ot, Ox_1,\dots Ox_n$. According to [37, §2.4.2] the operator L admits a Green bi-differential operator $\mathcal{G}_L$ of order $(2m-1)$ with respect to the space variables x, acting from $\mathbf{C}^{2m,1}_k(G_2)\times\mathbf{C}^{2m,1}_k(G_2)$ to the space of $(n+1)$-differential forms with coefficients from $C^1(G_2)$, i.e.

$$\int_{\partial G_3}\mathcal{G}_L(g,v)=(Lv,g)_{\mathbf{L}^2_k(G_3)}-(v,L^*g)_{\mathbf{L}^2_k(G_3)} \text{ for all } g,v\in\mathbf{C}^{2m,1}_k(\overline{G}_3) \quad (10)$$

and any domain $G_3\Subset G_2$ with piecewise smooth boundary. The exposition in [37, §2.4.2] was done in the category of operators with C^∞-smooth coefficients, but the construction of $\mathcal{G}_L$ is transparent and [37, formula (2.4.12)] provides that only partial derivatives $\partial_t L_\alpha$ or $\partial^\beta_x L_\alpha$ with $|\beta|\le|\alpha|$ of the coefficients L_α, $|\alpha|>0$, may be used calculating (10) (recall that the regularity of $\partial^\beta_x L_\alpha$ is granted by assumption (α4)). Then the Green operator for $\mathcal{L}$ is given as follows:

$$\mathcal{G}_{\mathcal{L}}(g,v)=g^*v\,dx-\mathcal{G}_L(g,v),$$

and, similarly to (10), we have

$$\int_{\partial G_3} \mathcal{G}_{\mathcal{L}}(g,v) = (\mathcal{L}v,g)_{\mathbf{L}^2_k(G_3)} - (v,\mathcal{L}^*g)_{\mathbf{L}^2_k(G_3)} \text{ for all } g,v \in \mathbf{C}^{2m,1}_k(\overline{G}_3). \tag{11}$$

Now, fix a point $y_0 \in K_0$. Then any vector column $U_j(x,t)$, $1 \leq j \leq k$, of the fundamental matrix $\Phi(x,y_0,t,t_0)$ belongs to the space $S_{\mathcal{L}}(G_1)$.

First, we assume that

$$\cup_{t\in(T_1(G_1),T_2(G_1))} K(t) \Subset \cup_{t\in(T_1(G_1),T_2(G_1))} \big(G_1(t) \cup K(t)\big). \tag{12}$$

In this case we may choose a bounded domain G_3 with a piecewise smooth boundary ∂G_3 such that $(y_0,t_0) \in G_3 \Subset G_2$ and $\partial G_3 \Subset G_1$. If the vector function $U_j(x,t)$ can be approximated in $\mathbf{C}^{2m,1}_k(G_1)$ by a sequence $\{u^{(i)}_j\}_{i\in\mathbb{N}}$ from the space $S_{\mathcal{L}}(G_2)$ then the sequences of the partial derivatives $\{\partial^\alpha_x \partial^j_t u^{(i)}_j\}$, $|\alpha| + 2mj \leq 2m$, converge uniformly on ∂G_3. On the other hand, (the first) Green formula (11) and the normality property (2) of the fundamental solution Φ imply (the second) Green formula:

$$u^{(i)}_j(x,t) = -\int_{\partial G_3} \mathcal{G}_{\mathcal{L}}(\Phi(x,t,y,\tau), u^{(i)}_j(y,\tau)) \text{ for all } (x,t) \in G_3. \tag{13}$$

Note that there is no need to assume that G_3 is a cylinder domain because this Green formula is a corollary of the *local* reproducing property of the fundamental solution. Now, passing to the limit with respect to $i \to +\infty$ in (13) we obtain

$$U_j(x,t) = -\int_{\partial G_3} \mathcal{G}_{\mathcal{L}}(\Phi(x,t,y,\tau), U_j(y,\tau)) \text{ for all } (x,t) \in G_3 \cap G_1. \tag{14}$$

However, since Φ is a fundamental solution to $\mathcal{L}$ then the right-hand side of formula (14) belongs to $S_{\mathcal{L}}(G_3)$. Therefore the vector function U_j extends as a solution V_j to equation (3) from $G_1 \cap G_3$ to G_3, i.e. to a neighbourhood of the point (y_0,t_0). In particular, assumption (UCP) for the operator $\mathcal{L}$ with respect to the space variables implies that this extension is unique on $G_3 \setminus (y_0,t_0)$. This means the vector function $V_j \in S_{\mathcal{L}}(G_3)$ coincide with the j-th vector column U_j of the fundamental matrix $\Phi(x,t,y_0,t_0)$ in $G_3 \setminus (y_0,t_0)$). Thus, we obtain a contradiction because for the matrix $V(x,t)$ with columns V_j, $1 \leq j \leq k$ we have $\mathcal{L}V = 0$ in G_3 but $\mathcal{L}\Phi(x,y_0,t,t_0)$ coincides with the δ-functional concentrated at the point (y_0,t_0).

Finally, if (12) is not fulfilled we consider the domain

$$\tilde{G}_1 = \Big(\cup_{t\in(T_1(G_1),T_2(G_1))} \big(G_1(t) \cup K(t)\big)\Big) \setminus K(t_0) \subset G_2.$$

containing the domain G_1. By the discussion above, the space $S_{\mathcal{L}}(G_2)$ is not everywhere dense in $S_{\mathcal{L}}(\tilde{G}_1)$ and hence there is functional $F_0 \in (S_{\mathcal{L}}(\tilde{G}_1))^*$ that annihilates the space $S_{\mathcal{L}}(G_2)$ but does not annihilate $S_{\mathcal{L}}(\tilde{G}_1)$. On the other hand, $S_{\mathcal{L}}(\tilde{G}_1) \subset S_{\mathcal{L}}(G_1)$ and by Hahn-Banach there is an extension

$F_1 \in (S_{\mathcal{L}}(G_1))^*$ of the functional F_0. Then, by the construction, the functional F_1 annihilates the space $S_{\mathcal{L}}(G_2)$ but does not annihilate the space $S_{\mathcal{L}}(\tilde{G}_1) \subset S_{\mathcal{L}}(G_1)$, i.e. $S_{\mathcal{L}}(G_2)$ can not be everywhere dense in $S_{\mathcal{L}}(G_1)$. □

2 The approximation in the mean

In this section we discuss an approximation theorem for solutions to the operator $\mathcal{L}$ belonging to the Lebesgue spaces. Actually, it is quite similar to the approximation theorems for elliptic operators mentioned in the introduction and the approximation theorem with uniform convergence on compact subsets for parabolic systems proved in the previous section. Also, they are known for the heat equation or for the parabolic Lamé system in cylinder domains with rather regular lateral surfaces, see, for instance, [33] or [41].

