ON RUNGE TYPE THEOREMS FOR SOLUTIONS TO STRONGLY UNIFORMLY PARABOLIC OPERATORS

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ABSTRACT. Let G_1, G_2 be domains in $\mathbb{R}^{n+1}$, $n \geq 2$, such that $G_1 \subset G_2$ and the domain G_1 have rather regular boundary. We investigate the problem of approximation of solutions to strongly uniformly $2m$-parabolic system $\mathcal{L}$ in the domain G_1 by solutions to the same system in the domain G_2. First, we prove that the space $S_{\mathcal{L}}(G_2)$ of solutions to the system $\mathcal{L}$ in the domain G_2 is dense in the space $S_{\mathcal{L}}(G_1)$, endowed with the standard Fréchet topology of the uniform convergence on compact subsets in G_1, if and only if the complements $G_2(t) \setminus G_1(t)$ have no non-empty compact components in $G_2(t)$ for each $t \in \mathbb{R}$, where $G_j(t) = \{x \in \mathbb{R}^n : (x,t) \in G_j\}$. Next, under additional assumptions on the regularity of the bounded domains G_1 and $G_1(t)$, we prove that solutions from the Lebesgue class $L^2(G_1) \cap S_{\mathcal{L}}(G_1)$ can be approximated by solutions from $S_{\mathcal{L}}(G_2)$ if and only if the same assumption on the complements $G_2(t) \setminus G_1(t)$, $t \in \mathbb{R}$, is fulfilled.

INTRODUCTION

Apparently, the Approximation Theory in Analysis begins with the famous Weierstraß theorem for continuous functions on segments of the real line, see [36], where polynomials were used as the approximating set. It appears that the matter has become significantly more complicated for complex functions of complex variable if one wants to approximate them by the *holomorphic* polynomials, see [25]. Actually, C. Runge [25] proposed to use the Cauchy kernel for the so-called rational approximation of holomorphic functions on open connected sets (plane domains) in $\mathbb{C}$ in the topology related to the uniform convergence on compact subsets of the domain. The notion of Runge's pair $\omega \subset \Omega$ of plane domains, for which the space $\mathcal{O}(\Omega)$ of holomorphic functions in Ω is dense in the space $\mathcal{O}(\omega)$ (endowed with the topology discussed above) gives rise to the investigation of topological/geometrical conditions on the pair. The analysis of continuous functions approximable on compact sets lead to the notion of the analytic capacity and the creation of the related theory, see, for instance, the paper by A.G. Vitushkin [35]. The matters were successfully extended to the theory of harmonic functions, see [21], and even for solutions to general elliptic operators with the uniqueness condition in small, see for instance, [31, ch. 4, 5], where the bilateral fundamental solutions to the related operators were used instead of the Cauchy kernel. Of course, one should mention the theorem for (non-necessarily elliptic) operators with constant coefficients related to the uniform approximation by exponential-polynomial solutions in convex domains, see [17] (the elliptic case), [18], [23, Ch. VI].

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As for the parabolic equations, the problem of the uniform approximation in the context of Runge's pairs was successfully solved for the heat equation in the papers [10], [4] (see also [8] for "the rational approximation" by the functions constructed with the use of the heat kernel).

On the other hand, approximation theorems in various function spaces, where behaviour of the elements are controlled up to the boundary of the considered sets, appeared to be more important for applications, see, for instance, the V.P. Havin [9] for the analytic functions or the monograph [30, Ch. 5–8] for the Sobolev solutions to systems of differential equations with surjective/injective symbols. For $L^2(G_1)$-solutions to the heat equation or to the parabolic Lamé type operator in a cylinder domain $G_1 = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$ we refer to recent papers [26] and [34], respectively.

Naturally, one should explain why the similar approximation theorems were overlooked in the framework of general theory of parabolic equations. Actually, there are two principal reasons for this fact. The first one is the above mentioned theorem on the uniform approximation of solutions to operators with the constant coefficients in convex domains by the exponential-polynomial solutions covering many needs of approximation results in the Theory of Partial Differential Equations. The second reason is that typical problems for parabolic operators (such as the Cauchy problem in $\mathbb{R}^n \times [0, T)$ or the problems in the cylinder domains related with the Dirichlet or Neumann data on the lateral side, combined with the Cauchy data on the base of the domain) were handled mostly with the Fourier method of separation of variables in the Bochner-Sobolev spaces, where approximation with respect to the functions of the space variables is crucial, see, for example, [16], [32], or by the integral representation method in the Hölder spaces where the approximation is not usually used, see, for instance, [7], [15].

On the other hand, the non-standard Cauchy problem for solutions to the parabolic equations in cylinder domains with the Cauchy data on a part of the lateral side of the cylinder behaves much more like the ill-posed Cauchy problem for solutions to elliptic systems, see, for instance, [27], [30, Ch. 10] for the elliptic theory and [14], [20], [34], [24], for the parabolic theory in the Sobolev type spaces. The approximation theorems for Runge's pairs in the Lebesgue type spaces are crucial for this type of problems because they provide both dense solvability and a possibility to construct the approximate solutions to the problems, see [21], [19], [27, Theorem 7.6], and [34, Corollary 3.5], respectively. Actually, the non-standard ill-posed Cauchy problem for the parabolic operators plays essential role in the development of non-invasive methods of Cardiology, see, for instance, [11, §4]. Please, note that formulations of such problems do not usually use initial data at a suitable time $t = t_0$ and hence we do not need to pose them in the cylinder domains.

In the present paper we consider approximation theorems for solutions to the strongly uniformly parabolic matrix differential operator $\mathcal{L} = \partial_t + L$ on a strip $\mathbb{R}^n \times \mathcal{I}$, where $\mathcal{I}$ is an interval on the time axis, L is a strongly elliptic operator with bounded regular coefficients on the strip admitting the bilateral fundamental solution and possessing the Unique Continuation Property with respect to the space variables. As far as the topic targets the problems without initial data, the presented approach is fit for both the parabolic and backwards parabolic operators.

Section §1 is devoted to the uniform approximation on compact subsets of the domain $G_1 \subset \mathbb{R}^{n+1}$ of the elements of the space $S_{\mathcal{L}}(G_1)$ consisting of continuous solutions to the equation $\mathcal{L}u = 0$ in G_1 by the elements of the space $S_{\mathcal{L}}(G_2)$ where the domain G_2 contains G_1. Similarly to the case of the heat equation, the space $S_{\mathcal{L}}(G_2)$ is dense in $S_{\mathcal{L}}(G_1)$ if and only if for each $t \in \mathcal{I}$ the complement of the set $G_1(t) = \{x \in \mathbb{R}^n : (x,t) \in G_1\}$ in the set $G_2(t)$ has no compact (non-empty) components. Actually, taking in accounts a small gap in the proof of the approximation theorem for the heat equation in [4] (that was discovered by [8]) concerned with rather general assumptions on the structure of the domains considered as Runge's pairs in this particular situation, we additionally assume some regularity of the domains' boundaries, see assumption (A), in section §1.

In the section §2 we consider a more subtle problem of approximation of solutions to parabolic operator $\mathcal{L}$ from the Lebesgue class $L^2(G_1)$ by more regular solutions in a bigger domain G_2. We present a solution to this problem in the case where G_1 is a bounded domain with piece-wise smooth boundary with additional geometric restrictions (see $(A1)$, $(A2)$). Finally, as a by-product, we obtain the theorem on existence of bases with double orthogonality property in Sobolev type spaces of solutions to parabolic systems.

1. The uniform approximation

Let $\mathbb{R}^n$, $n \geq 1$, be the n-dimensional Euclidean space with the coordinates $x = (x_1, \dots, x_n)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain (open connected set). As usual, denote by $\overline{\Omega}$ the closure of Ω, and by $\partial\Omega$ its boundary.

We consider functions over $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. As usual, for $s \in \mathbb{Z}_+$ we denote by $C^s(\Omega)$ and $C^s(\overline{\Omega})$ the spaces of all s times continuously differentiable functions on Ω and $\overline{\Omega}$, respectively. Next, for $p \in [0,1)$, we denote by $C^{s,p}(\overline{\Omega})$ the standard Hölder spaces. The spaces $C^{s,p}(\overline{\Omega})$ are known to be Banach spaces with the standard norms and the $C^{s,p}(\Omega)$ are the Fréchet spaces with the standard semi-norms, see, for instance, [13], [15].

