

Report on the article

On Runge type theorems for solutions to strongly uniformly parabolic operators

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In the paper under review the authors study Runge approximation results for strongly uniformly parabolic operators. More precisely, the approximability of continuous solutions u to a square $(k \times k)$ homogeneous strongly uniformly parabolic system of partial differential equations of order $2m$,

$$(1) \quad \mathcal{L}u = 0 \text{ in a domain } G_1 \subset \mathbb{R}^{n+1},$$

by (restrictions to G_1 of) continuous solutions to the same equation in a larger domain $G_2 \supset G_1$ in space-time, is studied, where $\mathcal{L} = \partial_t + \sum_{|\alpha| \leq 2m} L_\alpha(x, t) \partial_x^\alpha$ with $L_\alpha(x, t)$ being $(k \times k)$ -matrices of suitable regular functions defined for $(x, t) \in \mathbb{R}^n \times \mathcal{I}$ with an open interval $\mathcal{I} \subset \mathbb{R}$, and where solutions are understood in the distributional sense. Additionally, approximability refers to uniform convergence on compact subsets of G_1 on the one hand, and to L^2 -convergence on the other hand, where for the latter one has to consider continuous L^2 -solutions u to (1) in the first place, of course. Such pairs (G_1, G_2) of open domains are then called \mathcal{L} -Runge pairs.

An analogous problem was first studied by Runge for solutions to the Cauchy-Riemann equation on \mathbb{R}^2 , i.e. for holomorphic functions, generalized independently by Lax [La56] and Malgrange [Ma55] to arbitrary elliptic operators in the 1950s, and reached its peak with the work of Browder [Br62] in the 1960s. Since then, Runge approximation theorems for PDEs are a classical topic in analysis. Recently, this subject resurfaced due to its many applications like in the deep work of Enciso, Peralta-Salas on 3D Euler and Navier-Stokes equations [EP15] and in the context of inverse problems, where quantitative Runge approximation results turned out to be extremely useful which were first proved in the seminal work of Rüländ, Salo [RS19] (see [GRS22] and the references therein).

While all the above mentioned results deal with elliptic differential operators, much less is known for non-elliptic operators. As mentioned by the authors, for the heat operator, the corresponding question has been studied by Jones [10] and Diaz [4]¹. However, contrary to the claim of the authors on p. 2, l. 12, these results have not been overlooked. Indeed, in recent years, Runge approximation results have been established for variable coefficients parabolic operators of second degree by Enciso, García-Ferrero, Peralta-Salas [EGP19] and for constant coefficient partial differential operators with a single characteristic direction (in particular, parabolic operators of arbitrary order) by Kalmes [Ka21]. Additionally, these results have been complemented by Enciso, Peralta-Salas [EP21] and Debrouwere, Kalmes [DK22].

As mentioned above, the manuscript considers strongly uniformly parabolic square systems. The main results characterize, in particular under suitable regularity assumptions on the boundary ∂G_1 of the smaller domain G_1 , \mathcal{L} -Runge pairs (Theorem 1.1 and Theorem 2.2) in geometric terms. As is the case for many approximation results, the core argument of the proofs relies on a standard application of the Hahn-Banach Theorem combined with an explicit representation of

¹Numerical references refer to the list of references from the manuscript while alphabetical ones refer to the list below.

the topological dual space of the space in which the approximation result should hold true. The results do perhaps not go extremely far beyond the state-of-the-art and are not surprising for experts working in this field, but they certainly provide more clarity and a greater level of generality than previous known results.

However, unfortunately, the paper is not very well written. Apart from a rather draft-like character with respect to the style of the presentation in places, there are some issues with the proofs of the main results, see the list of remarks below. As a consequence, a final decision on a positive recommendation for publication in *Siberian electronic mathematical reports* will heavily depend on a thorough and careful revision of the manuscript.