Investigating spaces of solutions to $2m$-parabolic equation, we need the anisotropic Sobolev spaces $H^{2ms,s}(G)$, $s \in \mathbb{Z}_+$, in a domain $G \subset \mathbb{R}^n \times \mathcal{I}$ with the standard inner product,

$$(u, v)_{H^{2ms,s}(G)} = \sum_{|\alpha|+2mj \le 2ms} (\partial_x^\alpha \partial_t^j u, \partial_x^\alpha \partial_t^j v)_{L^2(G)}.$$

Also, for $\gamma \in \mathbb{Z}_+$, we denote by $H^{\gamma,2sm,s}(G)$ the set of all functions $u \in H^{2sm,s}(G)$ such that $\partial_x^\beta u \in H^{2ms,s}(G)$ for all $|\beta| \le \gamma$. As before, it is convenient to denote by $\mathbf{H}_k^{2ms,s}(G)$ the space of all the k-vector functions with the components from $H^{2ms,s}(G)$, and similarly for the spaces $\mathbf{H}_k^{\gamma,2ms,s}(G)$, etc.

We also will use the so-called Bochner spaces of functions depending on (x, t) from the strip $\mathbb{R}^n \times [T_1, T_2]$ with finite numbers $T_1 < T_2$. Namely, for a Banach space $\mathcal{B}$ (for example, the space of functions on a sub-domain of $\mathbb{R}^n$) and $p \ge 1$, we denote by $L^p(I, \mathcal{B})$ the Banach space of all the measurable mappings $u : [T_1, T_2] \to \mathcal{B}$ with the finite norm

$$\|u\|_{L^p([T_1,T_2],\mathcal{B})} := \| \|u(\cdot, t)\|_{\mathcal{B}} \|_{L^p([T_1,T_2])},$$

see, for instance, [22, ch. §1.2], [40, ch. III, § 1].

The space $C([T_1, T_2], \mathcal{B})$ is introduced with the use of the same scheme; this is the Banach space of all the continuous mappings $u : [T_1, T_2] \to \mathcal{B}$ with the finite norm

$$\|u\|_{C([T_1,T_2],\mathcal{B})} := \sup_{t \in [T_1,T_2]} \|u(\cdot, t)\|_{\mathcal{B}}.$$

Let $\mathbf{H}_{k,\mathcal{L}}^{\gamma,2sm,s}(G) = \mathbf{H}_k^{\gamma,2sm,s}(G) \cap S_{\mathcal{L}}(G)$, $s \in \mathbb{Z}_+$, $\gamma \in \mathbb{Z}_+$. By the discussion in Section §1, the space $\mathbf{H}_{k,\mathcal{L}}^{2m,1}(G)$ is a closed subspace of the Sobolev space $\mathbf{H}_k^{2m,1}(G)$. Similarly, if coefficients of the operator $\mathcal{L}$ are smooth, bounded and real analytic with respect to the variables x for each $t \in \mathcal{I}$, then $\mathbf{H}_{k,\mathcal{L}}^{\gamma,2sm,s}(G)$, $\mathbf{C}_{k,\mathcal{L}}^{2ms,s}(\overline{G}) = \mathbf{C}_k^{2ms,s}(\overline{G}) \cap S_{\mathcal{L}}(G)$, $\mathbf{C}_{k,\mathcal{L}}^{\infty}(\overline{G}) = \mathbf{C}_k^{\infty}(\overline{G}) \cap S_{\mathcal{L}}(G)$ are closed subspaces, consisting of solutions to equation (3), in the spaces $\mathbf{H}_k^{\gamma,2sm,s}(G)$, $\mathbf{C}_k^{2ms,s}(\overline{G})$ and $\mathbf{C}_k^{\infty}(\overline{G})$, respectively.

Also, we need the space $S_{\mathcal{L}}(\overline{G_2})$, defined as follows:

$$\cup_{G'\supset\overline{G}} S_{\mathcal{L}}(G'),$$

where the union is with respect to all the domains $G' \subset \mathbb{R}^n \times (t_1, t_2)$, containing the closure of the domain G. It follows from the a priori estimates referred to in §1 that the following (continuous) embeddings

$$S_{\mathcal{L}}(\overline{G}) \subset \mathbf{C}^{2m,1}_{k,\mathcal{L}}(\overline{G}) \subset \mathbf{H}^{2m,1}_{k,\mathcal{L}}(G) \tag{15}$$

are fulfilled. Of course, if we additionally know that the coefficients of the operator $\mathcal{L}$ are constant or smooth and real analytic with respect to the space variables then the operator is hypoelliptic and the following (continuous) embeddings

$$S_{\mathcal{L}}(\overline{G}) \subset \mathbf{C}^{\infty}_{k,\mathcal{L}}(\overline{G}) \subset \mathbf{H}^{\gamma,2ms,s}_{k,\mathcal{L}}(G) \tag{16}$$

hold true $\gamma, s \in \mathbb{Z}_+$.

To prove an approximation theorem for the spaces of the Lebesgue solutions to $\mathcal{L}$, we need more regularity of ∂G:

(A1) For each $t \in [T_1, T_2]$, the sets $G(t) = \{x \in \mathbb{R}^n : (x,t) \in G\}$, are domains in $\mathbb{R}^n$ with C^{2m}-boundaries if $n \geq 2$ or the union of a finite numbers of intervals if $n = 1$.