Let also $L^2(\Omega)$ be the Lebesgue space over Ω with the standard inner product $(u,v)_{L^2(\Omega)}$ and let $H^s(\Omega)$, $s \in \mathbb{N}$, be the Sobolev space with the standard inner product $(u,v)_{H^s(\Omega)}$. As usual, we consider the Sobolev space $H^{-s}(\Omega)$, $s \in \mathbb{N}$, as the dual space $H^s_0(\Omega)$ where $H^s_0(\Omega)$ is the closure of the space $C^\infty_{\text{comp}}(\Omega)$ consisting of smooth functions with compact supports in Ω.

For a natural number k, it is convenient to denote by $\mathbf{C}^{s,p}_k(\Omega)$ the space of k-vector functions with components of the class $C^{s,p}(\Omega)$ and, similarly, for the space $\mathbf{L}^2_k(\Omega)$, etc.

Let $\mathcal{I}$ be a finite or infinite interval on the time axis and let L be a $(k \times k)$-matrix differential operator with continuous coefficients in a strip $\mathbb{R}^n \times \mathcal{I}$ of an even order $2m$:

$$L = \sum_{|\alpha| \leq 2m} L_\alpha(x,t)\partial^\alpha_x$$

where $L_\alpha(x,t)$ are $(k \times k)$-matrices with entries as described above and such that $L^*_\alpha(x,t) = L_\alpha(x,t)$ for all multi-indexes $\alpha \in \mathbb{N}^n$ with $|\alpha| = 2m$, all $t \in \mathcal{I}$ and all $x \in \mathbb{R}^n$. Consider strongly uniformly (Petrovsky) $2m$-parabolic operator

$$\mathcal{L} = \partial_t + L,$$

see, for instance, [6], [29]. More precisely, this means that there is a positive constant c_0 such that

$$(-1)^m w^* \Big(\sum_{|\alpha|=2m} L_\alpha(x,t)\zeta^\alpha \Big) w \geq c_0 |w|^2 |\zeta|^{2mk}$$

for all $t \in \mathcal{I}$, all $x \in \mathbb{R}^n$, all $\zeta \in \mathbb{R}^n \setminus \{0\}$ and all $w \in \mathbb{C}^k \setminus \{0\}$; here w^* is the transposed and complex adjoint vector for the complex vector $w \in \mathbb{C}^k$. In particular, for each fixed $t_0 \in \mathcal{I}$ the operator L is (Petrovsky) elliptic with respect to the space variables x, i.e.

$$\det \Big(\sum_{|\alpha|=2m} L_\alpha(x,t_0)\zeta^\alpha \Big) \neq 0$$

for all $x \in \mathbb{R}^n$ and all $\zeta \in \mathbb{R}^n \setminus \{0\}$.

Following [6, pp. 245-246], we assume also that the operator $\mathcal{L}$ satisfies the following assumptions:

- (α_1) the coefficients $L_\alpha(x,t)$ are uniformly continuous over the strip $\mathbb{R}^n \times \mathcal{I}$ for $|\alpha| = 2m$;
- (α_2) the coefficients $L_\alpha(x,t)$ are uniformly bounded over the strip $\mathbb{R}^n \times \mathcal{I}$;
- (α_3) the coefficients $L_\alpha(x,t)$ satisfy the Hölder condition with respect to the space variables x uniformly over the strip $\mathbb{R}^n \times \mathcal{I}$;
- (α_4) each coefficient $L_\alpha(x,t)$ has partial derivative $\partial_x^\alpha L_\alpha(x,t)$ the strip $\mathbb{R}^n \times \mathcal{I}$ satisfying assumptions (α_1), (α_2), (α_3).

Under these assumptions on the coefficients of the operator it admits a unique fundamental solution $\Phi(x,y,t,\tau)$ possessing standard estimates [6, formulas (2.16), (2.17)]) and the normality property ([6, Property 2.2]), i.e.

$$\mathcal{L}_{x,t}\Phi(x,y,t,\tau) = I_k \delta(x-y,t-\tau), \tag{1.1}$$

where $\delta(x,t)$ is the Dirac functional supported at the point (x,t) and

$$\mathcal{L}^*_{y,\tau}\Phi^*(x,y,t,\tau) = I_k \delta(x-y,t-\tau). \tag{1.2}$$

We are going to investigate solutions to $2m$-parabolic equations in non-cylinder domains of special type, see for instance [29, §22]. Namely let G be a domain in the strip $\mathbb{R}^n \times \mathcal{I}$ and let

$$T_1(G) = T_1 = \inf_{(x,t)\in G} t, \; T_2(G) = T_2 = \sup_{(x,t)\in G} t.$$

Consider bounded sets $G(t) = \{x \in \mathbb{R}^n : (x,t) \in G\}$, $t \in \mathcal{I}$, playing an essential role in the sequel. We assume that the boundary of G satisfy the following property.

- (A) For any numbers t_3, t_4 such that $T_1 < t_3 < t_4 < T_2$ the set $\Gamma_{t_3,t_4} = \cup_{t\in[t_3,t_4]}\partial G(t)$ is a Lipschitz surface in $\mathbb{R}^{n+1}$.

For us, the primary interest for studying parabolic equations in such domains is a possibility of applications in the Cardiology, where $G(t) \subset \mathbb{R}^3 \times (T_1,T_2)$ is the shape of the human myocardium at the time t (see, for instance, [2] for the related mathematical models or [11] for the particular bi-domain model).

With this purpose we need the standard Banach anisotropic spaces $C^{2sm,s}(\overline{G})$, see for instance, [29, §22], with the norm:

$$\|u\|_{C^{2ms,s}(\overline{G})} = \sum_{|\alpha|+2mj\leq 2ms} \|\partial_x^\alpha \partial_t^j u\|_{C(\overline{G})}.$$

The corresponding anisotropic spaces $C^{2sm,s}(G)$ are Fréchet spaces with the topology of the uniform convergence on compact subsets of G with all the derivatives $\partial_x^\alpha \partial_t^j$, $|\alpha| + 2mj \leq 2ms$, see, for instance, [28].

Now, $S_{\mathcal{L}}(G)$ be the set of all the continuous k-vector functions on G, satisfying (homogeneous) equation

$$\mathcal{L}u = 0 \text{ in } G \tag{1.3}$$

in the sense of distributions. We endow this space with the standard topology of the uniform convergence on compact subsets of G. Next, we note that estimates [6, formulas (2.16), (2.17)]) for the fundamental solution imply the standard interior a priori estimates for solutions of the operator $\mathcal{L}$, see, for instance, [29, §19], or [7, Ch. 4, §2] for the second order operators. This means that all the continuous solutions to equation (1.3) are $(2m, 1)$-differentiable on their domain, i.e. the following continuous embedding holds true:

$$S_{\mathcal{L}}(G) \subset \mathbf{C}^{2m,1}_k(G).$$

In particular, this means that $S_{\mathcal{L}}(G)$ is a closed subspace in $\mathbf{C}_k(G)$ and it is a Fréchet space itself (with the standard Fréchet topology inducing the standard uniform convergence together with all the derivatives on compact subsets of G).

We also need one more assumption on the operator $\mathcal{L}$: the Unique Continuation Property with respect to the space variables. Namely,

(UCP) For any solution $u \in S_{\mathcal{L}}(G)$ and any $t_0 \in (T_1, T_2)$, if $u(x, t_0) = 0$ for all x from an open subset $\omega \subset G(t_0)$ then $u \equiv 0$ in the open connected component of $G(t_0)$, containing ω.