Remarks and suggestions:

- The additional references give below should be included in the list of references. Additionally, the authors should spend some lines on explaining the connection of their results to the closely related ones contained in [EGP19], [Ka21], and [DK22]. As a consequence, a reformulation of the third paragraph on p. 2 should be made.
- p. 2, l. -9: $\mathcal{L} = \partial_t + L$ should be replaced with $\partial_t - L$ in order to be consistent with [6,29] and the general notation for parabolic operators/systems.
- p. 2, l. -8: \mathcal{I} should be an open interval; at least, this is important in the proof of Theorem 2.2 where δ comes into play, i.e. on p. 11, l. 11.
- p. 4, l. 20: In view of [6, Proposition 2.2], for the reader's convenience, it should be mentioned that Φ^* denotes the Hermitian adjoint of Φ (i.e. $\Phi^* = (\overline{\Phi}^{ji})$ if $\Phi = (\Phi^{ij})$ where \bar{z} denotes complex conjugation) and that Φ^* is then a fundamental matrix (normality property, as stated) of the formal adjoint operator $\mathcal{L}_{y,\tau}^* = -\partial_\tau - \sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} \partial_y^\alpha (L_\alpha^*(y, \tau))$. However, there seems to be subtlety to be taken care of related to the proof of Theorem 1.1; see below.
- p. 5, l. 17: Condition (UCP) should be formulated for $S_{\mathcal{L}}(G)$ as well as for $S_{\mathcal{L}^*}(G)$. In fact, the latter is needed in the proof of sufficiency in Theorem 1.1; see below.
- p. 5, Theorem 1.1: In the reviewer's opinion, there are some hypotheses missing on G_2 required in the proof. It seems that $G_2 \neq \mathbb{R}^{n+1}$ is needed, for otherwise there is no $(y, \tau) \notin G_2$ in (1.5) on p. 6. Additionally, in the reviewer's opinion, one needs that every bounded component of $\mathbb{R}^n \setminus G_2(t)$ which is contained in a bounded component of $\mathbb{R}^n \setminus G_1(t)$ has a non-empty interior. This is needed on p. 6, l. 19 because, contrary to what is stated there, it is not true that $\mathbb{R}^n \setminus G_1(t)$ intersects $\mathbb{R}^n \setminus G_2(t)$ in a non-empty open set. Take for example $G_1 = (\mathbb{R}^n \setminus \{x \in \mathbb{R}^n; |x| \leq 1\}) \times \mathbb{R}$ and $G_2 = (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$. Then, $\mathbb{R}^n \setminus G_1(t) = \{x \in \mathbb{R}^n; |x| \leq 1\}$ is not contained in $G_2(t) = \mathbb{R}^n \setminus \{0\}$ but it is not true that the intersection of $\mathbb{R}^n \setminus G_1(t) = \{x \in \mathbb{R}^n; |x| \leq 1\}$ with $\mathbb{R}^n \setminus G_2(t) = \{0\}$ contains an open set.
- p. 5, l. -9: The proof of sufficiency is very close to the original proof of Malgrange and its generalization by Browder (see also Hörmander: *The Analysis of Linear Partial Differential Operators I*, Springer, Theorem 4.4.5) which should be mentioned.
- p. 6, l. 6: The argument starting here has nothing to do with the backwards parabolicity of \mathcal{L}^* . In order to avoid confusion of the reader, the sentence "Of course, the operator $\mathcal{L}^* \dots$ parabolic equation $(\partial_\tau - L_y^*)w = 0$." should be removed.

- p. 6, l. 10: For the reader's convenience, the authors should write something like the following when defining W . "Let W be the vector function with components obtained by applying the functional F to the corresponding columns of the matrix $(x, t) \mapsto \Phi(x, y, t, \tau)$ which by (1.1) belong to $S_{\mathcal{L}}(G_2)$ for $(y, \tau) \notin G_2$. For obvious reasons, we write $W(y, \tau) = \langle \Phi^*(x, y, t, \tau), d\mu_F(x, t) \rangle$." However, the latter is only true if the entries in Φ are real valued! (Otherwise, one has to replace Φ^* by Φ^t , the transposed matrix, i.e. $\Phi^t = (\Phi^{ji})$ if $\Phi = (\Phi^{ij})$.) Obviously, this is the case if the entries of the $L_{\alpha}(x, t)$'s are all real valued.