(A2) The boundary ∂G of G is the union $G(T_1) \cup G(T_2) \cup \Gamma$, where

$$\Gamma = \cup_{t\in(T_1,T_2)} \partial G(t)$$

is a $C^{2m,1}$-smooth surface without points where the tangential planes are parallel to the coordinate plane $\{t = 0\}$, i.e. we have

$$\sum_{j=1}^{n} (\nu_j(x,t))^2 \geq \varepsilon_0 \text{ for all } (x,t) \in \Gamma$$

with a positive number ε_0.

Under these assumptions we easily see that functions from the space $H^{2m,1}(G)$ and some of their partial derivatives have reasonable traces on ∂G. The definition, the uniqueness and an existence theorem for traces of functions from anisotropic spaces can be found in [2, §10]. In our particular situation we may specify the traces by hands (cf. [28, Ch.3, §7] for anisotropic spaces in cylinder domains). With this purpose, we denote by $Y_{s-1/2}(\Gamma)$ the completion of $C^s(\Gamma)$, $s \in \mathbb{N}$, with respect to the norm

$$\|\cdot\|_{Y_{s-1/2}(\Gamma)} = \left(\int_{T_1}^{T_2} \|\cdot\|^2_{H^{s-1/2}(\partial G(t))} dt\right)^{1/2};$$

by construction, these are Hilbert spaces embedded continuously into $L^2(\Gamma)$.

Of course, if $G = \Omega \times (T_1, T_2)$ is a cylinder domain in $\mathbb{R}^{n+1}$ with the base Ω being a domain with the boundary of class C^s, then $\Gamma = \partial\Omega \times (T_1, T_2)$ and $Y_{s-1/2}(\Gamma)$ coincides with the Bochner space $L^2([T_1, T_2], H^{s-1/2}(\partial\Omega))$.

Lemma 1. *Let G be a relatively compact domain in $\mathbb{R}^n \times \mathcal{I}$ satisfying* (A1), (A2). *Then any $w \in H^{2m,1}(G)$ has well-defined trace in ∂G. Besides, its derivatives $\partial^\alpha v$ have traces on Γ of the class $Y_{2m-|\alpha|-1/2}(\Gamma)$ for each $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le 2m-1$.*

Proof. First, we note that the space $H^{2m,1}(G)$ is continuously embedded into the isotropic Sobolev space $H^1(G)$ and hence the trace $w_{|\partial G}$ is well-defined. Actually, it belongs to $H^{1/2}(\partial G)$, see, for instance, [2].

Next, by the structure of the domain and the Fubini theorem, we have

$$\sum_{|\alpha|\le 2m} \|\partial^\alpha_y w\|^2_{L^2(G)} = \sum_{|\alpha|\le 2m} \int_{T_1}^{T_2} \int_{G(\tau)} |\partial^\alpha_y v(w,\tau)|^2 dy d\tau =$$

$$\int_{T_1}^{T_2} \|w(\cdot,\tau)\|^2_{H^{2m}(G(\tau))} d\tau$$

for any $w \in C^{2m}(\overline{G})$.

Now the standard Trace Theorem for the Sobolev spaces applied for the spaces $H^{2m}(G(t))$ yields

$$\|\partial^\alpha_y h\|_{H^{2m-|\alpha|-1/2}(\partial G(t))} \le C_{G(t),\alpha} \|\partial^\alpha_y h\|_{H^{2m-|\alpha|}(G(t))}$$

for all $|\alpha| \le 2m-1$, for all $h \in H^{2m}(G(t))$ and each $t \in (T_1,T_2)$. Hence

$$\int_{T_1}^{T_2} \sum_{|\alpha|\le 2m-1} \|\partial^\alpha_y w(\cdot,\tau)\|^2_{H^{2m-|\alpha|-1/2}(\partial G(t))} d\tau \le C \int_{T_1}^{T_2} \|w(\cdot,\tau)\|^2_{H^{2m}(G(\tau))} d\tau \tag{17}$$

for any $w \in C^{2m}(\overline{G})$; here the positive constant

$$C = C(G) = \sup_{|\alpha|\le 2m-1} \sup_{t\in[T_1,T_2]} C_{G(t),\alpha}$$

is finite because the related constants $C_{G(t),\alpha}$ may be chosen to depend on the $(n-1)$-measure of the domains $\partial G(t)$.

According to [2, §14], any function $w \in H^{2m,1}(G)$ may be approximated by functions from $C^\infty_{\mathrm{comp}}(\mathbb{R}^{n+1})$ in the topology of the space $H^{2m,1}(G)$. Pick a sequence $w^{(s)} \subset C^\infty_{\mathrm{comp}}(\mathbb{R}^{n+1})$ approximating w in $H^{2m,1}(G)$. Then (17) yields that the sequence $\{\partial^\alpha_y w^{(s)}\}$ is a Cauchy sequence in $Y_{2m-|\alpha|-1/2}(\Gamma)$ for each α with $|\alpha| \le 2m-1$. As $Y_{2m-|\alpha|-1/2}(\Gamma)$ is complete we conclude that for each α with $|\alpha| \le 2m-1$ it converges to an element w_α, i.e. there is a well-defined trace $\partial^\alpha_y w_{|\Gamma} = w_\alpha \in Y_{2m-|\alpha|-1/2}(\Gamma)$ of the function $\partial^\alpha_y w$ on the surface Γ. □

Now we formulate the main result of this section.

Theorem 3. *Let $\mathcal{L}$ satisfy assumptions (α_1), (α_2), (α_3), (α_4) and $G_1 \subset G_2$ be domains in the strip $\mathbb{R}^n \times \mathcal{I}$ such that $G_2 \ne \mathbb{R}^n \times \mathcal{I}$. If* (UCP) *is fulfilled for $\mathcal{L}^*$, domain G_2 satisfies assumption* (A), *bounded domain G_1 satisfies* (A1), (A2), *and the pair G_1, G_2 satisfies assumption* (B), *then $S_{\mathcal{L}}(\overline{G}_2)$ is everywhere dense in the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$.*

Proof. The proof is similar to the proof of Theorem 1. However we need a more regular boundary for the domain G_1.