Of course, if the $\mathcal{L}$ is the operator with constant coefficients then all the assumptions (α_1), (α_2), (α_3), (α_4) and (UCP) are fulfilled. Besides, they hold true if coefficients of the operator $\mathcal{L}$ are smooth, bounded and real analytic with respect to the variables x for each $t \in \mathcal{I}$. In this case, the elements of $S_{\mathcal{L}}(G)$ are actually smooth in G and they are real analytic with respect to the space variable $x \in G(t)$ for all $t \in (T_1, T_2)$, see, for instance [6]. Moreover, the following continuous embedding holds true in this particular situation:

$$S_{\mathcal{L}}(G) \subset \mathbf{C}^{\infty}_k(G).$$

Next, following C. Runge [25] and [4], we call domains $G_1 \subset G_2 \subset \mathbb{R}^{n+1}$ the $\mathcal{L}$-Runge's pair if $S_{\mathcal{L}}(G_2)$ is everywhere dense in $S_{\mathcal{L}}(G_1)$. The following approximation theorem is quite similar to the corresponding statement for the heat equation, see [10] for the case $G_2 = \mathbb{R}^{n+1}$ or [4] for the general case.

Theorem 1.1. *Let $\mathcal{L}$ satisfy assumptions (α_1), (α_2), (α_3), (α_4), (UCP). Let also $G_1 \subset G_2$ be domains in the strip $\mathbb{R}^n \times \mathcal{I}$ and G_1 satisfy assumption (A). Then $S_{\mathcal{L}}(G_2)$ is everywhere dense in $S_{\mathcal{L}}(G_1)$ if and only if for each $t \in \mathcal{I}$ the complement of the set $G_1(t)$ has no compact (non-empty) components in the set $G_2(t)$.*

Proof. We slightly modify the proof from [4] for the solutions to the heat equation, using the duality theorems from modern functional analysis, see, for instance, [5] and the classical a priori estimates for parabolic equations, see, for example, [6], [7].

As we have noted above, the space $S_{\mathcal{L}}(G_1)$ is a closed subspace of the space $\mathbf{C}_k(G_1)$, endowed with the standard topology of the uniform convergence on compact subsets of G_1. Then the Khan-Banach Theorem implies that G_1, G_2 is a $\mathcal{L}$-Runge's pair if and only if any continuous functional F on $\mathbf{C}_k(G_1)$ annihilating the space $S_{\mathcal{L}}(G_2)$ also annihilates the space $S_{\mathcal{L}}(G_1)$.

On the other hand, according the Riesz Theorem, see, for instance, [5, Theorem 4.10.1], any element F of the dual space $\mathbf{C}^*_k(G_1)$ for $\mathbf{C}_k(G_1)$ can be presented with the use of a (k-vector valued) Radon measure μ_F with the compact support $K(\mu_F)$ in G_1 , i.e.

$$F(u) = \langle u, d\mu_F \rangle \text{ for all } u \in S_{\mathcal{L}}(G_1). \tag{1.4}$$

Of course, the operator $\mathcal{L}^*$ is backwards-parabolic and, for any solution $v(y,\tau)$ to the equation $\mathcal{L}^* v = 0$, the vector $w(y,\tau) = v(y,-\tau)$ is a solution to parabolic equation $(\partial_\tau + L^*_y) w = 0$. Thus, if $F \in \mathbf{C}^*_k(G_1)$ annihilates the space $S_{\mathcal{L}}(G_2)$ then, according to (1.1), for the following vector W we have

$$W(y,\tau) = \langle \Phi^*(x,y,t,\tau), d\mu_F(x,t) \rangle = 0 \text{ for all } (y,\tau) \notin G_2. \tag{1.5}$$

But (1.2) implies that

$$\mathcal{L}^* W = d\mu_F \text{ in } \mathbb{R}^n \times \mathcal{I}, \tag{1.6}$$

in the sense of distributions and, in particular,

$$\mathcal{L}^* W = 0 \text{ in } \Big(\mathbb{R}^n \times \mathcal{I} \Big) \setminus K(\mu_F). \tag{1.7}$$

Therefore $W(y,\tau)$ is C^{2m}-smooth with respect to y in $\mathbb{R}^n \setminus K(\mu_F)(\tau)$ for each $\tau \in \mathcal{I}$ where $K(\mu_F)(\tau) = \{x \in \mathbb{R}^n : (x,\tau) \in K(\mu_F)\}$.

However the complement $G_2(t) \setminus G_1(t)$ has no compact components in $G_2(t)$ and hence each component of $\mathbb{R}^n \setminus G_1(t)$ intersects with $\mathbb{R}^n \setminus G_2(t)$ by a non-empty open set for each $t \in (T_1, T_2)$. Thus, (1.5) and the Unique Continuation Property (UCP) imply that

$$W(y,\tau) = 0 \text{ in } \mathbb{R}^n \setminus \overline{G_1(\tau)} \text{ for all } \tau \in \mathcal{I},$$

i.e. the vector W is supported in $\overline{G}_1$. Following [10], we may even say that W is supported in

$$\cup_{\tau \in (T_1,T_2)} \hat{K}_{G_1}(\mu_F)(\tau) \subset \overline{G}_1,$$

where $\hat{K}_{G_1}(\mu_F)(\tau)$ is the union of $K(\mu_F)(\tau)$ with all the the components of $G_1(\tau) \setminus K(\mu_F)(\tau)$ that are relatively compact in $G_1(\tau)$.

Of course, if W is compactly supported in G_1 then we may take a function $\varphi \in C^\infty(G_2)$ compactly supported in G_1 such that $\varphi \equiv 1$ on a neighbourhood of $\operatorname{supp}(W)$. Then, by the Leibniz rule, for each $u \in S_{\mathcal{L}}(G_1)$ we have

$$\mathcal{L}(\varphi u) = \varphi \mathcal{L} u + (\partial_t \varphi) u + \sum_{|\alpha| \le 2m} L_\alpha(x) \sum_{\beta+\gamma=\alpha, |\gamma| \ge 1} C_{\beta,\gamma} (\partial^\gamma \varphi)(\partial^\beta u) =$$

$$(\partial_t \varphi) u + \sum_{|\alpha| \le 2m} L_\alpha(x) \sum_{\beta+\gamma=\alpha, |\gamma| \ge 1} C_{\beta,\gamma} (\partial^\gamma \varphi)(\partial^\beta u),$$

where $C_{\beta,\gamma}$ are binomial type coefficients; in particular,

$$\operatorname{supp}(\mathcal{L}(\varphi u)) \cap \operatorname{supp}(W) = \emptyset.$$

Therefore (1.4) and (1.6) yield

$$F(u) = \langle u, d\mu_F \rangle = \langle u, \mathcal{L}^* W \rangle = \langle (\varphi u), \mathcal{L}^* W \rangle = \langle \mathcal{L}(\varphi u), W \rangle = 0, \tag{1.8}$$

i.e. F annihilates $S_{\mathcal{L}}(G_1)$, too.

As Diaz [4, p. 644] noted, complications may arise if W is not compactly supported in G_1 even in the case where $\mathcal{L}$ is the heat operator. Nevertheless he considered more general type of domains and for this reason the related proof was rather complicated.

In our more simple situation we see that, as μ_F is compactly supported in G_1, there are numbers T_3, T_4 such that $T_1 < T_3 < T_4 < T_2$ and $K(\mu_F)(\tau) = \emptyset$ for all $\tau < T_3$ and $\tau > T_4$, i.e. W is supported in $\cup_{\tau\in[T_3,T_4]}\hat{K}_{G_1}(\mu_F)(\tau)$. Moreover by the the assumptions $(\alpha_1) - (\alpha_4)$ on the operator $\mathcal{L}$, the vector W belongs to $\mathbf{C}^{2m,1}_k(\Big(\mathbb{R}^n \times \mathcal{I}\Big) \setminus K(\mu_F))$ and, in particular, $W = 0$ in $\mathbf{C}^{2m,1}_k(\Big(\mathbb{R}^n \times \mathcal{I}\Big) \setminus G_1)$.