Additionally, the authors should give a short argument why W can be extended as a distribution to $\mathbb{R}^n \times \mathcal{I}$; obviously, one has $W \in C_k((\mathbb{R}^n \times \mathcal{I}) \setminus K(\mu_f))$.

- p. 6, l. 19: It has already been mentioned that some additional hypotheses on G_2 are required here. Once this has been done, it will follow from (1.6) and (UCP) (for \mathcal{L}^* !) that W vanishes identically in every bounded connected component of $\mathbb{R}^n \setminus G_1(t)$ for every t . However, some additional arguments are needed to show that W also vanishes identically in the unbounded connected component of $\mathbb{R}^n \setminus G_1(t)$. Once this is done, it will follow $\text{supp } W \subseteq \mathbb{R}^{n+1} \setminus \overline{G_1}$.

In the reviewer's opinion, the proof can then be considerably simplified by the following argument. For $(x, t) \in \partial G_1$ with $t \in (T_1, T_2)$ it holds $x \in \partial G_1(t)$. By hypothesis (A) and $\text{dist}(\partial G_1, K(\mu_f)) > 0$ it follows immediately that W vanishes in an open subset of $\mathbb{R}^n \times \mathcal{I}$ of the connected component of $(\mathbb{R}^n \times \mathcal{I}) \setminus K(\mu_f)$ which contains (x, t) . Therefore, by (UCP) for \mathcal{L}^* and (1.7) it follows that W vanishes in a neighborhood of (x, t) . Moreover, for $(x, t) \in \partial G_1$ with $t \in \{T_1, T_2\}$ it is even simpler to conclude that W vanishes in a neighborhood of (x, t) . Hence, $\text{supp } W$ is contained in G_1 (and not only in $\overline{G_1}$). Now, by the arguments from [10], to which the authors are already alluding to, should yield that $\text{supp } W$ is a compact subset of G_1 - for the reader's convenience, the authors should sketch these arguments explicitly. Once this is accomplished, the last paragraph on p. 6 will complete the proof of sufficiency.

- p. 7, l. 27: To the reviewer it is completely unclear how a priori estimates for parabolic systems can be used to establish the convergence of the sequence $\{u^{(i)}\}_{i \in \mathbb{N}}$ in a neighborhood of (y_0, t_0) . This should be elaborated by the authors in detail.
- p. 10, proof of Theorem 2.2: Essentially everything mentioned above concerning the proof of Theorem 1.1 applies to the proof of Theorem 2.2 as well - except the simplification of the argument concerning the support of W . Additionally, in equation (2.5) one indeed has to insert Φ^* - and not Φ^t .
- p. 11, l. -20: What is the purpose of the two paragraphs starting with "Of course, ..."? In the reviewer's opinion, these two paragraphs should be removed.
- p. 15 Corollary 2.4: The assertion that $\mathbf{H}_{k, \mathcal{L}}^{\gamma, 2s, s}(G_2)$ is everywhere dense in $\mathbf{L}_{k, \mathcal{L}}^2(G_1)$ requires G_2 to be bounded. Otherwise, $S_{\mathcal{L}}(\overline{G_2})$ is not a subspace of $\mathbf{H}_{k, \mathcal{L}}^{\gamma, 2s, s}(G_2)$.