Clearly, the set $S_{\mathcal{L}}(\overline{G}_2)$ is everywhere dense in $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$ if and only if the following relations

$$(u, w)_{\mathbf{L}^2_k(G_1)} = 0 \text{ for all } w \in S_{\mathcal{L}}(\overline{G_2}) \tag{18}$$

means precisely for the vector $u \in \mathbf{L}^2_{k,\mathcal{L}}(G_1)$ that $u \equiv 0$ in G_1.

As in the proof of Theorem 1, we use the fact that the operator $\mathcal{L}$ admits the bilateral fundamental solution possessing the normality property (2).

Let for the vector $u \in \mathbf{L}^2_{k,\mathcal{L}}(G_1)$ relation (18) be fulfilled. Consider an auxiliary vector function

$$v(y,\tau) = \int_{G_1} \Phi^*(x, y, t, \tau) u(x, t) dx\, dt. \tag{19}$$

Again, the function v is well defined for $(y, \tau) \notin G_1$, but, arguing as in the proof of Theorem 1, we easily see that it extends as a distrubution to the strip $\mathbb{R}^n \times \mathcal{I}$. According to (1) we have $\Phi(x, y, t, \tau) \in S_{\mathcal{L}}(\overline{G}_2)$ with respect to variables (x, t) for each fixed pair $(y, \tau) \notin \overline{G}_2$ and, then relation (18) implies

$$v(y, \tau) = 0 \text{ in } \Big(\mathbb{R}^n \times \mathcal{I}\Big) \setminus \overline{G}_2. \tag{20}$$

On the other hand, (2) yields

$$\mathcal{L}^* v = \chi_{G_1} u \text{ in } \mathbb{R}^n \times \mathcal{I}, \tag{21}$$

where χ_{G_1} is the characteristic function of the domain G_1. Obviously,

$$\mathcal{L}^*_{y,\tau} v(y, \tau) = 0 \text{ in } \Big(\mathbb{R}^n \times \mathcal{I}\Big) \setminus \overline{G}_1,$$

and then, by the discussed above properties of the fundamental solution, the vector function v is $C^{2m,1}_k$-vector function in $\Big(\mathbb{R}^n \times \mathcal{I}\Big) \setminus \overline{G}_1$.

Since both G_1 and G_2 satisfy assumption (A) then, in the same way as in the proof of Theorem 1, assumption (B), formula (8) and the Unique Continuation Property (UCP) for the operator $\mathcal{L}^*$ imply that

$$v(y, \tau) = 0 \text{ in } \Big(\mathbb{R}^n \times \mathcal{I}\Big) \setminus \overline{G}_1. \tag{22}$$

Then, in addition, (21), (22) mean that the vector v is a solution to the Cauchy problem

$$\begin{cases} \mathcal{L}^* v = \chi_{G_1} u \text{ in } \mathbb{R}^n \times (T_1 - \delta, T_2 + \delta), \\ v(y, T_2 + \delta) = 0 \text{ on } \mathbb{R}^n \end{cases}$$

with a sufficiently small $\delta > 0$ such that $[T_1 - \delta, T_2 + \delta] \subset \mathcal{I}$. Taking in account the natural relation between parabolic and backward-parabolic operators and using arguments from [18, ch. 2, §5 theorem 3], we may conclude that $v \in \mathbf{H}^{2m,1}_k(\mathbb{R}^n \times (T_1 - \delta, T_2 + \delta))$ and the solution is unique in this class. The regularity of this unique solution to the Cauchy problem can be expressed in term of the Bochner classes, too. Namely, $v \in C([T_1 - \delta, T_2 + \delta], \mathbf{H}^m_k(\mathbb{R}^n)) \cap$

$L^2([T_1-\delta,T_2+\delta],\mathbf{H}^{2m}_k(\mathbb{R}^n))$, see, for instance, [22], [40, ch. 3, §1], where similar linear problems for parabolic equations were considered. In particular, the vector function v belongs to the space

$$C([T_1-\delta,T_2+\delta],\mathbf{H}^{m}_k(\mathbb{R}^n))\cap\mathbf{H}^{2m,1}_k(\mathbb{R}^n\times(T_1-\delta,T_2+\delta)). \tag{23}$$

Lemma 2. *Any vector v of type (19), satisfying (22), can be approximated by the vectors from $\mathbf{C}^\infty_{k,\mathrm{comp}}(G_1)$ in the topology of the Hilbert space $\mathbf{H}^{2m,1}_k(G_1)$.*

Proof. The approximability of Sobolev functions with compact support in a domain is closely related to the notion of trace. However, (23) implies that traces $v_{|\overline{G(T_j)}}$ of v on the closures of $G(T_j)$ belong to $\mathbf{H}^m_k(G(T_j))$, $1\le j\le 2$, and they are equal to zero because of (22). Moreover, according to Lemma 1, the traces of $\partial^\alpha v_{|\Gamma}\in\mathbf{Y}^{2m-|\alpha|-1/2}_k(\Gamma)$ equal to zero on Γ for all $|\alpha|\le 2m-1$.

Now the statement of the lemma easily follows with the use of the standard regularisation, see, for instance, [28, ch. 3, §5, §7,] for the isotropic Sobolev spaces, cf. see [33] for cylinder domains.

Indeed, denote by $h^{(n)}_\delta(x)$ the standard compactly supported function with the support in the ball $B(x,\delta)\subset\mathbb{R}^n$ with the centre at the point x and of the radius $\delta>0$:

$$h^{(n)}_\delta(x)=\begin{cases}0, & \text{if } \ |x|\ge\delta,\\ c(\delta)\exp\left(1/(|x|^2-\delta^2)\right), & \text{if } \ |x|<\delta,\end{cases}$$

where $c(\delta)$ is the constant providing equality

$$\int_{\mathbb{R}^n}h^{(n)}_\delta(x)\,dx=1.$$

Then, as it is well known, for any function $w\in L^1(K)$ on a measurable compact K in $\mathbb{R}^n$, the standard regularisation

$$(R^{(n)}_\delta w)(x)=\int_{\mathbb{R}^n}h^{(n)}_\delta(x-y,t-\tau)w(y,\tau)\,dyd\tau$$

belongs to the space $C^\infty_0(\mathbb{R}^n)$ for any positive number δ and the support of $R^{(n)}_\delta w$ lies in a δ-neighbourhood of K.