Now, according to assumption (A), the union $\Gamma_{T_3,T_4} = \cup_{t\in[T_3,T_4]}\partial G(t)$ is a Lipschitz surface in $\mathbb{R}^{n+1}$. If necessary, increasing T_4 and decreasing T_3 we see that there is a domain $\tilde{G} \subset G_1$ with Lipschitz boundary consisting of the surface Γ_{T_3,T_4} and the sets $G(T_3)$ and $G(T_4)$ such that the measure μ_F is compactly supported in $\tilde{G}$. Let $\tilde{G}_\varepsilon$ be the sets in $\tilde{G}$ such that the distance between the boundaries of $\tilde{G}_\varepsilon$ and $\tilde{G}$ be less than ε. Then there is a number $\varepsilon_0 > 0$ such that

- the set $\tilde{G}_{\varepsilon_0}$ is again is a domain with Lipschitz boundary and the measure μ_F is compactly supported in $\tilde{G}_\varepsilon$.
- $\tilde{G} \setminus \tilde{G}_{\varepsilon_0}$ has no compact components in $\tilde{G}$.

Replacing the support of measure μ_F with $\overline{\tilde{G}_{\varepsilon_0}}$, we see that $\cup_{\tau\in[T_3,T_4]}\hat{K}_{G_1}(\mu_F)(\tau) \subset \overline{\tilde{G}_{\varepsilon_0}}$ and hence W is compactly supported in both $\tilde{G}$ and G_1.

Back, let there be a number $t_0 \in \mathbb{R}$ such that the set $G_1(t_0)$ have a compact (non-empty) component K_0 in the set $G_2(t_0)$. in $\mathbb{R}^n$. Fix a point $y_0 \in K_0$. Then any vector column $U_j(x,t)$, $1 \le j \le k$, of the fundamental matrix $\Phi(x - y_0, t - t_0)$ belongs to the space $S_{\mathcal{L}}(G_1)$. If the vector function $U_j(x,t)$ can be approximated in $\mathbf{C}^{2m,1}_k(G_1)$ by a sequence $\{u^{(i)}\}_{i\mathbb{N}}$ from the space $S_{\mathcal{L}}(G_2)$ then this sequence should satisfy the Cauchy criterion in the metrizable space $\mathbf{C}^{2m,1}_k(G_1 \setminus G_3)$ with a neighbourhood G_3 of the point (y_0,t_0). Then by the standard a priori estimates for parabolic systems we conclude that sequence $\{u^{(i)}\}_{i\mathbb{N}}$ converges, actually, to a vector function $U(x,t)$ in the space of smooth k-vector functions over a neighbourhood of the point (y_0,t_0). In particular, assumption (UCP) for the system $\mathcal{L}$ with respect to the space variables implies that the vector function $U_j(x,t) \in S_{\mathcal{L}}((\mathbb{R}^n \times \mathcal{I}) \setminus (y_0,t_0))$ uniquely extends to a neighbourhood G_4 of the point (y_0,t_0) as a solution $\tilde{U}_j(x,t)$ to $\mathcal{L}$. Thus, we obtain a contradiction because for the matrix $\tilde{\Phi}(x,t)$ with columns $\tilde{U}_j(x,t)$, $1 \le j \le k$, we have $\mathcal{L}\tilde{\Phi}(x,t) = 0$ in G_4 but $\mathcal{L}\Phi(x - y_0, t - t_0)$ coincides with the δ-functional concentrated at the point (y_0,t_0). □

2. The approximation in the mean

In this section we discuss an approximation theorem for solutions to the operator $\mathcal{L}$ belonging to the Lebesgue spaces. Actually, it is quite similar to the approximation theorems for elliptic operators mentioned in the introduction and the approximation theorem with uniform convergence on compact subsets for parabolic systems proved in the previous section. Also, they are known for the heat equation or for the parabolic Lamé system in cylinder domains with rather regular lateral surfaces, see, for instance, [26] or [34].

Investigating spaces of solutions to $2m$-parabolic equation, we need the anisotropic Sobolev spaces than $H^{2ms,s}(G)$, $s \in \mathbb{Z}_+$, in a domain $G \subset \mathbb{R}^n \times \mathcal{I}$ with the standard inner product,

$$(u,v)_{H^{2ms,s}(G)} = \sum_{|\alpha|+2mj\leq 2ms} (\partial_x^\alpha \partial_t^j u, \partial_x^\alpha \partial_t^j v)_{L^2(G)}$$

Also, for $\gamma \in \mathbb{Z}_+$, we denote by $H^{\gamma,2sm,s}(G)$ the set of all functions $u \in H^{2sm,s}(G)$ such that $\partial_x^\beta u \in H^{2ms,s}(G)$ for all $|\beta| \leq \gamma$. As before, it is convenient to denote by $\mathbf{H}_k^{2ms,s}(G)$ the space of all the k-vector functions with the components from $H^{2ms,s}(G)$, and similarly for the spaces $\mathbf{H}_k^{\gamma,2ms,s}(G)$, etc.

We also will use the so-called Bocher spaces of functions depending on (x,t) from the strip $\mathbb{R}^n \times [T_1,T_2]$ with finite numbers $T_1 < T_2$. Namely, for a Banach space $\mathcal{B}$ (for example, the space of functions on a sub-domain of $\mathbb{R}^n$) and $p \geq 1$, we denote by $L^p(I,\mathcal{B})$ the Banach space of all the measurable mappings $u : [T_1,T_2] \to \mathcal{B}$ with the finite norm

$$\|u\|_{L^p([T_1,T_2],\mathcal{B})} := \| \|u(\cdot,t)\|_{\mathcal{B}} \|_{L^p([T_1,T_2])},$$

see, for instance, [16, ch. §1.2], [32, ch. III, § 1].

The space $C([T_1,T_2],\mathcal{B})$ is introduced with the use of the same scheme; this is the Banach space of all the continuous mappings $u : [T_1,T_2] \to \mathcal{B}$ with the finite norm

$$\|u\|_{C([T_1,T_2],\mathcal{B})} := \sup_{t\in[T_1,T_2]} \|u(\cdot,t)\|_{\mathcal{B}}.$$

Let $\mathbf{H}_{k,\mathcal{L}}^{\gamma,2sm,s}(G) = \mathbf{H}_k^{\gamma,2sm,s}(G) \cap S_{\mathcal{L}}(G)$, $s \in \mathbb{Z}_+$, $\gamma \in \mathbb{Z}_+$. By the discussion in Section §1, the space $\mathbf{H}_{k,\mathcal{L}}^{2m,1}(G)$ is a closed subspace of the Sobolev space $\mathbf{H}_k^{2m,1}(G)$. Similarly, if coefficients of the operator $\mathcal{L}$ are smooth, bounded and real analytic with respect to the variables x for each $t \in \mathcal{I}$, then $\mathbf{H}_{k,\mathcal{L}}^{\gamma,2sm,s}(G)$, $\mathbf{C}_{k,\mathcal{L}}^{2ms,s}(\overline{G}) = \mathbf{C}_k^{2ms,s}(\overline{G}) \cap S_{\mathcal{L}}(G)$, $\mathbf{C}_{k,\mathcal{L}}^{\infty}(\overline{G}) = \mathbf{C}_k^{\infty}(\overline{G}) \cap S_{\mathcal{L}}(G)$ are closed subspaces, consisting of solutions to equation (1.3), in the spaces $\mathbf{H}_k^{\gamma,2sm,s}(G)$, $\mathbf{C}_k^{2ms,s}(\overline{G})$ and $\mathbf{C}_k^{\infty}(\overline{G})$, respectively.

Also, we need the space $S_{\mathcal{L}}(\overline{G_2})$, defined as follows:

$$\cup_{G'\supset\overline{G}} S_{\mathcal{L}}(G'),$$

where the union is with respect to all the domains $G' \subset \mathbb{R}^n \times (t_1,t_2)$, containing the closure of the domain G. It follows from the a priori estimates discussed in §1 that the following (continuous) embeddings

$$S_{\mathcal{L}}(\overline{G}) \subset \mathbf{C}_{k,\mathcal{L}}^{2m,1}(\overline{G}) \subset \mathbf{H}_{k,\mathcal{L}}^{2m,1}(G) \tag{2.1}$$

are fulfilled. Of course, if we additionally know that the coefficients of the operator $\mathcal{L}$ are constant or smooth and real analytic with respect to the space variables then the operator is hypoelliptic and the following (continuous) embeddings

$$S_{\mathcal{L}}(\overline{G}) \subset \mathbf{C}_{k,\mathcal{L}}^{\infty}(\overline{G}) \subset \mathbf{H}_{k,\mathcal{L}}^{\gamma,2ms,s}(G_2) \tag{2.2}$$

hold true $\gamma, s \in \mathbb{Z}_+$.