A non-exhaustive list of typos:

- p. 1, abstract l. 6: replace "...standard Fréchet topology of the uniform ..." by "...standard Fréchet topology of uniform ..."
- p. 1, l. 5 of the introduction: remove "the" before holomorphic polynomials
- p. 1, l. -7: replace "for" by "to"
- p. 2, l. -7: replace "admitting the bilateral" by "admitting a bilateral"

- p. 3, l. -16: insert "of" after "dual space"
- p. 3, l. -11: insert "open" before "interval"; see above
- p. 3, l. -10: replace " a" by "the" before strip
- p. 3, l. -5: insert "the" after "Consider"
- p. 4, conditions (α_1) , (α_2) , and (α_2) : replace "over" by "on"
- p. 4, condition (α_3) : replace "the Hölder" by "a Hölder"
- p. 4, condition (α_4) : there seems to be something off with $\partial_x^\alpha L_\alpha(x, t)$ - same α twice - please check.
- p. 4, formula (1.1): The right hand side equals I_k times the Dirac distribution at the point (x, t) which is commonly written as $\delta(x - y, t - \tau)$ where δ denotes the Dirac distribution at the origin.
- p. 4, l. -14: Why are the sets $G(t)$ considered to be bounded? I guess this is a typo.
- p. 4, l. -13: replace "satisfy" by "satisfies"
- p. 5, l. 1: insert "the" after "satisfying"
- p. 5, l. 7: replace "the operator \mathcal{L} " by a reference to equation (1.1)
- p. 5, l. 20: "Of course, if \mathcal{L} is an operator . . ."
- p. 5, l. 22: insert "the" before "coefficients"
- p. 5, l. -17: remove "the" before " \mathcal{L} -Runge's pair"
- p. 5, l. -3: Replace "Khan" by "Hahn"
- p. 6, l. 3: remove "the" before "compact"
- p. 6, l. 7: insert "the" before "parabolic"
- p. 7, l.1: remove "than"
- p. 8, l. 7: It is "Bochner"
- p. 8, l. -9: replace "discussed" by "referred to"
- p. 8, l. -7: replace " $\mathbf{H}_{k,\mathcal{L}}^{\gamma,2ms,s}(G_2)$ " by " $\mathbf{H}_{k,\mathcal{L}}^{\gamma,2ms,s}(G)$ "
- p. 8, l. -2: ". . . the union of a finite number of . . ."
- p. 9, l. 1: insert G between "of" and "is"
- p. 9, l. 3: " $C^{2m,1}$ -smooth surface without points where the tangent space is parallel . . ."
- p. 9, l. 13: replace "closure" by "completion"
- p. 9, l. 15: remove "the" before "construction"
- p. 9, l. 18: there seems to be something off with the phrase " for each α with $|\alpha| \leq s - 1$ ", please check
- p. 9, l. 21: remove "vector"
- p. 10,l. 6: remove "in" before "by vectors"
- p. 10, l. 8: replace "fundamental" by "Cauchy sequence"
- p. 10, l. -14: remove "of the statement"
- p. 10, l. -12: replace "is" by "be"
- p. 10, starting from l. -10: replace " $\Phi^{(*)}(x - y, t - \tau)$ " by " $\Phi^{(*)}(x, y, t, \tau)$ " throughout the rest of the paper
- p. 11, l. 2: replace " $C_{loc}^{2m,1}$ " by " $C_k^{2m,1}$ "
- p. 11, l. 17: there is something missing between "linear problems for" and "are considered" - I guess...
- p. 12, l. 6: at the end of the line there is a closing bracket which has no opening counterpart
- p. 12, l. 8: "Next according to assumption (A2) . . ."
- p. 12, l. 10: The boundary is the zero set of such a function $\rho!$
- p. 15, l. 6: replace the reference to (2.1) by a reference to (2.2)
- p. 15, l. 9: insert "problem" after "Cauchy"
- p. 15, l. -14: we have " $R^*R : \mathbf{H}_{k,\mathcal{L}}^{2m,1}(G_2) \rightarrow \mathbf{H}_{k,\mathcal{L}}^{2m,1}(G_2)$ (G_2 twice!)"

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