According to assumption (A2) there is a real valued function $\rho(x,t)$ of the class $C^{2m,1}$ in a neighbourhood U of the surface Γ and such that

$$\partial G_1(t)=\{(x,t)\in\mathbb{R}^n\times[T_1,T_2]:\,\rho(x,t)=0\},\ \nabla\rho\neq 0 \text{ in } U\}.$$

Hence, for all sufficiently small numbers $\varepsilon>0$ the sets

$$G^\varepsilon_1=\{(x,t)\in\mathbb{R}^n:\,\rho(x,t)<-\varepsilon\}$$

are domains with boundaries of class $C^{2m,1}$ and

$$G^\varepsilon_1\Subset G^{\varepsilon'}_1\Subset G_1,$$

if $0<\varepsilon'<\varepsilon$, and

$$\lim_{\varepsilon\to+0}\mathrm{mes}(G_1\setminus\overline{G^\varepsilon_1})=0.$$

Moreover the sets $G_1(t)$ are domains with boundaries of class C^{2m} and for the Lebesgue measure of the domain $G_1(t) \setminus \overline{G_1^\varepsilon(t)}$ we have

$$G_1^\varepsilon(t) \Subset G_1^{\varepsilon'}(t) \Subset G_1(t),\ t \in [T_1, T_2],$$
$$\lim_{\varepsilon \to +0} \mathrm{mes}(G_1(t) \setminus \overline{G_1^\varepsilon(t)}) = 0,$$

uniformly with respect to $t \in [T_1, T_2]$. According to [28, ch. 3, §5, lemma 1], there is a positive constant $C_1(G_1)$, depending on the square of the surfaces $G_1(T_j)$, $1 \le j \le 2$, only, and such that

$$C_1(G_1)\, \varepsilon^{-2} \Big(\int_{T_1}^{T_1+\varepsilon} \|\tilde v\|^2_{L^2(G(t))} dt + \int_{T_2-\varepsilon}^{T_2} \|\tilde v\|^2_{L^2(G(t))} dt \Big) \le \tag{24}$$
$$\int_{T_1}^{T_1+\varepsilon} (\|\tilde v\|^2_{L^2(G(t))} + \|\partial_t \tilde v\|^2_{L^2(G(t))}) dt + \int_{T_2-\varepsilon}^{T_2} (\|\tilde v\|^2_{L^2(G(t))} + \|\partial_t \tilde v\|^2_{L^2(G(t))}) dt$$

for any function $\tilde v \in H^1(G_1)$ with zero traces $\tilde v_{|G_1(T_j)}$ on $G_1(T_j)$, $1 \le j \le 2$.

Similarly, under assumptions (A1), $(A2)$, if $\partial G(t) \in C^{2m}$ then there is a constant $C_0(\Gamma)$, depending on the square of the surface Γ, only, and such that

$$\Big(\int_{T_1}^{T_2} \|\partial^\alpha \tilde v\|^2_{L^2(G(t) \setminus \overline{G^\varepsilon}(t))} dt \Big)^{1/2} \le C_0(\Gamma)\, \varepsilon^{2m-|\alpha|} \Big(\int_{T_1}^{T_2} \|\tilde v\|^2_{H^{2m}(G(t) \setminus \overline{G^\varepsilon}(t))} dt \Big)^{1/2} \tag{25}$$

for any function $\tilde v \in H^{2m,1}(G_1)$ with zero traces $\partial^\alpha \tilde v_{|\Gamma}$ on Γ corresponding to $|\alpha| \le 2m-1$.

Set

$$R_\varepsilon(x,t) = \int_{T_1+\varepsilon/5}^{T_2-\varepsilon/5} h^{(1)}_{\varepsilon/7}(t-\tau) \int_{G_1^{\varepsilon/5}(\tau)} h^{(n)}_{\varepsilon/7}(x-y)\, dy\, d\tau.$$

Using assumptions on G_1 we may choose some constants c_α, c_i independent on x and t such that

$$0 \le R_\varepsilon(x,t) \le 1, \tag{26}$$
$$|\partial^\alpha_x R_\varepsilon(x,t)| \le c_\alpha\, \varepsilon^{-|\alpha|},\ |\partial^i_t R_\varepsilon(x,t)| \le c_i\, \varepsilon^{-i}, \tag{27}$$

for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\alpha \in \mathbb{Z}^n_+$, $i \in \mathbb{Z}_+$, cf., for instance, [28, ch. 3, §5].

Fix a sequence $\{v_j\} \subset \mathbf{C}^\infty_{0,k}(\mathbb{R}^{n+1})$, converging to v in the space $\mathbf{H}^{2m,1}_k(\mathbb{R}^{n+1})$ (as we mentioned above, the existence of such a sequence follows from [2, §14]). Then the functional sequence

$$\{v_{j,\varepsilon}(x,t) = R_\varepsilon(x,t) v_j(x,t)\}$$

lies to $\mathbf{C}^\infty_{k,0}(G_1)$.