To prove an approximation theorem for the spaces of the Lebesgue solutions to $\mathcal{L}$, we need more regularity of ∂G:

(A1) For each $t \in [T_1,T_2]$, the sets $G(t) = \{x \in \mathbb{R}^n : (x,t) \in G\}$, are domains in $\mathbb{R}^n$ with C^{2m}-boundaries if $n \geq 2$ or the union of finite numbers of intervals if $n = 1$.

(A2) The boundary ∂G of is the union $G(T_1) \cup G(T_2) \cup \Gamma$, where

$$\Gamma = \cup_{t\in(T_1,T_2)} \partial G(t)$$

is a $C^{2m,1}$-smooth surface of with no points where the tangential planes are parallel to the coordinate plane $\{t = 0\}$, i.e. we have

$$\sum_{j=1}^{n} (\nu_j(x,t))^2 \geq \varepsilon_0 \text{ for all } (x,t) \in \Gamma$$

with a positive number ε_0.

Under these assumptions we easily see that functions from the space $H^{2m,1}(G)$ and some of their partial derivatives have reasonable traces on ∂G. The definition, the uniqueness and an existence theorem for traces of functions from anisotropic spaces can be found in [3, §10] (see also [33, Ch 2.] for non-regular cases $n = 2$). In our particular situation we may specify the traces by hands (cf. [22, Ch.3, §7] for anisotropic spaces in cylinder domains). With this purpose, we denote by $Y_{s-1/2}(\Gamma)$ the closure of $C^s(\Gamma)$, $s \in \mathbb{N}$, with respect to the norm

$$\| \cdot \|_{Y_{s-1/2}(\Gamma)} = \int_{T_1}^{T_2} \| \cdot \|_{H^{s-1/2}_k(\partial G(t))} dt;$$

by the construction, these are Hilbert spaces embedded continuously into $L^2(\Gamma)$.

Of course, if $G = \Omega \times (T_1, T_2)$ is a cylinder domain in $\mathbb{R}^{n+1}$ with the base Ω being a domain with the boundary of class C^s, then $\Gamma = \partial\Omega \times (T_1, T_2)$ and $Y_{s-1/2}(\Gamma)$ coincides with the Bochner space $L^2([T_1, T_2], H^{s-1/2}(\partial\Omega))$ for each α with $|\alpha| \leq s - 1$.

Lemma 2.1. *Let G be a relatively compact domain in $\mathbb{R}^n \times \mathcal{I}$ satisfying (A1), (A2). Then any vector $w \in H^{2m,1}(G)$ has well-defined trace in ∂G. Besides, its derivatives $\partial^\alpha v$ have traces on Γ of the class $Y_{2m-|\alpha|-1/2}(\Gamma)$ for each $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq 2m - 1$.*

Proof. First, we note that the space $H^{2m,1}(G)$ is continuously embedded into the isotropic Sobolev space $H^1(G)$ and hence the trace $w_{|\partial G}$ is well-defined. Actually, it belongs to $H^{1/2}(\partial G)$, see, for instance, [3].

Next, by the structure of the domain and the Fubini theorem, we have

$$\sum_{|\alpha|\leq 2m} \|\partial^\alpha_y w\|^2_{L^2(G)} = \sum_{|\alpha|\leq 2m} \int_{T_1}^{T_2} \int_{G(\tau)} |\partial^\alpha_y v(w,\tau)|^2 dy d\tau =$$

$$\int_{T_1}^{T_2} \|w(\cdot,\tau)\|^2_{H^{2m}(G(\tau))} d\tau$$

for any $w \in C^{2m}(\overline{G})$.

Now the standard Trace Theorem for the Sobolev spaces applied for the spaces $H^{2m}(G(t))$ yields

$$\|\partial^\alpha_y h\|_{H^{2m-|\alpha|-1/2}(\partial G(t))} \leq C_{G(t),\alpha} \|\partial^\alpha_y h\|_{H^{2m-|\alpha|}(G(t))}$$

for all $|\alpha| \leq 2m - 1$, for all $h \in H^{2m}(G(t))$ and each $t \in (T_1, T_2)$. Hence

(2.3)

$$\int_{T_1}^{T_2} \sum_{|\alpha|\leq 2m-1} \|\partial^\alpha_y w(\cdot,\tau)\|_{H^{2m-|\alpha|-1/2}(\partial G(t))} d\tau \leq C(G) \int_{T_1}^{T_2} \|w(\cdot,\tau)\|^2_{H^{2m}(G(\tau))} d\tau$$

for any $w \in C^{2m}(\overline{G})$; here the positive constant

$$C(G) = \sup_{|\alpha|\le 2m-1} \sup_{t\in[T_1,T_2]} C_{G(t),\alpha}$$

is finite because the related constants $C_{G(t),\alpha}$ may be chosen to depend on the $(n-1)$-measure of the domains $\partial G(t)$.

According to [3, §14], any vector function $w \in H^{2m,1}(G)$ may be approximated in by vectors from $C^\infty_{\rm comp}(\mathbb{R}^{n+1})$ in the topology of the space $H^{2m,1}(G)$. Pick a sequence $w^{(s)} \subset C^\infty_{\rm comp}(\mathbb{R}^{n+1})$ approximating w in $H^{2m,1}(G)$. Then (2.3) yields that the sequence $\{\partial^\alpha_y w^{(s)}\}$ is fundamental in $Y_{2m-|\alpha|-1/2}(\Gamma)$ for each α with $|\alpha| \le 2m-1$. As $Y_{2m-|\alpha|-1/2}(\Gamma)$ is complete we conclude that for each α with $|\alpha|\le 2m-1$ it converges to an element w_α, i.e. there is a well-defined trace $\partial^\alpha_y w_{|\Gamma} = w_\alpha \in Y_{2m-|\alpha|-1/2}(\Gamma)$ of the function $\partial^\alpha_y w$ on the surface Γ. □

Now we formulate the main result of this section.

Theorem 2.2. *Let $\mathcal{L}$ satisfy assumptions (α_1), (α_2), (α_3), (α_4), (UCP). Let $G_1 \subset G_2$ be domains in $\mathbb{R}^n\times\mathcal{I}$ such that bounded domain G_1 satisfy $(A1)$, $(A2)$. Then $S_{\mathcal{L}}(\overline{G}_2)$ is everywhere dense in $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$ if and only if for each $t\in\mathcal{I}$ the complement of the set $G_1(t)$ has no compact (non-empty) components in the set $G_2(t)$.*

Proof. The proof is similar to the proof of Theorem 1.1. However we need a more regular boundary for the domain G_1.

Sufficiency. Clearly, the set $S_{\mathcal{L}}(\overline{G}_2)$ is everywhere dense in $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$ if and only if the following relations

$$(u,w)_{\mathbf{L}^2(G_1)} = 0 \text{ for all } w \in S_{\mathcal{L}}(\overline{G_2}) \tag{2.4}$$

means precisely for the vector $u \in \mathbf{L}^2_{k,\mathcal{L}}(G_1)$ that $u\equiv 0$ in G_1.

Assume that the complement for each $t\in\mathcal{I}$ the complement of the set $G_1(t)$ has no compact (non-empty) components in the set $G_2(t)$. In order to prove the sufficiency of the statement we will again use the fact that the operator $\mathcal{L}$ admits the bilateral fundamental solution possessing the normality property (1.2).

Let for the vector $u\in\mathbf{L}^2_{\mathcal{L}}(G_1)$ relation (2.4) is fulfilled. Consider an auxiliary vector function

$$v(y,\tau) = \int_{G_1} \Phi^*(x-y,t-\tau)u(x,t)dx\,dt. \tag{2.5}$$

According to (1.1) we have $\mathcal{L}_{x,t}\Phi(x-y,t-\tau)=0$, if $(x,t)\ne(y,\tau)$. That is why $\mathcal{L}_{x,t}\Phi(x-y,t-\tau)=0$ in G_2 for each fixed pair $(y,\tau)\notin\overline{G}_2$, i.e. $\Phi(x-y,t-\tau)\in S_{\mathcal{L}}(\overline{G}_2)$ with respect to variables (x,t) for each fixed pair $(y,\tau)\notin\overline{G}_2$. In particular, relations (2.4) imply

$$v(y,\tau) = 0 \text{ in } \Big(\mathbb{R}^n\times\mathcal{I}\Big)\setminus\overline{G}_2. \tag{2.6}$$

On the other hand, (1.2) yields

$$\mathcal{L}^* v = \chi_{G_1} u \text{ in } \mathbb{R}^n\times\mathcal{I}, \tag{2.7}$$

where χ_{G_1} is the characteristic function of the domain G_1. Obviously,

$$\mathcal{L}^*_{y,\tau} v(y,\tau) = 0 \text{ in } \Big(\mathbb{R}^n\times\mathcal{I}\Big)\setminus\overline{G}_1,$$

and then, by the discussed above properties of the fundamental solution, the vector function v is $C^{2m,1}_{\rm loc}$-vector function in $\Big(\mathbb{R}^n \times \mathcal{I}\Big) \setminus \overline{G}_1$.

Since the complement $G_2(t) \setminus G_1(t)$ has no compact components in $G_2(t)$ then each component of $\mathbb{R}^n \setminus G_1(t)$ intersects with $\mathbb{R}^n \setminus G_2(t)$ by a non-empty open set for all $t \in (T_1, T_2)$. Thus, (2.6) and the Unique Continuation Property (UCP) imply that

$$v(y,\tau) = 0 \text{ in } \Big(\mathbb{R}^n \times \mathcal{I}\Big) \setminus \overline{G}_1. \tag{2.8}$$

Then, in addition, (2.7), (2.8) mean that the vector v is a solution to the Cauchy problem

$$\left\{ \begin{array}{l} \mathcal{L}^* v = \chi_{G_1} u \text{ in } \mathbb{R}^n \times (T_1-\delta, T_2+\delta), \\ v(y, T_2+\delta) = 0 \text{ on } \mathbb{R}^n \end{array} \right.$$

with a sufficiently small $\delta > 0$ such that $[T_1-\delta, T_2+\delta] \subset \mathcal{I}$. Taking in account the natural relation between parabolic and backward-parabolic operators and using arguments from [12, ch. 2, §5 theorem 3], we may conclude that $v \in \mathbf{H}^{2m,1}_k(\mathbb{R}^n \times (T_1-\delta, T_2+\delta))$ and the solution is unique in this class. The regularity of this unique solution to the Cauchy problem can be expressed in term of the Bochner classes, too. Namely, $v \in C([T_1-\delta, T_2+\delta], \mathbf{H}^m_k(\mathbb{R}^n)) \cap L^2([T_1-\delta, T_2+\delta], \mathbf{H}^{2m}_k(\mathbb{R}^n))$, see, for instance, [16], [32, ch. 3, §1], where similar linear problems for are considered. In particular, the vector function v belongs to the space

$$C([T_1-\delta, T_2+\delta], \mathbf{H}^m_k(\mathbb{R}^n)) \cap \mathbf{H}^{2m,1}_k(\mathbb{R}^n \times (T_1-\delta, T_2+\delta)). \tag{2.9}$$

Lemma 2.3. *Any vector v of type* (2.5)*, satisfying* (2.8)*, can be approximated by the vectors from $\mathbf{C}^\infty_{k,{\rm comp}}(G_1)$ in the topology of the Hilbert space $\mathbf{H}^{2m,1}_k(G_1)$.*

Proof. The approximability of Sobolev functions with compact support in a domain is closely related to the notion of trace.

Of course, as $\mathbf{H}^{2m,1}_k(\mathbb{R}^n \times (T_1-\delta, T_2+\delta))$ is embedded continuously into $\mathbf{H}^1_k(G_1)$, we see that $v \in \mathbf{H}^1_{k,0}(G_1)$, (i.e. the trace $v_{|\partial G}) \in \mathbf{H}^{1/2}_k(\partial G_1)$ in the sense of isotropic Sobolev spaces and it equals to zero on ∂G) and then it can be approximated by the vectors from $\mathbf{C}^\infty_{k,{\rm comp}}(G_1)$ in the topology of the Hilbert space $\mathbf{H}^1_k(G_1)$. However, it is not we actually need.

On the other hand, as the boundary of the domain G_1 is rather regular, it satisfies [1, Condition A] and we may use results by [1] to conclude that v may be approximated in $\mathbf{H}^{2m,1}_k(G_1)$ by vectors from the space $\mathbf{C}^\infty_{k,{\rm comp}}(\overline{G}_1)$. Again, it is not precisely what we want.

However, (2.9) implies that traces $v_{|\overline{G(T_j)}}$ of v on the closures of $G(T_j)$ belong to $\mathbf{H}^m_k(G(T_j))$, $1 \le j \le 2$, and they are equal to zero because of (2.8). Moreover, according to Lemma 2.1, the traces of $\partial^\alpha v_{|\Gamma} \in \mathbf{Y}^{2m-|\alpha|-1/2}_k(\Gamma)$ equal to zero on Γ for all $|\alpha| \le 2m-1$.

Now the statement of the lemma easily follows with the use of the standard regularisation, see, for instance, [22, ch. 3, §5, §7,] for the isotropic Sobolev spaces, cf. see [26] for cylinder domains.

Indeed, denote by $h^{(n)}_\delta(x)$ the standard compactly supported function with the support in the ball $B(x,\delta) \subset \mathbb{R}^n$ with the centre at the point x and of the radius $\delta > 0$:

$$h^{(n)}_\delta(x) = \left\{ \begin{array}{ll} 0, & \text{if} \quad |x| \ge \delta, \\ c(\delta) \exp\left(1/(|x|^2-\delta^2)\right), & \text{if} \quad |x| < \delta, \end{array} \right.$$

where $c(\delta)$ is the constant providing equality

$$\int_{\mathbb{R}^n} h^{(n)}_\delta (x)\, dx = 1.$$

Then, as it is well known, for any function $w \in L^1(K)$ on a measurable compact K in $\mathbb{R}^n$, the standard regularisation

$$(R^{(n)}_\delta w)(x) = \int_{\mathbb{R}^n} h^{(n)}_\delta (x-y, t-\tau) w(y,\tau)\, dy d\tau$$

belongs to the space $C^\infty_0 (\mathbb{R}^n)$ for any positive number δ and the support of $R^{(n)}_\delta w)$ lies in a δ-neighbourhood of K.

Next, according t assumptions $(A1)$, $(A2)$ there is a real valued function $\rho(x,t)$ of the class $C^{2m,1}$ in a neighbourhood U of the surface Γ and such that

$$\partial G_1 (t) = \{(x,t) \in \mathbb{R}^n \times [T_1,T_2] : \rho(x,t) < 0\},\ \nabla \rho \neq 0 \text{ in } U\}.$$

Hence, for all sufficiently small numbers $\varepsilon > 0$ the sets

$$G_1^\varepsilon = \{(x,t) \in \mathbb{R}^n : \rho(x,t) < -\varepsilon\}$$

are domains with boundaries of class $C^{2m,1}$ and

$$G_1^\varepsilon \Subset G_1^{\varepsilon'} \Subset G_1,$$

if $0 < \varepsilon' < \varepsilon$, and

$$\lim_{\varepsilon \to +0} \mathrm{mes} (G_1 \setminus \overline{G_1^\varepsilon}) = 0.$$