By the triangle inequality,

$$\|v - v_{j,\varepsilon}\|_{\mathbf{H}^{2m,1}_k(G_1)} \le \|v - v_j\|_{\mathbf{H}^{2m,1}_k(G_1)} + \|v_j - v_{j,\varepsilon}\|_{\mathbf{H}^{2m,1}_k(G_1)}. \tag{28}$$

As

$$\lim_{j \to +\infty} \|v - v_j\|_{\mathbf{H}^{2m,1}_k(G_1)} = 0, \tag{29}$$

then we need to estimate the second summand in the right hand side of formula (28), only. However,

$$v_j(x,t) - v_{j,\varepsilon}(x,t) = (1 - R_\varepsilon(x,t))v_j(x,t)$$

and, in particular,

$$v_j(x,t) - v_{j,\varepsilon}(x,t) = 0 \text{ for all } (x,t) \in \cup_{t=T_1+\varepsilon}^{T_2-\varepsilon} G_1^\varepsilon(t). \quad (30)$$

Hence,

$$2^{-1}\|v_j - v_{j,\varepsilon}\|^2_{\mathbf{H}_k^{2m,1}(G_1)} \leq \|(1-R_\varepsilon)\partial_t v_j\|^2_{\mathbf{L}_k^2(G_1)} + \quad (31)$$
$$\Big\|(\partial_t R_\varepsilon) v_j\Big\|^2_{\mathbf{L}_k^2(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_2]} G_1^\varepsilon(t))} +$$
$$\sum_{|\alpha|\leq 2m} \|\partial^\alpha((1-R_\varepsilon)v_j)\|^2_{\mathbf{L}_k^2(\cup_{t\in[T_1,T_2]} G_1(t)\setminus G_1^\varepsilon(t))}.$$

Then (24), (25), (26), (27), (30) and the Fubini theorem imply that

$$\sum_{|\alpha|\leq 2m} \|\partial^\alpha(1-R_\varepsilon v_j)\|^2_{\mathbf{L}_k^2(G_1)} \leq \quad (32)$$
$$\sum_{|\alpha|\leq 2m} \|(1-R_\varepsilon)(\partial^\alpha v_j\|^2_{\mathbf{L}_k^2(\cup_{t\in[T_1,T_2]} G_1(t)\setminus G_1^\varepsilon(t))} +$$
$$\sum_{|\beta+\gamma|\leq 2m, \beta\neq 0} \|(\partial^\beta R_\varepsilon)(\partial^\gamma v_j\|^2_{\mathbf{L}_k^2(\cup_{t\in[T_1,T_2]} G_1(t)\setminus G_1^\varepsilon(t))} \leq$$
$$\sum_{|\alpha|\leq 2m} \|(1-R_\varepsilon)(\partial^\alpha v_j\|^2_{\mathbf{L}_k^2(\cup_{t\in[T_1,T_2]} G_1(t)\setminus G_1^\varepsilon(t))} +$$
$$\sum_{|\beta+\gamma|\leq 2m, \beta\neq 0} \varepsilon^{2m-|\gamma+\beta|}\|v_j\|^2_{\mathbf{H}_k^{2m,0}(\cup_{t\in[T_1,T_2]} G_1(t)\setminus G_1^\varepsilon(t))} \leq$$
$$\|v\|_{\mathbf{H}_k^{2m,1}(\cup_{t\in[T_1,T_2]} G_1(t)\setminus \overline{G_1^\varepsilon(t)})}$$

with constants $C, \tilde{C}$, independent on v and ε.

The boundaries of the domains $\cup_{t\in[T_1,T_1+\varepsilon]} G_1(t)$ and $\cup_{t\in[T_2-\varepsilon,T_2]} G_1(t)$ are not smooth, but combining results [28, ch. 3, §5] related to a function v, having the trace vanishing on surfaces $G_1(T_1)$ and $G_2(T_2)$, with bounds (26), (27), we see that

$$\|(\partial_t R_\varepsilon)v\|^2_{\mathbf{L}_k^2(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_2]} G_1^\varepsilon(t))} \leq \quad (33)$$
$$\varepsilon^{-1}\cdot\varepsilon C \left(\|_t v\|^2_{\mathbf{L}_k^2(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_2]} G_1^\varepsilon(t))} + \|\partial_t v\|^2_{\mathbf{L}_k^2(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_2]} G_1^\varepsilon(t))}\right) \leq$$
$$\|v\|_{\mathbf{H}_k^{2m,1}(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_1]} G_1(t)\setminus \overline{G_1^\varepsilon(t)})}$$

with a constant C, independent on v and ε.

Besides, according to (26), (27),

$$\|(1-R_\varepsilon)\partial_t^i v\|^2_{\mathbf{L}_k^2(G_1)} \leq C\|v\|_{\mathbf{H}_k^{2m,1}(\cup_{t\in[T_1,T_2]} G_1(t)\setminus \overline{G_1^\varepsilon(t)})} \quad (34)$$

with a constant C, independent on v and ε.

Using the continuity of the Lebesgue integral with respect to the measure of the integration set, we conclude that

$$\lim_{\varepsilon\to+0} \|v\|_{\mathbf{H}^{2m,1}_k(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_1]} G_1(t))} = 0, \tag{35}$$

$$\lim_{\varepsilon\to+0} \|v\|_{\mathbf{H}^{2m,1}_k(\cup_{t\in[T_1,T_2]} G_1(t)\setminus\overline{G_1^\varepsilon(t)})} = 0. \tag{36}$$

Finally, combining estimates (31)–(34) and taking in account (35), (36), we conclude that the statement of the lemma holds true. □

Next, using lemma 2 and fixing a sequence $\{v_k\} \subset \mathbf{C}^\infty_{k,\mathrm{comp}}(G_1)$ converging to the vector v in $\mathbf{H}^{2m,1}_k(G_1)$, we see that

$$\|u\|^2_{\mathbf{L}^2_k(G_1)} = (u, \mathcal{L}^* v)_{\mathbf{L}^2_k(G_1)} = \lim_{i\to+\infty} (u, \mathcal{L}^* v_i)_{\mathbf{L}^2_k(G_1)} = 0,$$

because $\mathcal{L}u = 0$ in G_1 in the sense of distributions. Thus, $u \equiv 0$ in G_1, i.e. the sufficiency is proved. □

Predictably, statement similar to Theorem 2 also holds true.

Theorem 4. *Let $\mathcal{L}$ satisfy assumptions (α_1), (α_2), (α_3), (α_4) and $G_1 \subset G_2$ be domains in the strip $\mathbb{R}^n \times \mathcal{I}$ satisfying assumption* (A). *If* (UCP) *is fulfilled for $\mathcal{L}$ and for the coefficients L_α, $0 < |\alpha| \leq 2m$, the partial derivatives $\partial_t L_\alpha$ are continuous on $\mathbb{R}^n \times \mathcal{I}$ then assumption* (B) *is neccessary for $S_{\mathcal{L}}(\overline{G}_2)$ to be everywhere dense in the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$.*

Proof of Threorem 4. By the interior a priori estimates for parabolic systems, the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$ is imbedded continouosly to $S_{\mathcal{L}}(G_1)$. Then we actually we may use the same arguments as in the proof of Theorem 2. □

Next, we obtain the following useful statement.