Moreover the sets $G_1(t)$ are domains with boundaries of class C^{2m} and for the Lebesgue measure of the domain $G_1(t) \setminus \overline{G_1^\varepsilon (t)}$ we have

$$G_1^\varepsilon (t) \Subset G_1^{\varepsilon'} (t) \Subset G_1 (t),\ t \in [T_1,T_2],$$

$$\lim_{\varepsilon \to +0} \mathrm{mes} (G_1(t) \setminus \overline{G_1^\varepsilon(t)}) = 0$$

uniformly with respect to $t \in [T_1,T_2]$. According to [22, ch. 3, §5, lemma 1], there is a positive constant $C_1(G_1)$, depending on the square of the surfaces $G_1(T_j)$, $1 \leq j \leq 2$, only, and such that

$$C_1 (G_1)\, \varepsilon^{-2} \Big(\int_{T_1}^{T_1+\varepsilon} \|\tilde v\|^2_{L^2(G(t))} dt + \int_{T_2-\varepsilon}^{T_2} \|\tilde v\|^2_{L^2(G(t))} dt \Big) \leq \tag{2.10}$$

$$\int_{T_1}^{T_1+\varepsilon} (\|\tilde v\|^2_{L^2(G(t))} + \|\partial_t \tilde v\|^2_{L^2(G(t))}) dt + \int_{T_2-\varepsilon}^{T_2} (\|\tilde v\|^2_{L^2(G(t))} + \|\partial_t \tilde v\|^2_{L^2(G(t))}) dt$$

for any function $\tilde v \in H^1(G_1)$ with zero traces $\tilde v_{|G_1(T_j)}$ on $G_1(T_j)$, $1 \leq j \leq 2$.

Similarly, under assumptions $(A1)$, $(A2)$, if $\partial G(t) \in C^{2m}$ then there is a constant $C_0(\Gamma)$, depending on the square of the surface Γ, only, and such that

$$\Big(\int_{T_1}^{T_2} \|\partial^\alpha \tilde v\|^2_{L^2(G(t)\setminus \overline{G^\varepsilon}(t))} dt \Big)^{1/2} \leq C_0 (\Gamma)\, \varepsilon^{2m-|\alpha|} \Big(\int_{T_1}^{T_2} \|\tilde v\|^2_{H^{2m}(G(t)\setminus \overline{G^\varepsilon}(t))} dt \Big)^{1/2} \tag{2.11}$$

for any function $\tilde v \in H^{2m,1}(G_1)$ with zero traces $\partial^\alpha \tilde v_{|\Gamma}$ on Γ corresponding to $|\alpha| \leq 2m-1$.

Set

$$R_\varepsilon (x,t) = \int_{T_1+\varepsilon/5}^{T_2-\varepsilon/5} h^{(1)}_{\varepsilon/7} (t-\tau) \int_{G_1^{\varepsilon/5}(\tau)} h^{(n)}_{\varepsilon/7} (x-y)\, dy\, d\tau.$$

Using assumptions on G_1 we may choose some constants c_α, c_i independent on x and t such that

$$0 \leq R_\varepsilon(x,t) \leq 1, \tag{2.12}$$

$$|\partial^\alpha_x R_\varepsilon(x,t)| \leq c_\alpha\, \varepsilon^{-|\alpha|},\ |\partial^i_t R_\varepsilon(x,t)| \leq c_i\, \varepsilon^{-i}, \tag{2.13}$$

for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\alpha \in \mathbb{Z}^n_+$, $i \in \mathbb{Z}_+$, cf., for instance, [22, ch. 3, §5].

Fix a sequence $\{v_j\} \subset C^\infty_0(\mathbb{R}^{n+1})$, converging to v in the space $\mathbf{H}^{2m,1}_k(\mathbb{R}^{n+1})$ (as we mentioned above, the existence of such a sequence follows from [3, §14]). Then the functional sequence

$$\{v_{j,\varepsilon}(x,t) = R_\varepsilon(x,t) v_j(x,t)\}$$

lies to $\mathbf{C}^\infty_{k,0}(G_1)$.

By the triangle inequality,

$$\|v - v_{j,\varepsilon}\|_{\mathbf{H}^{2m,1}_k(G_1)} \leq \|v - v_j\|_{\mathbf{H}^{2m,1}_k(G_1)} + \|v_k - v_{j,\varepsilon}\|_{\mathbf{H}^{2m,1}_k(G_1)}. \tag{2.14}$$

As

$$\lim_{j\to+\infty} \|v - v_j\|_{\mathbf{H}^{2m,1}_k(G_1)} = 0, \tag{2.15}$$

then we need to estimate the second summand in the right hand side of formula (2.14), only. However,

$$v_j(x,t) - v_{j,\varepsilon}(x,t) = (1 - R_\varepsilon(x,t)) v_j(x,t)$$

and, in particular,

$$v_j(x,t) - v_{j,\varepsilon}(x,t) = 0 \text{ for all } (x,t) \in \cup_{t=T_1+\varepsilon}^{T_2-\varepsilon} G^\varepsilon_1(t). \tag{2.16}$$

Hence,

$$\begin{gathered} 2^{-1}\|v_j - v_{j,\varepsilon}\|^2_{H^{2m,1}(G_1)} \leq \|(1-R_\varepsilon)\partial_t v_j\|^2_{L^2(G_1)} + \\ \Big\|(\partial_t R_\varepsilon) v_j\Big\|^2_{L^2(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_2]} G^\varepsilon_1(t))} + \\ \sum_{|\alpha|\leq 2m} \|\partial^\alpha((1-R_\varepsilon)v_j)\|^2_{L^2(\cup_{t\in[T_1,T_2]} G_1(t)\setminus G^\varepsilon_1(t))}. \end{gathered} \tag{2.17}$$

Then (2.10), (2.11), (2.12), (2.13), (2.16) and the Fubini theorem imply that

$$\begin{gathered} \sum_{|\alpha|\leq 2m} \|\partial^\alpha(1 - R_\varepsilon v_j)\|^2_{L^2(G_1)} \leq \\ \sum_{|\alpha|\leq 2m} \|(1-R_\varepsilon)(\partial^\alpha v_j\|^2_{L^2(\cup_{t\in[T_1,T_2]} G_1(t)\setminus G^\varepsilon_1(t))} + \\ \sum_{|\beta+\gamma|\leq 2m, \beta\neq 0} \|(\partial^\beta R_\varepsilon)(\partial^\gamma v_j\|^2_{L^2(\cup_{t\in[T_1,T_2]} G_1(t)\setminus G^\varepsilon_1(t))} \leq \\ \sum_{|\alpha|\leq 2m} \|(1-R_\varepsilon)(\partial^\alpha v_j\|^2_{L^2(\cup_{t\in[T_1,T_2]} G_1(t)\setminus G^\varepsilon_1(t))} + \\ \sum_{|\beta+\gamma|\leq 2m, \beta\neq 0} \varepsilon^{2m-|\gamma+\beta|} \|v_j\|^2_{H^{2m,0}(\cup_{t\in[T_1,T_2]} G_1(t)\setminus G^\varepsilon_1(t))} \leq \\ \|v\|_{H^{2m,1}(\cup_{t\in[T_1,T_2]} G_1(t)\setminus \overline{G^\varepsilon_1(t)})} \end{gathered} \tag{2.18}$$

with constants $C, \tilde{C}$, independent on v and ε.

The boundaries of the domains $\cup_{t\in[T_1,T_1+\varepsilon]}G_1(t)$ and $\cup_{t\in[T_2-\varepsilon,T_2]}G_1(t)$ are not smooth, but combining results [22, ch. 3, §5] related to a function v, having the trace vanishing on surfaces $G_1(T_1)$ and $G_2(T_2)$, with bounds (2.12), (2.13), we see that

$$\|(\partial_t R_\varepsilon)v\|^2_{L^2(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_2]}G_1^\varepsilon(t))} \leq \tag{2.19}$$

$$\varepsilon^{-1}\cdot\varepsilon C\left(\|_t v\|^2_{L^2(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_2]}G_1^\varepsilon(t))} + \|\partial_t v\|^2_{L^2(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_2]}G_1^\varepsilon(t))}\right) \leq$$
$$\|v\|_{H^{2m,1}(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_1]}G_1(t)\setminus\overline{G_1^\varepsilon(t)})}$$

with a constant C, independent on v and ε.