Corollary 1. *Let the coefficients of the operator $\mathcal{L}$ be smooth, bounded and real analytic with respect to the variables x for each $t \in \mathcal{I}$. Let also $s, \gamma \in \mathbb{Z}_+$, $G_1 \subset G_2$ be domains in $\mathbb{R}^n \times \mathcal{I}$ such that $G_2 \neq \mathbb{R}^n \times \mathcal{I}$, domain G_2 satisfy* (A)*, and bounded domain G_1 satisfy* (A1), (A2). *If* (B) *holds true then the space $\mathbf{C}^\infty_{k,\mathcal{L}}(\overline{G}_2)$ is everywhere dense in $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$; in particular, $\mathbf{H}^{\gamma,2s,s}_{k,\mathcal{L}}(G_2)$ is everywhere dense in $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$ if G_2 is bounded, too.*

Proof. Follows immediately from Theorem 3, because of embeddings (16). □

Finally, Theorem 3 allows to prove the existence of a basis with the double orthogonality property in the spaces of solutions to the operator $\mathcal{L}$ that is very useful to investigate the non-standard ill-posed Cauchy problem for elliptic and parabolic equations, see, [38, Ch. 12], [34], [41].

Corollary 2. *Let $\mathcal{L}$ satisfy assumptions (α_1), (α_2), (α_3), (α_4), and let* (UCP) *hold for both $\mathcal{L}$ and $\mathcal{L}^*$. Let also $G_1 \subset G_2$ be bounded domains in $\mathbb{R}^n \times \mathcal{I}$, such that $T_1(G_1) = T_1(G_2)$, $T_2(G_1) = T_2(G_2)$, G_2 satisfy* (A) *and G_1 satisfy* (A1), (A2). *If* (B) *holds true then there is an orthonormal basis*

$\{b_\nu\}$ is the space $\mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2)$ such that its restriction $\{b_{\nu|G_1}\}$ to G_1 is an orthonormal basis in the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$.

Proof. By the definition, the space $\mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2)$ is embedded continuously into the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$. We denote by R the natural embedding operator

$$R : \mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2) \to \mathbf{L}^2_{k,\mathcal{L}}(G_1).$$

As the numbers T_1 and T_2 are the same for the domains G_1 and G_2, the Unique Continuation Property (UCP) for the operator $\mathcal{L}$ with respect to the space variables implies that the operator R is injective. Besides, it follows from Theorem 3 that the range of the operator R is everywhere dense in the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$.

By the definition, the anisotropic Sobolev space $\mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2)$ is embedded continuously to the isotropic Sobolev space $\mathbf{H}^1_{k,\mathcal{L}}(G_2)$. Besides, by Rellich-Kondrashov theorem the embedding $\mathbf{H}^1_k(G_1) \to \mathbf{L}^2_k(G_1)$ is compact. Thus, taking in account the continuous embedding $\mathbf{H}^1_k(G_2) \to \mathbf{H}^1_k(G_1)$, we see that the natural embedding operator R is compact.

Finally, [34, example 1.9] implies that the complete system of eigen-vectors of the compact self-adjoint operator $R^*R : \mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2) \to \mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2)$ is the basis looked for; here R^* is the adjoint operator for R in the sense of the Hilbert space theory. □

Corollary 3. *Let the coefficients of the operator $\mathcal{L}$ be smooth, bounded and real analytic with respect to the variables x for each $t \in \mathcal{I}$. Let also $s, \gamma \in \mathbb{Z}_+$, $G_1 \subset G_2$ be bounded domains $\mathbb{R}^n \times \mathcal{I}$, such that $T_1(G_1) = T_1(G_2)$, $T_2(G_1) = T_2(G_2)$, domain G_2 satisfy* (A)*, and domain G_1 satisfy* (A1)*,* (A2)*. If* (B) *holds true then there is an orthonormal basis $\{b_\nu\}$ is the space $\mathbf{H}^{\gamma,2ms,s}_{k,\mathcal{L}}(G_2)$ such that its restriction $\{b_{\nu|G_1}\}$ to G_1 is an orthonormal basis in the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$.*

References

[1] M. Beheshti, S. Umapathy, S. Krishnan, *Electrophysiological Cardiac Modeling: A Review*. Crit Rev Biomed Eng. **44**:1–2 (2016), 99–122.

[2] O.V. Besov, V.P. Il'in, S.M. Nikol'skii, *Integral representation of functions and imbedding theorems*, V.1. Washington, DC, V.H. Winston & Sons, 1978.

[3] F.E.Browder, *Approximation by solutions of partial differential equations*, Amer. J. Math. **84** (1962), 134–160.

[4] A. Debrouwere, T. Kalmes, *Quantitative Runge type approximation theorems for zero solutions of certain partial differential operators*, arxiv.org/abs/2209.10794.

[5] R. Diaz, *A Runge theorem for solutions of the heat equation*, Proc. Amer. Math. Soc. **80**:4 (1980), 643–646.

[6] R.E. Edwards, *Functional anlysis*, Holt, Rinehart and Winston, London, 1965.

[7] S.D. Eidel'man, *Parabolic equations*, Partial differential equations – 6, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., **63** (1990), VINITI, Moscow, 201–313.

[8] A. Enciso, D. Peralta-Salas, *Existence of knotted vortex tubes in steady Euler flows*, Acta Math. **214** (2015), 61–134.

[9] A. Enciso, M.A. García-Ferrero, D. Peralta-Salas, *Approximation theorems for parabolic equations and movement of local hot spots*, Duke Math. J. **168**:5 (2019), 897–939.

[10] A. Enciso A., Peralta-Salas D., *Approximation theorems for the Schr?dinger equation and Quantum Vortex Reconnection*, Comm. Math. Physics, **387** (2021), 1111–1149.