Besides, according to (2.12), (2.13),

$$\|(1-R_\varepsilon)\partial_t^i v\|^2_{L^2(G_1)} \leq C\|v\|_{H^{2m,1}(\cup_{t\in[T_1,T_2]}G_1(t)\setminus\overline{G_1^\varepsilon(t)})} \tag{2.20}$$

with a constant C, independent on v and ε.

Using the continuity of the Lebesgue integral with respect to the measure of the integration set, we conclude that

$$\lim_{\varepsilon\to+0}\|v\|_{H^{2m,1}(\cup_{t\in[T_1,T_1+\varepsilon]\cup[T_2-\varepsilon,T_1]}G_1(t))} = 0, \tag{2.21}$$

$$\lim_{\varepsilon\to+0}\|v\|_{H^{2m,1}(\cup_{t\in[T_1,T_2]}G_1(t)\setminus\overline{G_1^\varepsilon(t)})} = 0. \tag{2.22}$$

Finally, combining estimates (2.17)–(2.20) and taking in account (2.21), (2.22), we conclude that the statement of the lemma holds true. □

Next, using lemma 2.3 and fixing a sequence $\{v_k\}\subset\mathbf{C}^\infty_{k,\mathrm{comp}}(G_1)$ converging to the vector v in $\mathbf{H}^{2m,1}_k(G_1)$, we see that

$$\|u\|^2_{\mathbf{L}^2_k(G_1)} = (u,\mathcal{L}^*v)_{\mathbf{L}^2_k(G_1)} = \lim_{i\to+\infty}(u,\mathcal{L}^*v_i)_{\mathbf{L}^2_k(G_1)} = 0,$$

because $\mathcal{L}u=0$ in G_1 in the sense of distributions. Thus, $u\equiv 0$ in G_1, i.e. the sufficiency is proved.

Necessity. Actually, we may use the arguments similar to that in the proof of the necessity of Theorem 1.1. Indeed, let there be a number $t_0\in\mathbb{R}$ such that the set $G_1(t_0)$ have a compact (non-empty) component K_0 in the set $G_2(t_0)$. By the assumptions on the boundaries of the set $G_1(t)$, we may consider K_0 as the closure of a domain D_0 with Lipschitz boundary in $\mathbb{R}^n$. Fix a point $y_0\in D_0$. Then any vector column $U_j(x,t)$, $1\leq j\leq k$, of the fundamental matrix $\Phi(x-y_0,t-t_0)$ belongs to the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$. If the vector function $U_j(x,t)$ can be approximated in $\mathbf{L}^2_k(G_1)$ by a sequence $\{u^{(i)}\}_{i\mathbb{N}}$ from the space $S_{\mathcal{L}}(\overline{G}_2)$ then this sequence should satisfy the Cauchy criterion in the Banach space $\mathbf{L}^2_k(G_1\setminus G_3)$ with a neighbourhood G_3 of the point (y_0,t_0). Then by the standard a priori estimates for parabolic systems we conclude that sequence $\{u^{(i)}\}_{i\mathbb{N}}$ converges, actually, to a vector function $U(x,t)$ in the space of smooth k-vector functions over a neighbourhood of the point (y_0,t_0). In particular, assumption (UCP) to the system $\mathcal{L}$ with respect to the space variables implies that the vector function $U_j(x,t)\in S_{\mathcal{L}}((\mathbb{R}^n\times(0,+\infty))\setminus(y_0,t_0))$ uniquely extends to a neighbourhood G_4 of the point (y_0,t_0) as a solution $\tilde U_j(x,t)$ to $\mathcal{L}$. Thus, we obtain a contradiction because for the matrix $\tilde\Phi(x,t)$ with columns $\tilde U_j(x,t)$, $1\leq j\leq k$, we have $\mathcal{L}\tilde\Phi(x,t)=0$ in G_4 but $\mathcal{L}\Phi(x-y_0,t-t_0)$ coincides with the δ-functional concentrated at the point (y_0,t_0). □

Next, we obtain the following useful statement.

Corollary 2.4. *Let the coefficients of the operator $\mathcal{L}$ be smooth, bounded and real analytic with respect to the variables x for each $t \in \mathcal{I}$. Let also $s, \gamma \in \mathbb{Z}_+$, $G_1 \subset G_2$ be domains in $\mathbb{R}^n \times \mathcal{I}$ such that bounded G_1 satisfy $(A1)$, $(A2)$. If for each $t \in \mathcal{I}$ the complement of the set $G_1(t)$ has no compact (non-empty) components in the set $G_2(t)$ then the spaces $\mathbf{C}^\infty_{k,\mathcal{L}}(\overline{G}_2)$ and $\mathbf{H}^{\gamma,2s,s}_{k,\mathcal{L}}(G_2)$ are everywhere dense in $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$.*

Proof. Follows immediately from theorem 2.2, because of embeddings (2.1). □

Finally, Theorem 2.2 allows to prove the existence of a basis with the double orthogonality property in the spaces of solutions to the operator $\mathcal{L}$ that is very useful to investigate the non-standard ill-posed Cauchy for elliptic and parabolic equations, see, [30, Ch. 12], [27], [34].

Corollary 2.5. *Let $\mathcal{L}$ satisfy assumptions (α_1), (α_2), (α_3), (α_4), (UCP). Let $G_1 \subset G_2$ be domains in $\mathbb{R}^n \times \mathcal{I}$, such that $T_1(G_1) = T_1(G_2)$, $T_2(G_1) = T_2(G_2)$ and bounded domain G_1 satisfy $(A1)$, $(A2)$. If for each $t \in \mathcal{I}$ the complement of the set $G_1(t)$ has no compact (non-empty) components in the set $G_2(t)$ then there is an orthonormal basis $\{b_\nu\}$ is the space $\mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2)$ such that its restriction $\{b_{\nu|G_1}\}$ to G_1 is on orthonormal basis in the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$.*

Proof. By the definition, the space $\mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2)$ is embedded continuously into the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$. We denote by R the natural embedding operator

$$R : \mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2) \to \mathbf{L}^2_{k,\mathcal{L}}(G_1).$$

As the numbers T_1 and T_2 are the same for the domains G_1 and G_2, the Unique Continuation Property (UCP) for the operator $\mathcal{L}$ with respect to the space variables implies that the operator R is injective. Besides, it follows from Theorem 2.2 that the range of the operator R is everywhere dense in the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$.

By the definition, the anisotropic Sobolev space $\mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2)$ is embedded continuously to the isotropic Sobolev space $\mathbf{H}^1_{k,\mathcal{L}}(G_2)$. Besides, by Rellich-Kondrashov theorem the embedding $\mathbf{H}^1_k(G_1) \to \mathbf{L}^2_k(G_1)$ is compact. Thus, taking in account the continuous embedding $\mathbf{H}^1_k(G_2) \to \mathbf{H}^1_k(G_1)$, we see that the natural embedding operator R is compact.

Finally, [27, example 1.9] implies that the complete system of eigen-vectors of the compact self-adjoint operator $R^*R : \mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_2) \to \mathbf{H}^{2m,1}_{k,\mathcal{L}}(G_1)$ is the basis looked for; here R^* is the adjoint operator for R in the sense of the Hilbert space theory. □

Corollary 2.6. *Let the coefficients of the operator $\mathcal{L}$ be smooth, bounded and real analytic with respect to the variables x for each $t \in \mathcal{I}$. Let also $s, \gamma \in \mathbb{Z}_+$, $G_1 \subset G_2$ be domains $\mathbb{R}^n \times \mathcal{I}$, such that $T_1(G_1) = T_1(G_2)$, $T_2(G_1) = T_2(G_2)$ and bounded domain G_1 satisfy $(A1)$, $(A2)$. If for each $t \in \mathcal{I}$ the complement of the set $G_1(t)$ has no compact (non-empty) components in the set $G_2(t)$ then there is an orthonormal basis $\{b_\nu\}$ is the space $\mathbf{H}^{\gamma,2ms,s}_{k,\mathcal{L}}(G_2)$ such that its restriction $\{b_{\nu|G_1}\}$ to G_1 is on orthonormal basis in the space $\mathbf{L}^2_{k,\mathcal{L}}(G_1)$.*

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