[11] A. Friedman, *Partial differential equations of parabolic type*, Englewood Cliffs, NJ, Prentice-Hall, Inc., 1964.

[12] M.A. García-Ferrero, A. Rüland, W. Zaton, *Runge Approximation and Stability Improvement for a Partial Data Calderon Problem for the Acoustic Helmholtz Equation*, Inverse Problems & Imaging **16** (2022), 251–281.

[13] P.M. Gauthier, N. Tarkhanov, *Rational approximation and universality for a quasi-linear parabolic equation*, Journal of Contemporary Mathematical Analysis, **43** (2008), 353–364.

[14] V.P. Havin, *Approximation by analytic functions in the mean*, Dokl. Akad. Nauk SSSR, **178**:5 (1968), 1025–1028.

[15] B.F. Jones, Jr., *An approximation theorem of Runge type for the heat equation*, Proc. Amer. Math. Soc. **52** (1975), no. 1, 289–292.

[16] V. Kalinin, A.A. Shlapunov, K. Ushenin, *On uniqueness theorems for the inverse problem of Electrocardiography in the Sobolev spaces*, Z. Angew. Math. Mech. **103**:1 (2023), e202100217.

[17] T. Kalmes, *An approximation theorem of Runge type for kernels of certain non-elliptic partial differential operators*, Bull. Sci. Math. **170** (2021), 103012.

[18] N.V. Krylov, *Lectures on elliptic and parabolic equations in Sobolev spaces.* Graduate Studies in Math. V. 96, AMS, Providence, Rhode Island, 2008.

[19] I.A. Kurilenko, A.A. Shlapunov, *On Carleman-type Formulas for Solutions to the Heat Equation*, Journal of Siberian Federal University, Math. and Phys., **12**:4 (2019), 421–433.

[20] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N.Ural'tseva, *Linear and quasilinear equations of parabolic type*, M., Nauka, 1967.

[21] P.D. Lax, *A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations*, Comm. Pure Appl. Math. 9 (1956), 747–766.

[22] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéare*, Dunod/Gauthier-Villars, Paris, 1969.

[23] B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Annales de l'Institut Fourier, **6** (1955/56), 271–355.

[24] B. Malgrange, *Sur les systémes differentiels á coefficients constants*, Colloq. internat. Centre nat. rech. scient., **117** (1963), 113–122.

[25] V.G. Maz'ya, V.P. Havin, *The solutions of the Cauchy problem for the Laplace equation (uniqueness, normality, approximation)*, Tr. Mosk. Mat. Obs., **30** (1974), 61–114.

[26] K.O. Makhmudov, O.I. Makhmudov, N.N. Tarkhanov, *A Nonstandard Cauchy Problem for the Heat Equation*, Math. Notes, **102**:2 (2017), 250–260.

[27] S.N. Mergelyan, *Harmonic approximation and approximate solution of the Cauchy problem for the Laplace equation*, Uspekhi Mat. Nauk, **11**:5 (71) (1956), 3–26.

[28] V.P. Mikhailov, *Partial differential equations*, M., Nauka, 1976.

[29] V.P. Palamodov, *Linear Differential Operators with Constant*, Coefficients, Grundlehren der Mathematischen Wissenschaften 168. Springer, Berlin, 1970.

[30] R.E. Puzyrev, A.A. Shlapunov, *On a mixed problem for the parabolic Lamé type operator*. J. Inv. Ill-posed Problems, **23**:6 (2015), 555–570.

[31] A. Rüland, M. Salo, *Quantitative Runge Approximation and Inverse Problems*, International Mathematics Research Notices **20** (2019), 6216–6234.

[32] C. Runge, *Zur Theorie der eindeutigen analytischen Funktionen*, Acta Math. **6** (1885), 229–244.

[33] A.A. Shlapunov, *On approximation of solutions to the heat equation from Lebesgue class L^2 by more regular solutions*, Math. Notes, **111**:5 (2022), 778–794.

[34] A.A. Shlapunov, N. Tarkhanov, *Bases with double orthogonality in the Cauchy problem for systems with injective symbols.* Proc. London. Math. Soc., **71**:1 (1995), 1–54.

[35] H. Schaefer, *Topological vector spaces*, Springer-Verlag, Berlin, 1971.

[36] V.A. Solonnikov, *On boundary value problems for linear parabolic systems of differential equations of general form*, Proc. Steklov Inst. Math., **83** (1965), 1–184.

[37] N. Tarkhanov, *Complexes of differential operators*, Kluwer Academic Publishers, Dordrecht, NL, 1995.

[38] N. Tarkhanov, *The Cauchy Problem for Solutions of Elliptic Equations*, Akademie-Verlag, Berlin, 1995.

[39] N. Tarkhanov, *The Analysis of Solutions of Elliptic Equations*, Kluwer Academic Publishers, Dordrecht, NL, 1997.

[40] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, North Holland Publ. Comp., Amsterdam, 1979.

[41] P.Yu. Vilkov, I.A. Kurilenko, A.A. Shlapunov, *Approximation of solutions to parabolic Lamé type operators in cylinder domains and Carleman's formulas for them*, Siberian Math. J., **63**:6, 2022, pp. 1049–1059.

[42] A.G. Vitushkin, *The analytic capacity of sets in problems of approximation theory*, Uspekhi Mat. Nauk, **22**:6 (138) (1967), 141–199.

[43] K. Weierstraß, *Über die analytische Darstellbarkeit sogenannter willkurlicher Functionen einer reellen Veranderlichen Sitzungsberichteder*, Sitzungsberichte der Akademie zu Berlin, 1885, pp. 63–639 and 789–805.

Alexander Anatolievich Shlapunov
Siberian Federal University,
pr. Svobodnyi, 79,
660041, Krasnoyarsk, Russia
Sirius Mathematics Center, Sirius University of Science and Technology,
Olimpiyskiy ave. b.1,
354349 Sochi, Russia
E-mail address: ashlapunov@sfu-kras.ru

Pavel Yur'evich Vilkov
Siberian Federal University,
pr. Svobodnyi, 79,
660041, Krasnoyarsk, Russia
E-mail address: pavel_vilkov17@mail.ru